THE NEW EXACT SOLITARY SOLUTIONS FOR THE (3+1)-DIMENSIONAL ZAKHAROV-KUZNETSOV EQUATION USING THE RICCATI EQUATION

by

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In this paper, a non-linear (3+1)-dimensional Zakharov-Kuznetsov equation is investigated by employing the subsidiary equation method, which arises in quantum magneto plasma. The periodic solutions, rational wave solutions, soliton solutions for the quantum Zakharov-Kuznetsov equation which play an important role in mathematical physics are obtained with the help of the Riccati equation expansion method. Meanwhile, the electrostatic potential can be accordingly obtained. Compared to the other methods, the exact solutions obtained will extend on earlier reports by using the Riccati equation.

Key words: *Zakharov-Kuznetsov equation, subsidiary equation, exact solitary, Riccati equation*

Introduction

Over the past decades, the non-linear PDE which arise in various fields of non-linear science are vastly studied to reveal the physical systems of all kinds of situations [1-11]. In the study of mathematical models, we pay more attention to find the exact solution and solitary solution of these equations. Based on these solutions, researchers can better understand physical mechanism behind the non-linear model [12].

The Zakharov-Kuznetsov (ZK) equation from many scientific problems including fluid mechanics is investigated in recent years [13]. Here, we will utilize the subsidiary equation method to construct exact solutions for the non-linear (3+1)-dimensional ZK equation. Hence, the results of this paper are an extension of the earlier reports.

The non-linear (3+1)-dimensional quantum ZK equation is given:

$$\frac{\partial A}{\partial T}+\alpha A\frac{\partial A}{\partial Z}+\beta\frac{\partial^3 A}{\partial Z^3}+\gamma\frac{\partial}{\partial Z}\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right)A=0 \tag{1}$$

where α,β,γ are constants and A is the electrostatic potential.

Method description

In this part, we are going to find soliton solutions, rational function solutions and periodic solutions of the eq. (1) by using the simplest equation method. Now, we briefly illustrate

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the method. Firstly, we assume that the unknown function of a partial differential equation is $A(x,y,z,t)$ and use traveling transformation $\xi = kx + ly + mz - ct$ (here k,l,m are wave numbers, c is wave speed). The PDE can be converted to a non-linear ODE with the help of the traveling transformation:

$$P\left(F(\xi), \frac{dF}{d\xi}, \frac{d^2F}{d\xi^2}, \cdots\right) = 0 \tag{2}$$

then we assume that the exact solution of the eq. (2) can be constructed as finite series:

$$F(\xi) = \sum_{i=0}^{L} a_i [Y(\xi)]^i \tag{3}$$

where a_i is a constant, and $Y(\xi)$ – the exact solution of the well-known equation as the simplest equation and the general solution is known. The L is a constant and its value is obtained by using a balance in the eq. (2). At last, substituting of the finite-series of the eq. (3) into eq. (2) and as a result, a system about polynomial of $y(\xi)$ is obtained. Equating all the coefficients of the power of polynomials $y(\xi)$ to zero yields an equation set. Then, the exact solitary solutions of eq. (2) are obtained, from which the coefficients a_i are obtained.

Subsequently, the Riccati equation is set as the simplest equation to solve the exact solitary solutions.

The Riccati equation is:

$$Y_\xi = a[Y(\xi)]^2 + bY(\xi) + c \tag{4}$$

where a,b,c are arbitrary constants.

Riccati equation as the simplest equation method

For the eq. (1), the equation can be transformed by using $A(\xi) = A(X,Y,Z,T)$ and $\xi = kX + lY + mZ - cT$ as follow:

$$-c\frac{dA}{d\xi} + \alpha mA\frac{dA}{d\xi} + \beta m^3\frac{d^3A}{d\xi^3} + \gamma m\frac{d}{d\xi}\left(k^2\frac{d^2}{d\xi^2} + l^2\frac{d^2}{d\xi^2}\right)A = 0 \tag{5}$$

integrating once with respect to ξ yields:

$$-cA + \alpha m\frac{1}{2}A^2 + m(\beta m^2 + \gamma k^2 + \gamma l^2)\frac{d^2A}{d\xi^2} = 0 \tag{6}$$

balancing between the highest order non-linear term and the highest order derivative term in the eq. (5) yields $n = 2$. So, we assume the solution of eq. (6) as follow:

$$A(\xi) = a_0 + a_1Y(\xi) + a_2Y^2(\xi) \tag{7}$$

Here, the first-order derivative of $Y(\xi)$ meets the Riccati equation. Then substituting eqs. (7) and (3) into eq. (6) and setting like power coefficient of $Y(\xi)$ to zero, we derive an equation set of algebraic relationship between the parameters of the solution and the parameters of the solved equation class.

$$-c\left(a_0 + a_1Y + a_2Y^2\right) + \frac{\alpha m}{2}\left(a_0 + a_1Y + a_2Y^2\right)^2 +$$
$$+m\left(\beta m^2 + \gamma k^2 + \gamma l^2\right)\left(2a_1aY + a_1b + 6a_2aY^2 + 4a_2bY + 2a_2c\right)Y_\xi = 0 \tag{8}$$

further calculation, we have:

$$Y^0 : -ca_0 + \frac{m\alpha}{2} a_0{}^2 + (\beta m^3 + \gamma mk^2 + \gamma ml^2)\left(a_1 b + 2a_2 c\right)c = 0 \tag{9}$$

$$Y^1 : -ca_1 + m\alpha a_0 a_1 + (\beta m^3 + \gamma mk^2 + \gamma ml^2)\left(2a_1 ac + 6a_2 bc + a_1 b^2\right) = 0 \tag{10}$$

$$Y^2 : -ca_2 + \frac{m\alpha}{2}(2a_0 a_2 + a_1{}^2) + (\beta m^3 + \gamma mk^2 + \gamma ml^2)(8a_2 ac + 3a_1 ab + 4a_2 b^2) = 0 \tag{11}$$

$$Y^3 : \alpha m a_1 a_2 + (\beta m^3 + \gamma mk^2 + \gamma ml^2)(10a_2 ab + 2a_1 a^2) = 0 \tag{12}$$

$$Y^4 : \frac{\alpha m}{2} a_2^2 + (\beta m^3 + \gamma mk^2 + \gamma ml^2) 6a_2 a^2 = 0. \tag{13}$$

Solving the previous equations, we can obtain the following results:

$$a_0 = -\frac{\left(b^2 + 8ac\right)\left(\beta m^2 + \gamma k^2 + \gamma l^2\right)}{\alpha} + \frac{c}{\alpha m}, \quad a_1 = -\frac{12ab(\beta m^2 + \gamma k^2 + \gamma l^2)}{\alpha}$$

$$a_2 = -\frac{12a^2(\beta m^2 + \gamma k^2 + \gamma l^2)}{\alpha} \tag{14}$$

and we have restricted condition:

$$m^2(4ac - b^2)^2(m^2\beta + \gamma k^2 + \gamma l^2)^2 - c^2 = 0 \tag{15}$$

so, the solutions of the equation are:

Case 1. When $b^2 - 4ac > 0$ and $bc \neq 0$ (or $ac \neq 0$), the dark soliton of eq. (1):

$$A_1(\xi) = -\frac{\left(b^2 + 8ac\right)\left(\beta m^2 + \gamma k^2 + \gamma l^2\right)m + c}{\alpha m} +$$
$$+\frac{12ab\left(\beta m^2 + \gamma k^2 + \gamma l^2\right)}{\alpha}\left\{\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\tanh\left[\frac{\sqrt{b^2 - 4ac}}{2}\xi\right]\right\} -$$
$$-\frac{12a^2\left(\beta m^2 + \gamma k^2 + \gamma l^2\right)}{\alpha}\left\{\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\tanh\left[\frac{\sqrt{b^2 - 4ac}}{2}\xi\right]\right\}^2 \tag{16}$$

and a singular soliton:

$$A_2(\xi) = -\frac{\left(b^2 + 8ac\right)\left(\beta m^2 + \gamma k^2 + \gamma l^2\right)m + c}{\alpha m} +$$
$$+\frac{12ab\left(\beta m^2 + \gamma k^2 + \gamma l^2\right)}{\alpha}\left\{\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\coth\left[\frac{\sqrt{b^2 - 4ac}}{2}\xi\right]\right\} -$$
$$-\frac{12a^2\left(\beta m^2 + \gamma k^2 + \gamma l^2\right)}{\alpha}\left\{\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\coth\left[\frac{\sqrt{b^2 - 4ac}}{2}\xi\right]\right\}^2 \tag{17}$$

Case 2. When $b^2-4ac<0$ and $bc\neq 0$ (or $ac\neq 0$), the periodic solution of eq. (1):

$$A_3(\xi)=-\frac{\left(b^2+8ac\right)\left(\beta m^2+\gamma k^2+\gamma l^2\right)m+c}{\alpha m}+$$
$$+\frac{12ab\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{\alpha}\left\{-\frac{b}{2a}+\frac{\sqrt{4ac-b^2}}{2a}\tan\left[\frac{\sqrt{4ac-b^2}}{2}\xi\right]\right\}-$$
$$-\frac{12a^2\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{\alpha}\left\{-\frac{b}{2a}+\frac{\sqrt{4ac-b^2}}{2a}\tan\left[\frac{\sqrt{4ac-b^2}}{2}\xi\right]\right\}^2 \tag{18}$$

$$A_4(\xi)=-\frac{\left(b^2+8ac\right)\left(\beta m^2+\gamma k^2+\gamma l^2\right)m+c}{\alpha m}+$$
$$+\frac{12ab\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{\alpha}\left\{\frac{b}{2a}+\frac{\sqrt{4ac-b^2}}{2a}\cot\left[\frac{\sqrt{4ac-b^2}}{2}\xi\right]\right\}-$$
$$-\frac{12a^2\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{\alpha}\left\{\frac{b}{2a}+\frac{\sqrt{4ac-b^2}}{2a}\cot\left[\frac{\sqrt{4ac-b^2}}{2}\xi\right]\right\}^2 \tag{19}$$

Case 3. When $b^2-4ac>0$ and $bc\neq 0$ (or $ac\neq 0$), the soliton-like of eq. (1):

$$A_5(\xi)=-\frac{\left(b^2+8ac\right)\left(\beta m^2+\gamma k^2+\gamma l^2\right)m+c}{\alpha m}+$$
$$+\frac{12ab\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{\alpha}\left\{\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\left[\tan\left(\sqrt{b^2-4ac}\xi\right)\pm i\,\mathrm{sech}\left(\sqrt{b^2-4ac}\xi\right)\right]\right\}-$$
$$-\frac{12a^2\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{\alpha}\left\{\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\left[\tan\left(\sqrt{b^2-4ac}\xi\right)\pm i\,\mathrm{sech}\left(\sqrt{b^2-4ac}\xi\right)\right]\right\}^2 \tag{20}$$

$$A_6(\xi)=-\frac{\left(b^2+8ac\right)\left(\beta m^2+\gamma k^2+\gamma l^2\right)m+c}{\alpha m}+$$
$$+\frac{12ab\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{\alpha}\left\{\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\left[\coth\left(\sqrt{b^2-4ac}\xi\right)\pm \mathrm{csch}\left(\sqrt{b^2-4ac}\xi\right)\right]\right\}-$$
$$-\frac{12a^2\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{\alpha}\left\{\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\left[\coth\left(\sqrt{b^2-4ac}\xi\right)\pm \mathrm{csch}\left(\sqrt{b^2-4ac}\xi\right)\right]\right\}^2 \tag{21}$$

$$A_7(\xi)=-\frac{\left(b^2+8ac\right)\left(\beta m^2+\gamma k^2+\gamma l^2\right)m+c}{\alpha m}+$$

$$+\frac{12ab\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{2\alpha}\left\{b+\sqrt{b^2-4ac}\left[\tanh\left(\frac{\sqrt{b^2-4ac}}{4}\xi\right)\pm\coth\left(\frac{\sqrt{b^2-4ac}}{4}\xi\right)\right]\right\}-$$

$$-\frac{12a^2\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{2\alpha}\left\{b+\sqrt{b^2-4ac}\left[\tanh\left(\frac{\sqrt{b^2-4ac}}{4}\xi\right)\pm\coth\left(\frac{\sqrt{b^2-4ac}}{4}\xi\right)\right]\right\}^2 \quad (22)$$

Case 4. When $b^2-4ac<0$ and $bc\neq 0$ (or $ac\neq 0$), the periodic solution of eq. (1):

$$A_8(\xi)=-\frac{\left(b^2+8ac\right)\left(\beta m^2+\gamma k^2+\gamma l^2\right)m+c}{\alpha m}+$$

$$+\frac{12ab\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{\alpha}\left\{-\frac{b}{2a}+\frac{\sqrt{4ac-b^2}}{2a}\tan\left[\frac{\sqrt{4ac-b^2}}{2}\xi\right]\pm\sec\left(\frac{\sqrt{4ac-b^2}}{2}\xi\right)\right\}-$$

$$-\frac{12a^2\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{\alpha}\left\{-\frac{b}{2a}+\frac{\sqrt{4ac-b^2}}{2a}\tan\left[\frac{\sqrt{4ac-b^2}}{2}\xi\right]\pm\sec\left[\frac{\sqrt{4ac-b^2}}{2}\xi\right]\right\}^2 \quad (23)$$

$$A_9(\xi)=-\frac{\left(b^2+8ac\right)\left(\beta m^2+\gamma k^2+\gamma l^2\right)m+c}{\alpha m}+$$

$$+\frac{12ab\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{\alpha}\left\{\frac{b}{2a}+\frac{\sqrt{4ac-b^2}}{2a}\cot\left[\sqrt{4ac-b^2}\,\xi\right]\pm\csc\left[\sqrt{4ac-b^2}\,\xi\right]\right\}-$$

$$-\frac{12a^2\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{\alpha}\left\{\frac{b}{2a}+\frac{\sqrt{4ac-b^2}}{2a}\cot\left[\sqrt{4ac-b^2}\,\xi\right]\pm\csc\left[\sqrt{4ac-b^2}\,\xi\right]\right\}^2 \quad (24)$$

$$A_{10}(\xi)=-\frac{\left(b^2+8ac\right)\left(\beta m^2+\gamma k^2+\gamma l^2\right)m+c}{\alpha m}+$$

$$+\frac{12ab\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{\alpha}\left\{-b+\sqrt{4ac-b^2}\tan\left[\frac{\sqrt{4ac-b^2}}{4}\xi\right]-\coth\left[\frac{\sqrt{4ac-b^2}}{4}\xi\right]\right\}-$$

$$-\frac{12a^2\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{\alpha}\left\{-b+\sqrt{4ac-b^2}\tan\left[\frac{\sqrt{4ac-b^2}}{4}\xi\right]-\coth\left[\frac{\sqrt{4ac-b^2}}{4}\xi\right]\right\}^2 \quad (25)$$

Case 5. When $a\neq 0$ and $b=c=0$, the rational wave solution of eq. (1):

$$A_{11}(\xi)=\frac{-\left(b^2+8ac\right)\left(\beta m^2+\gamma k^2+\gamma l^2\right)m+c}{\alpha m}-\frac{12ab\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{\alpha a_3\xi}-$$

$$-\frac{12a^2\left(\beta m^2+\gamma k^2+\gamma l^2\right)}{\alpha a_3^2\xi^2} \quad (26)$$

Conclusion

In this work, a non-linear (3+1)-dimensional ZK equation is studied by employing the subsidiary equation. The exact eleven traveling wave solutions for the quantum ZK equation are obtained with the help of the Riccati equation expansion method. These solutions include the dark soliton solution, soliton-like solutions, periodic solution and rational wave solution. The exact solutions obtained will extend on earlier reports and some restrictions are presented.

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