THE VECTOR CALCULUS WITH RESPECT TO MONOTONE FUNCTIONS APPLIED TO HEAT CONDUCTION PROBLEMS

by

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This paper addresses the theory of the vector calculus with respect to monotone functions for the first time. The Green-like theorem, Stokes-like theorem, Gauss--like theorem, and Green-like identities are obtained with the aid of the notation of Gibbs. The results are used to model the heat-conduction problems arising in the complex phenomenon.

Key words: *vector calculus with respect to monotone functions, Green-like theorem, Stokes-like theorem, Gauss-like theorem, Green-like identities, heat-conduction problems*

Introduction

The classcial vector calculus [1] is the mathematical quantities of mathematical physics in 3-D space. A great many of the topics in the mathematical physics can be described with the aid of the techniques of vector calculus. There are many topics containing the electricity and magnetism [2], relativity [3], elasticity [4] and conduction of the heat in the solids [5], which can be considered as the description of vector and scalar quantities in the time-space domain.

Let us recall some important and fundamental mathematical ideas in vector analysis. Based on the notations of Gibbs [6, 7] and Heaviside [8], the Gauss's theorem which was proposed by Gauss in 1813 [9]. The Green's theorem was structured by Green in 1828 [10]. The Ostrogradski's theorem was proposed by Ostrogradski in 1827 and published in 1831 (see [11-13]). The Stokes's theorem was set up by Stokes in 1854 [14], and proved by Hankel in 1861 [15].

The relationships among the Gauss's theorem, Green's theorem, Ostrogradski's theorem, and Stokes's theorem were discussed in [11, 16, 17]. The calculus with respect to monotone function, as one of the general calculi, which includes the derivative with respect to monotone function (so-called Leibniz derivative), proposed by Leibniz in 1676 [18], and the integral with respect to monotone function (so-called Riemann-Stieltjes integral), proposed by Stieltjes in 1894 [19], based on the Riemann's work [20], was discussed in [21-23].

The difficulty arises from the fact that the statement of the classcial vector calculus is not used to explain the heat-conduction problems arising in the complex phenomenon based on the vector analysis related to the calculus with respect to monotone function. The main target of

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the paper is to propose theory of the vector calculus with respect to monotone functions and to present the potential applications in heat-conduction problems.

The calculus with respect to monotone function

Let $\psi_u(t) = (\psi \circ u)(t) = \psi[u(t)]$, where $u^{(1)}(t) > 0$, and let $\mathbb{N}$ be the set of the integer numbers.

The Leibniz derivative

The Leibniz derivative of the function $\psi_u(t)$ is defined [19, 22]:

$$D_{t,u(\cdot)}^{(1)} \psi_u(t) = \frac{1}{u^{(1)}(t)} \frac{\mathrm{d}\psi_u(t)}{\mathrm{d}t} \tag{1}$$

The total differential with respect to monotone function $u(t)$ of the function $\psi_u(t)$, denoted as $\mathrm{d}\psi_u(t) = \mathrm{d}[(\psi \circ u)(t)] = \mathrm{d}\psi[u(t)]$, can be given:

$$\mathrm{d}\psi_u(t) = \left[u^{(1)}(t) D_{t,u(\cdot)}^{(1)} \psi_u(t) \right] \mathrm{d}t \tag{2}$$

Let $\varphi_u(t) = (\varphi \circ u)(t) = \varphi[u(t)]$, where $u^{(1)}(t) > 0$.

The Riemann-Stieltjes integral

The Riemann-Stieltjes integral of the function $\psi_u(t)$ is defined [19, 22]:

$$ {}_a I_{b,u(\cdot)}^{(1)} \psi_u(t) = \int_a^b \psi_u(t) u^{(1)}(t) \, \mathrm{d}t \tag{3}$$

Here, eqs. (1) and (3) are called the calculus with respect to monotone function [22-24].
The properties of the calculus with respect to monotone function read as follows [22]:
(A1) The chain rule for the Leibniz derivative [22]:

$$D_{t,u(\cdot)}^{(1)} w\{\varphi[u(t)]\} = w^{(1)}(\varphi) \cdot D_{t,u(\cdot)}^{(1)} \varphi(t) \tag{4}$$

where $w^{(1)}(\varphi) = \mathrm{d}w(\varphi)/\mathrm{d}\varphi$ exists.

(A2) The first fundamental theorem of the Riemann-Stieltjes integral:

$$\varphi(t) - \varphi(a) = {}_a I_{t,u(\cdot)}^{(1)} \left[D_{t,u(\cdot)}^{(1)} \varphi(t) \right] \tag{5}$$

(A3) The change-of-variable theorem for the Riemann-Stieltjes integral:

$$\int_a^t w^{(1)}(\varphi) \cdot D_{t,u(\cdot)}^{(1)} \varphi(t) u^{(1)}(t) \mathrm{d}t = w\{\varphi[u(t)]\} - w\{\varphi[u(a)]\} \tag{6}$$

Remark: the eq. (5) was discovered in [24, 25] and further reviewed in [22].

The partial derivatives with respect to monotone functions

Let $g^{(1)}(x) > 0$, $h^{(1)}(y) > 0$, and $u^{(1)}(z) > 0$.

The partial derivatives with respect to monotone functions $g(x)$, $h(y)$, and $u(z)$ of $\Phi = \Phi(x, y, z) = \phi[g(x), h(y), u(z)]$ are defined:

$$\partial_{x,g}^{(1)} \Phi = \frac{1}{g^{(1)}(x)} \frac{\partial \Phi}{\partial x}, \quad \partial_{y,h}^{(1)} \Phi = \frac{1}{h^{(1)}(y)} \frac{\partial \Phi}{\partial y}, \quad \partial_{z,u}^{(1)} \Phi = \frac{1}{u^{(1)}(z)} \frac{\partial \Phi}{\partial z} \tag{7a,b,c}$$

respectively.

The total differential with respect to monotone functions $g(x)$, $h(y)$, and $u(z)$ of the scalar field $\Phi=\Phi[g(x),h(y),u(z)]$ is defined:

$$\mathrm{d}\Phi=\left[g^{(1)}(x)\partial_{x,g}^{(1)}\Phi\right]\mathrm{d}x+\left[h^{(1)}(y)\partial_{y,h}^{(1)}\Phi\right]\mathrm{d}y+\left[u^{(1)}(z)\partial_{z,u}^{(1)}\Phi\right]\mathrm{d}z \tag{8}$$

Taking $\Theta(t)=\Phi[x(t),y(t),z(t)]$, we have that:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t}=\left[g^{(1)}(x)\partial_{x,g}^{(1)}\Phi\right]\frac{\mathrm{d}x}{\mathrm{d}t}+\left[h^{(1)}(y)\partial_{y,h}^{(1)}\Phi\right]\frac{\mathrm{d}y}{\mathrm{d}t}+\left[u^{(1)}(z)\partial_{z,u}^{(1)}\Phi\right]\frac{\mathrm{d}z}{\mathrm{d}t} \tag{9}$$

The gradient with respect to monotone functions

The gradient with respect to monotone functions in a Cartesian co-ordinate system is defined:

$$\nabla_{(g,h,u)}=ig^{(1)}(x)\partial_{x,g}^{(1)}+jh^{(1)}(y)\partial_{y,h}^{(1)}+ku^{(1)}(z)\partial_{z,u}^{(1)} \tag{10}$$

Thus, the gradient with respect to monotone functions of a scalar field $\Phi=\Phi(x,y,z)$ in a Cartesian co-ordinate system, denoted by $\nabla_{(g,h,u)}\Phi$, is given:

$$\nabla_{(g,h,u)}\Phi=ig^{(1)}(x)\partial_{x,g}^{(1)}\Phi+jh^{(1)}(y)\partial_{y,h}^{(1)}\Phi+ku^{(1)}(z)\partial_{z,u}^{(1)}\Phi \tag{11}$$

With the aid of eqs. (8) and (11), it is not difficult to show that:

$$\mathrm{d}\Phi=\left[g^{(1)}(x)\partial_{x,g}^{(1)}\Phi\right]\mathrm{d}x+\left[h^{(1)}(y)\partial_{y,h}^{(1)}\Phi\right]\mathrm{d}y+\left[u^{(1)}(z)\partial_{z,u}^{(1)}\Phi\right]\mathrm{d}z\nabla=\nabla_{(g,h,u)}\Phi\bullet\mathbf{n}\mathrm{d}r \tag{12}$$

where $\boldsymbol{n}$ is the unit normal to the surface, r – the distance measured along the normal, dr – the distance measured along the normal, and $\mathbf{d}\boldsymbol{r}=\boldsymbol{n}\mathrm{d}r=i\mathrm{d}x+j\mathrm{d}y+k\mathrm{d}z$.

As the special case that $\nabla_{(g,h,u)}\Phi$ and $\boldsymbol{n}$ are parallel and $|\boldsymbol{n}|=1$, we have $\mathrm{d}\Phi=|\nabla_{(g,h,u)}\Phi|\mathrm{d}r$ such that $|\nabla_{(g,h,u)}\Phi|=\mathrm{d}\Phi/\mathrm{d}r$, which is the rate of change of Φ along the normal, and the direction derivative is defined as $\nabla_{(g,h,u)}\Phi\bullet\boldsymbol{n}=\mathrm{d}\Phi/\mathrm{d}r=\partial_n^{(g,h,u)}\Phi$.

The Laplace-like operator, denoted as $\Delta_{(g,h,u)}$, of the scalar field Φ is defined:

$$\Delta_{(g,h,u)}\Phi=\left[g^{(1)}(x)\partial_{x,g}^{(1)}\right]^2\Phi+\left[h^{(1)}(y)\partial_{y,h}^{(1)}\right]^2\Phi+\left[u^{(1)}(z)\partial_{z,u}^{(1)}\right]^2\Phi \tag{13}$$

The properties for the gradient with respect to monotone functions read:

$$\Delta_{(g,h,u)}\Phi=\nabla_{(g,h,u)}^2\Phi=\nabla_{(g,h,u)}\bullet\nabla_{(g,h,u)}\Phi \tag{14}$$

$$\nabla_{(g,h,u)}(\Theta\Phi)=\Theta\nabla_{(g,h,u)}(\Phi)+\Phi\nabla_{(g,h,u)}(\Theta) \tag{15}$$

$$\nabla_{(g,h,u)}\bullet\left[\Theta\nabla_{(g,h,u)}\Phi\right]=\Theta\Delta_{(g,h,u)}\Phi+\nabla_{(g,h,u)}\Phi\bullet\nabla_{(g,h,u)}\Theta \tag{16}$$

where Φ and Θ are the scalar fields.

Theory of the vector calculus with respect to monotone functions

In a Cartesian co-ordinate system, we consider the theory of the vector calculus with respect to monotone functions.

The subline Riemann-Stieltjes-type integral

The subline Riemann-Stieltjes-type integral with respect to monotone functions of the function $\boldsymbol{\Pi}[g(x),h(y),u(z)]$ along the subcurve $L(x,y,z)=L[g(x),h(y),u(z)]$, denoted by $\mathfrak{M}$, is defined:

$$\mathfrak{M} = \int\limits_{L(x,y,z)} \boldsymbol{\Pi}\big(g(x),h(y),u(z)\big)\bullet \mathbf{d}\boldsymbol{l} \tag{17}$$

where $\boldsymbol{\Pi}[g(x),h(y),u(z)] = \Pi_x i + \Pi_y j + \Pi_z k$, and the element of the vector line is:

$$\mathbf{d}\boldsymbol{l} = \boldsymbol{m}\mathrm{d}\ell = i\Big[g^{(1)}(x)\partial_{x,g}^{(1)}\ell\Big]dx + j\Big[h^{(1)}(y)\partial_{y,h}^{(1)}\ell\Big]\mathrm{d}y + k\Big[u^{(1)}(z)\partial_{z,u}^{(1)}\ell\Big]\mathrm{d}z \tag{18}$$

Let us consider that for $\widehat{\boldsymbol{l}} = ig[x(t)] + jh[y(t)] + ku[z(t)]$:

$$\frac{\mathbf{d}\boldsymbol{l}}{\mathrm{d}t} = i\Big[g^{(1)}(x)\partial_{x,g}^{(1)}\ell\Big]\frac{\mathrm{d}x}{\mathrm{d}t} + j\Big[h^{(1)}(y)\partial_{y,h}^{(1)}\ell\Big]\frac{\mathrm{d}y}{\mathrm{d}t} + k\Big[u^{(1)}(z)\partial_{z,u}^{(1)}\ell\Big]\frac{\mathrm{d}z}{\mathrm{d}t} \tag{19}$$

Thus, we have from eqs. (18) and (19) that:

$$\mathrm{d}\ell = \sqrt{\Big[g^{(1)}(x)\partial_{x,g}^{(1)}\ell\Big]^2(\mathrm{d}x)^2 + \Big[h^{(1)}(y)\partial_{y,h}^{(1)}\ell\Big]^2(\mathrm{d}y)^2 + \Big[u^{(1)}(z)\partial_{z,u}^{(1)}\ell\Big]^2(\mathrm{d}z)^2} \tag{20}$$

and

$$\boldsymbol{m} = \frac{\mathbf{d}\boldsymbol{l}}{\mathrm{d}\ell} = \frac{i\Big[g^{(1)}(x)\partial_{x,g}^{(1)}\ell\Big]\mathrm{d}x + j\Big[h^{(1)}(y)\partial_{y,h}^{(1)}\ell\Big]\mathrm{d}y + k\Big[\partial_{z,u}^{(1)}\ell u^{(1)}(z)\Big]\mathrm{d}z}{\sqrt{\Big[g^{(1)}(x)\partial_{x,g}^{(1)}\ell\Big]^2(\mathrm{d}x)^2 + \Big[h^{(1)}(y)\partial_{y,h}^{(1)}\ell\Big]^2(\mathrm{d}y)^2 + \Big[u^{(1)}(z)\partial_{z,u}^{(1)}\ell\Big]^2(\mathrm{d}z)^2}} \tag{21}$$

So, we have that:

$$\frac{\mathrm{d}\ell}{\mathrm{d}t} = \sqrt{\Big[g^{(1)}(x)\partial_{x,g}^{(1)}\ell\Big]^2\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \Big[h^{(1)}(y)\partial_{y,h}^{(1)}\ell\Big]^2\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \Big[u^{(1)}(z)\partial_{z,u}^{(1)}\ell\Big]^2\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2} \tag{22}$$

From eq. (22) there is:

$$d\ell = \sqrt{\Big[g^{(1)}(x)\partial_{x,g}^{(1)}\ell\Big]^2\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \Big[h^{(1)}(y)\partial_{y,h}^{(1)}\ell\Big]^2\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \Big[u^{(1)}(z)\partial_{z,u}^{(1)}\ell\Big]^2\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \tag{23}$$

which leads to the arc length $\ell = \int_0^\ell \mathrm{d}\ell$ from $t = a$ to $t = b$, given:

$$\ell = \int\limits_a^b \sqrt{\Big[g^{(1)}(x)\partial_{x,g}^{(1)}\ell\Big]^2\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \Big[h^{(1)}(y)\partial_{y,h}^{(1)}\ell\Big]^2\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \Big[u^{(1)}(z)\partial_{z,u}^{(1)}\ell\Big]^2\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \tag{24}$$

As a special case of eq. (24), we have that:

$$\ell = \int\limits_a^b \sqrt{\Big[g^{(1)}(x)\partial_{x,g}^{(1)}\ell\Big]^2 + \Big[h^{(1)}(y)\partial_{y,h}^{(1)}\ell\Big]^2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \Big[u^{(1)}(z)\partial_{z,u}^{(1)}\ell\Big]^2\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \tag{25}$$

Making use of eq. (18), we have:

$$\int\limits_{L(x,y,z)} \boldsymbol{\Pi}\big[g(x),h(y),u(z)\big]\bullet \mathbf{d}\boldsymbol{l} = \int\limits_{L(t)} \boldsymbol{\Pi}\{g[x(t)],h[y(t)],u[z(t)]\}\bullet\frac{\mathbf{d}\boldsymbol{l}}{\mathrm{d}t}\mathrm{d}t \tag{26}$$

since

$$\boldsymbol{\Pi}\big[g(x),h(y),u(z)\big]\bullet \mathbf{d}\boldsymbol{l} = \boldsymbol{\Pi}\{g[x(t)],h[y(t)],u[z(t)]\}\bullet\frac{\mathbf{d}\boldsymbol{l}}{\mathrm{d}t} \tag{27}$$

The vector field $\boldsymbol{\Pi}[g(x),h(y),u(z)]$ in $L(x,y,z)=L[g(x),h(y),u(z)]$ is said to be conservative if:

$$\oint_{L(x,y,z)} \boldsymbol{\Pi}\left[g(x),h(y),u(z)\right]\bullet \mathbf{d}\boldsymbol{l}=0 \tag{28}$$

Thus, eq. (17) can be given:

$$\int_{L(x,y,z)} \boldsymbol{\Pi}\bullet \mathbf{d}\boldsymbol{l}=\int_{L(x,y,z)} \Pi_x\left[g^{(1)}(x)\partial_{x,g}^{(1)}\ell\right]\mathrm{d}x+\Pi_x\left[h^{(1)}(y)\partial_{y,h}^{(1)}\ell\right]\mathrm{d}y+\Pi_x\left[u^{(1)}(z)\partial_{z,u}^{(1)}\ell\right]\mathrm{d}z \tag{29}$$

Riemann-Stieltjes-type double integral with respect to monotone functions

The Riemann-Stieltjes-type double integral with respect to monotone functions of the scalar field $\Phi[\ell_g(x),\zeta_h(y)]$ on the region $S(x,y)=S[\ell_g(x),\zeta_h(y)]$, denoted by $A(\Phi)$, is defined:

$$A(\Phi)=\iint_{S(x,y)} \Phi[g(x),h(y)]\,\mathrm{d}S \tag{30}$$

where $\mathrm{d}S=[g^{(1)}(x)\partial_{x,g}^{(1)}\ell][h^{(1)}(y)\partial_{y,h}^{(1)}\zeta]\mathrm{d}x\,\mathrm{d}y$.

When $\ell_g(x)=(\ell\circ g)(x)=\ell[g(x)]$ and $\zeta_h(x)=(\zeta\circ h)(y)=\zeta[h(y)]$, we have:

$$\mathrm{d}S=\left[g^{(1)}(x)\partial_{x,g}^{(1)}\ell\right]\left[h^{(1)}(y)\partial_{y,h}^{(1)}\zeta\right]\mathrm{d}x\,\mathrm{d}y=\mathrm{d}\ell_g(x)\,d\zeta_h(y) \tag{31}$$

It is shown from eqs. (30) and (31) that:

$$\iint_{S(x,y)} \Phi\left[\ell_g(x),\zeta_h(y)\right]\mathrm{d}S=\iint_{S(x,y)} \Phi\left[\ell_g(x),\zeta_h(y)\right]\mathrm{d}\ell_g(x)\,\mathrm{d}\zeta_h(y)=$$
$$=\int_c^d\left\{\int_a^b \Phi\left[\ell_g(x),k_h(y)\right]\mathrm{d}\ell_g(x)\right\}\mathrm{d}\zeta_h(y)=\int_a^b\left\{\int_c^d \Phi\left[\ell_g(x),k_h(y)\right]\mathrm{d}\zeta_h(y)\right\}\mathrm{d}\ell_g(x) \tag{32}$$

where $x\in[a,b]$ and $y\in[c,d]$.

Let us define the matrix by $\mathrm{d}\ell_g(x)\mathrm{d}\zeta_h(y)=\mathrm{M}\mathrm{d}X_\nu(\theta)\mathrm{d}Y_\rho(\vartheta)$, where $\ell_g(x)=$ $=\ell_g\{x[X_\nu(\theta),Y_\rho(\vartheta)]\}$ and $\zeta_h(y)=\zeta_h\{y[X_\nu(\theta),Y_\rho(\vartheta)]\}$ and:

$$\mathrm{M}=\begin{vmatrix}\dfrac{\partial\ell_g(x)}{\partial X_\nu(\theta)} & \dfrac{\partial\ell_g(x)}{\partial Y_\rho(\vartheta)}\\ \dfrac{\partial\zeta_h(y)}{\partial X_\nu(\theta)} & \dfrac{\partial\zeta_h(y)}{\partial Y_\rho(\vartheta)}\end{vmatrix} \tag{33}$$

Thus, we have:

$$\iint_{S(x,y)} \Phi\left[\ell_g(x),\zeta_h(y)\right]\mathrm{d}\ell_g(x)\,\mathrm{d}\zeta_h(y)=\iint_{S(\theta,\vartheta)} \Phi\left[X_\nu(\theta),Y_\rho(\vartheta)\right]\mathrm{M}\mathrm{d}X_\nu(\theta)\,\mathrm{d}Y_\rho(\vartheta) \tag{34}$$

Riemann-Stieltjes-type volume integral with respect to monotone functions

The Riemann-Stieltjes-type volume integral with respect to monotone functions of the scalar field $\Phi[\ell_g(x),\zeta_h(y),\xi_u(z)]$ is defined:

$$V(\Phi) = \iiint\limits_{\Omega(x,y,z)} \Phi\left[\ell_g(x), \zeta_h(y), \xi_u(z)\right]\mathrm{d}V \tag{35}$$

where $\mathrm{d}V = [g^{(1)}(x)\partial_{x,g}^{(1)}\ell][h^{(1)}(y)\partial_{y,h}^{(1)}\zeta][u^{(1)}(z)\partial_{z,u}^{(1)}\xi]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

On putting $\ell_g(x) = \ell[g(x)]$, $\zeta_h(x) = \zeta[h(y)]$, and $\xi_u(z) = \xi[u(z)]$, we have that:

$$\mathrm{d}V = \left[g^{(1)}(x)\partial_{x,g}^{(1)}\ell\right]\left[h^{(1)}(y)\partial_{y,h}^{(1)}\zeta\right]\left[u^{(1)}(z)\partial_{z,u}^{(1)}\xi\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \mathrm{d}\ell_g(x)\mathrm{d}\zeta_h(y)\mathrm{d}\xi_u(z) \tag{36}$$

Thus, we present:

$$\iiint\limits_{\Omega(x,y,z)} \Phi\left[\ell_g(x), \zeta_h(y), \xi_u(z)\right]\mathrm{d}V = \iiint\limits_{\Omega(x,y,z)} \Phi\left[\ell_g(x), \zeta_h(y), \xi_u(z)\right]\mathrm{d}\ell_g(x)\,\mathrm{d}\zeta_h(y)\,\mathrm{d}\xi_u(z) \tag{37}$$

Let us define:

$$\mathrm{d}\ell_g(x)\,\mathrm{d}\zeta_h(y)\,\mathrm{d}\xi_u(z) = \mathrm{K}\,\mathrm{d}X_\nu(\theta)\,\mathrm{d}Y_\rho(\vartheta)\,\mathrm{d}Z_\gamma(\eta) \tag{38}$$

where

$$\ell_g(x) = \ell_g\{x[X_\nu(\theta), Y_\rho(\vartheta), Z_\gamma(\eta)]\},\ \zeta_h(y) = \zeta_h\{y[X_\nu(\theta), Y_\rho(\vartheta), Z_\gamma(\eta)]\},$$
$$\xi_u(z) = \zeta_h\{y[X_\nu(\theta), Y_\rho(\vartheta), Z_\gamma(\eta)]\}$$

and

$$\mathrm{K} = \begin{vmatrix} \dfrac{\partial\ell_g(x)}{\partial X_\nu(\theta)} & \dfrac{\partial\ell_g(x)}{\partial Y_\rho(\vartheta)} & \dfrac{\partial\ell_g(x)}{\partial Z_\gamma(\eta)} \\ \dfrac{\partial\zeta_h(y)}{\partial X_\nu(\theta)} & \dfrac{\partial\zeta_h(y)}{\partial Y_\rho(\vartheta)} & \dfrac{\partial\zeta_h(y)}{\partial Z_\gamma(\eta)} \\ \dfrac{\partial\xi_u(z)}{\partial X_\nu(\theta)} & \dfrac{\partial\xi_u(z)}{\partial Y_\rho(\vartheta)} & \dfrac{\partial\xi_u(z)}{\partial Z_\gamma(\eta)} \end{vmatrix} \tag{39}$$

Thus, we have for

$$\Phi(x,y,z) = \Phi[\ell_g(x), \zeta_h(y), \xi_u(z)] \quad \text{and} \quad \Phi(\theta,\vartheta,\eta) = \Phi[X_\nu(\theta), Y_\rho(\vartheta), Z_\gamma(\eta)]$$

that:

$$\iiint\limits_{\Omega(x,y,z)} \Phi(x,y,z)\,\mathrm{d}\ell_g(x)\,\mathrm{d}\zeta_h(y)\,\mathrm{d}\xi_u(z) = \iiint\limits_{\Omega(\theta,\vartheta,\eta)} \Phi(\theta,\vartheta,\eta)\,\mathrm{K}\,\mathrm{d}X_\nu(\theta)\,\mathrm{d}Y_\rho(\vartheta)\,\mathrm{d}Z_\gamma(\eta) \tag{40}$$

Let us reconsider that:

$$(\mathbf{d}\boldsymbol{l} \times \mathbf{d}\boldsymbol{\zeta})\bullet\mathbf{d}\boldsymbol{\xi} = \mathrm{d}\ell_g(x)\,\mathrm{d}\zeta_h(y)\,\mathrm{d}\xi_u(z) \tag{41}$$

where

$$\mathbf{d}\boldsymbol{l} = i\mathrm{d}\ell_g(x), \quad \mathbf{d}\boldsymbol{\zeta} = j\mathrm{d}\zeta_h(y), \quad \mathbf{d}\boldsymbol{\xi} = k\mathrm{d}\xi_u(z) \tag{42a,b,c}$$

Here, (64) is the generalized case in [26].

Riemann-Stieltjes-type surface integral with respect to monotone functions

The Riemann-Stieltjes-type surface integral with respect to monotone functions of the vector field $\boldsymbol{\psi}[\ell_g(x), \zeta_h(y), \xi_u(z)]$ is defined:

$$\iint_{\boldsymbol{S}(x,y,z)} \boldsymbol{\psi}\left[\ell_g(x),\zeta_h(y),\xi_u(z)\right]\bullet\mathbf{d}\boldsymbol{S} = \iint_{\boldsymbol{S}(x,y,z)} \boldsymbol{\psi}\left[\ell_g(x),\zeta_h(y),\xi_u(z)\right]\bullet\boldsymbol{n}\mathrm{d}S \tag{43}$$

where $\boldsymbol{n} = \mathbf{d}\boldsymbol{S}/\mathrm{d}S$ is the unit normal vector to the surface $\boldsymbol{S}(x,y,z) = \boldsymbol{S}[\ell_g(x),\zeta_h(y),\xi_u(z)]$.

Let us consider that $\boldsymbol{n} = \mathbf{d}\mathrm{S}/|\mathbf{d}\boldsymbol{S}| = \mathbf{d}\boldsymbol{S}/\mathrm{d}S$, $\mathrm{d}S = |\mathbf{d}\boldsymbol{S}|$, and:

$$\mathbf{d}\boldsymbol{S} = \mathrm{d}\zeta_h(y)\mathrm{d}\ell_u(z)i + \mathrm{d}\ell_g(x)\mathrm{d}\zeta_u(z)j + \mathrm{d}\ell_g(x)\mathrm{d}\zeta_h(y)k \tag{44}$$

Thus, we may have from eqs. (43) and (44) that:

$$\iint_{\boldsymbol{S}(x,y,z)} \boldsymbol{\psi}\bullet\mathbf{d}\boldsymbol{S} = \iint_{S(x,y,z)} \psi_x \mathrm{d}\ell_h(y)\mathrm{d}\zeta_u(z) + \psi_y \mathrm{d}\ell_g(x)\mathrm{d}\zeta_u(z) + \psi_z \mathrm{d}\ell_g(x)\mathrm{d}\zeta_h(y) \tag{45}$$

where $\boldsymbol{\psi} = \boldsymbol{\psi}[\ell_g(x),\zeta_h(y),\xi_u(z)] = i\psi_x + j\psi_y + k\psi_y$.

The flux of the vector field $\boldsymbol{\psi}[\ell_g(x),\zeta_h(y),\xi_u(z)]$ across the surface $\mathbf{d}\boldsymbol{S}$, denoted by $\boldsymbol{\Phi}$, is defined:

$$\boldsymbol{\Phi} = \oint_{\boldsymbol{S}(x,y,z)} \boldsymbol{\psi}\bullet\mathbf{d}\boldsymbol{S} \tag{46}$$

The divergence with respect to monotone functions

The divergence with respect to monotone functions of the vector field $\boldsymbol{\psi}$ is defined:

$$\nabla_{(g,h,u)}\bullet\boldsymbol{\psi} = \lim_{\Delta V_m\to 0}\frac{1}{\Delta V_m}\oiint_{\Delta\boldsymbol{S}_m(x,y,z)} \boldsymbol{\psi}\bullet\mathbf{d}\boldsymbol{S} \tag{47}$$

where the volume V is divided into a large number of small subvolumes ΔV_m with surfaces $\Delta\boldsymbol{S}_m(x,y,z)$, $\boldsymbol{\psi}$ – the continuously differentiable vector field, and $\mathbf{d}\boldsymbol{S}$ – the element of the surface $\boldsymbol{S}(x,y,z)$ bounding the solid $\Omega(x,y,z)$.

With use of eqs. (10) and (47) can be written:

$$\nabla_{(g,h,u)}\bullet\boldsymbol{\psi} = g^{(1)}(x)\partial^{(1)}_{x,g}\psi_x + jh^{(1)}(y)\partial^{(1)}_{y,h}\psi_y + ku^{(1)}(z)\partial^{(1)}_{z,u}\psi_z \tag{48}$$

where $\boldsymbol{\psi} = \boldsymbol{\psi}[\ell_g(x),\zeta_h(y),\xi_u(z)] = i\psi_x + j\psi_y + k\psi_y$.

Gauss-like theorem

The Gauss-like theorem states that:

$$\iiint_{\Omega(x,y,z)} \nabla_{(g,h,u)}\bullet\boldsymbol{\psi}\mathrm{d}V = \oiint_{\boldsymbol{S}(x,y,z)} \boldsymbol{\psi}\bullet\boldsymbol{n}\,\mathrm{d}S \tag{49}$$

where $\boldsymbol{\psi}$ is a continuously differentiable vector field, $\mathrm{d}V$ – the element of volume $\Omega(x,y,z)$, $\boldsymbol{n}$ – the unit outward normal to $\boldsymbol{S}(x,y,z)$, and $\mathrm{d}S$ – an element of the surface area of the surface $\boldsymbol{S}(x,y,z)$ bounding the solid $\Omega(x,y,z)$.

Taking $\mathbf{d}\boldsymbol{S} = \boldsymbol{n}\mathrm{d}S$, we have from eq. (49) that:

$$\iiint_{\Omega(x,y,z)} \nabla_{(g,h,u)}\bullet\boldsymbol{\psi}\mathrm{d}V = \oiint_{\boldsymbol{S}(x,y,z)} \boldsymbol{\psi}\bullet\mathbf{d}\boldsymbol{S} \tag{50}$$

The curl with respect to monotone functions

The curl respect to monotone functions of the vector field $\boldsymbol{\Pi}$ is defined:

$$\nabla_{(g,h,u)} \times \boldsymbol{\Pi} = \lim_{\Delta S_m(x,y,z)\to 0} \frac{1}{\Delta S_m(x,y,z)} \oint_{\Delta L_m(x,y,z)} \boldsymbol{\Pi}\big(g(x),h(y),u(z)\big)\bullet \mathbf{d}\boldsymbol{l} \tag{51}$$

where $\boldsymbol{\Pi}$ be a continuously differentiable vector field, $\mathbf{d}\boldsymbol{l}$ – the element of the vector line, $\Delta S_m(x,y,z)$ – the small surface element perpendicular to $\boldsymbol{n}$, $\Delta L_m(x,y,z)$ – the closed curve of the boundary of $\Delta S_m(x,y,z)$, and $\boldsymbol{n}$ are oriented in a positive sense.

Similarly, eq. (51) can be expressed:

$$\nabla_{(g,h,u)} \times \boldsymbol{\Pi} = \begin{pmatrix} i & j & k \\ g^{(1)}(x)\partial_{x,g}^{(1)} & h^{(1)}(y)\partial_{y,h}^{(1)} & u^{(1)}(z)\partial_{z,u}^{(1)} \\ \Pi_x & \Pi_y & \Pi_z \end{pmatrix} \tag{52}$$

where $\boldsymbol{\Pi} = \boldsymbol{\Pi}[\ell_g(x), \zeta_h(y), \xi_u(z)] = i\Pi_x + j\Pi_y + k\Pi_y$.

Stokes-like theorem

The Stokes-like theorem states that:

$$\iint_{S(x,y,z)} \left[\nabla_{(g,h,u)} \times \boldsymbol{\psi}\right]\bullet \boldsymbol{n}\mathrm{d}S = \oint_{L(x,y,z)} \boldsymbol{\psi}\bullet \mathbf{d}\boldsymbol{l} \tag{53}$$

where $\boldsymbol{\psi}$ is a constant vector field, $S(x,y,z)$ denotes an open, two sided curve surface, $L(x,y,z)$ represents the closed contour bounding S, and $\mathbf{d}\boldsymbol{l}$ denotes the element of the vector line.

Taking $\mathbf{d}\boldsymbol{S} = \boldsymbol{n}\mathrm{d}S$, we show from eq. (53) that:

$$\iint_{S(x,y,z)} \left[\nabla_{(g,h,u)} \times \boldsymbol{\psi}\right]\bullet \boldsymbol{n}\mathrm{d}S = \oint_{L(x,y,z)} \boldsymbol{\psi}\bullet \mathbf{d}\boldsymbol{l} \tag{54}$$

Green-like theorem

The Green-like theorem states:

$$\oint_{L(x,y)} \boldsymbol{\psi}\bullet \mathbf{d}\boldsymbol{l} = \iint_{S(x,y)} \left[g^{(1)}(x)\partial_{x,g}^{(1)}Q - h^{(1)}(y)\partial_{y,h}^{(1)}P\right]\mathrm{d}S \tag{55}$$

or

$$\oint_{L(x,y)} P\mathrm{d}\ell_g(x) + Q\mathrm{d}\zeta_h(y) = \iint_{S(x,y)} \left[g^{(1)}(x)\partial_{x,g}^{(1)}Q - h^{(1)}(y)\partial_{y,h}^{(1)}P\right]\mathrm{d}\ell_g(x)\mathrm{d}\zeta_h(y) \tag{56}$$

where $\boldsymbol{\Pi} = iP + jQ$, $\mathbf{d}\boldsymbol{l} = i\,\mathrm{d}\ell_g(x) + j\,\mathrm{d}\zeta_h(y)$, $\mathrm{d}S = \mathrm{d}\ell_g(x)\mathrm{d}\zeta_h(y)$, and $S(x,y)$ is a domain bounded by a contour $L(x,y)$.

Green-like identities

Taking $\Phi = \Theta\nabla_{(g,h,u)}\Phi$ such that:

$$\nabla_{(g,h,u)}\bullet\left[\Theta\nabla_{(g,h,u)}\Phi\right] = \Theta\Delta_{(g,h,u)}\Phi + \nabla_{(g,h,u)}\Phi\bullet\nabla_{(g,h,u)}\Theta \tag{57}$$

and

$$\nabla_{(g,h,u)}\bullet\left[\Phi\nabla_{(g,h,u)}\Theta\right] = \Phi\Delta_{(g,h,u)}\Theta + \nabla_{(g,h,u)}\Phi\bullet\nabla_{(g,h,u)}\Theta \tag{58}$$

where Φ and Θ are the scalar fields.

Making the use of eq. (49), the Green-like identity of first type can be written:

$$\iiint\limits_{\Omega(x,y,z)} \nabla_{(g,h,u)} \bullet \left[\Theta \Delta_{(g,h,u)} \Phi + \nabla_{(g,h,u)} \Phi \bullet \nabla_{(g,h,u)} \Theta \right] \mathrm{d}V = \oiint\limits_{\boldsymbol{S}(x,y,z)} \Theta \partial_n^{(g,h,u)} \Phi \mathrm{d}S \tag{59}$$

In a similar way, we present:

$$\iiint\limits_{\Omega(x,y,z)} \nabla_{(g,h,u)} \bullet \left[\Phi \Delta_{(g,h,u)} \Theta + \nabla_{(g,h,u)} \Phi \bullet \nabla_{(g,h,u)} \Theta \right] \mathrm{d}V = \oiint\limits_{\boldsymbol{S}(x,y,z)} \Phi \partial_n^{(g,h,u)} \Theta \mathrm{d}S \tag{60}$$

which reduces to the Green-like identity of second type, given:

$$\iiint\limits_{\Omega(x,y,z)} \nabla_{(g,h,u)} \bullet \left[\Theta \Delta_{(g,h,u)} \Phi - \Phi \Delta_{(g,h,u)} \Theta \right] \mathrm{d}V = \oiint\limits_{\boldsymbol{S}(x,y,z)} \left[\Theta \partial_n^{(g,h,u)} \Phi - \Phi \partial_n^{(g,h,u)} \Theta \right] \bullet \boldsymbol{n} \mathrm{d}S \tag{61}$$

Taking $g(x) = x$, $h(y) = y$, and $h(z) = z$, the Gauss-like, Stokes-like and Green-like theorems and Green-like identities become the Gauss [9], Stokes [14], Green [10] theorems [27] and Green identities [10], respectively.

A new heat-conduction model

Let us consider the Fourier-like law for the heat fluid density, denoted as $\boldsymbol{q}$, expressed by:

$$\boldsymbol{q} = -\varpi \nabla_{(g,h,u)} \bullet T = -\varpi \left[i g^{(1)}(x) \partial_{x,g}^{(1)} T + j h^{(1)}(y) \partial_{y,h}^{(1)} T + k u^{(1)}(z) \partial_{z,u}^{(1)} T \right] \tag{62}$$

where ϖ is the thermal conductivity, $T = T[g(x), h(y), u(z), t]$ and $\boldsymbol{q} = \boldsymbol{q}[g(x), h(y), u(z), t]$.

The heat entering through $\boldsymbol{S}(x, y, z)$ at unit time, denoted by E_1, is written as follows:

$$\mathrm{E}_1 = \oiint\limits_{\boldsymbol{S}(x,y,z)} \boldsymbol{q} \bullet \mathbf{d}\boldsymbol{S} \tag{63}$$

The energy generation in the domain $\Omega(x, y, z)$, denoted by E_2, reads as follows:

$$\mathrm{E}_2 = \iiint\limits_{\Omega(x,y,z)} G \mathrm{d}V \tag{64}$$

where $G = G[g(x), h(y), u(z), t]$ is the energy generation at unit time and unit volume.

The changes in storage energy in the domain $\Omega(x, y, z)$, denoted by E_3, can be given:

$$\mathrm{E}_3 = \iiint\limits_{\Omega(x,y,z)} \hbar \wp \frac{\partial T}{\partial t} \mathrm{d}V \tag{65}$$

where $\hbar$ is the density and $\wp$ is the specific heat of the complex material.

The First law of thermodynamics at unit time states that:

$$\mathrm{E}_2 - \mathrm{E}_1 = \mathrm{E}_3 \tag{66}$$

which can be rewritten:

$$-\oiint\limits_{\boldsymbol{S}(x,y,z)} \boldsymbol{q} \bullet \mathbf{d}\boldsymbol{S} + \iiint\limits_{\Omega(x,y,z)} G \mathrm{d}V = \iiint\limits_{\Omega(x,y,z)} \hbar \wp \frac{\partial T}{\partial t} \mathrm{d}V \tag{66}$$

Making use of the Gauss-like theorem, we have from eq. (62) that:

$$\oiint\limits_{\boldsymbol{S}(x,y,z)} \boldsymbol{q} \bullet \mathbf{d}\boldsymbol{S} = -\varpi \iiint\limits_{\Omega(x,y,z)} \nabla_{(g,h,u)}^2 \bullet T \mathrm{d}V \tag{67}$$

Substituting eq. (67) into eq. (66), we have:

$$\iiint\limits_{\Omega(x,y,z)} \left[\varpi\nabla^2_{(g,h,u)} \bullet T + G - \hbar\wp\frac{\partial T}{\partial t} \right] \mathrm{d}V = 0 \tag{68}$$

Thus, we have from eq. (68):

$$\varpi\nabla^2_{(g,h,u)} \bullet T + G = \hbar\wp\frac{\partial T}{\partial t} \tag{69}$$

which leads to:

$$\varpi\nabla^2_{(g,h,u)} \bullet T + G = \hbar\wp\frac{\partial T}{\partial t} \tag{70}$$

Considering that $G = 0$, we have the transient heat-condition equation:

$$\varpi\nabla^2_{(g,h,u)} \bullet T = \hbar\wp\frac{\partial T}{\partial t} \tag{71}$$

Similarly, taking $\partial T/\partial t = 0$, we have the Laplace-like equation, *e. g.*:

$$\varpi\nabla^2_{(g,h,u)} \bullet T = 0 \tag{72}$$

or

$$\varpi\left[g^{(1)}(x)\partial^{(1)}_{x,g}\right]^2 T + \left[h^{(1)}(y)\partial^{(1)}_{y,h}\right]^2 T + \left[u^{(1)}(z)\partial^{(1)}_{z,u}\right]^2 T = 0 \tag{73}$$

From eq. (71) the transient heat-condition equation in the 2-D space reads:

$$\varpi\left[g^{(1)}(x)\partial^{(1)}_{x,g}\right]^2 T + \left[h^{(1)}(y)\partial^{(1)}_{y,h}\right]^2 T = \hbar\wp\frac{\partial T}{\partial t} \tag{74}$$

The transient heat-condition equation in the 1-D space can be written:

$$\varpi\left[g^{(1)}\left(x\right)\partial^{(1)}_{x,g}\right]^2 T = \hbar\wp\frac{\partial T}{\partial t} \tag{75}$$

From eq. (69) we obtain the Poisson-like equation, *e. g.*:

$$\varpi\nabla^2_{(g,h,u)} \bullet T = -G \tag{76}$$

On putting $g(x) = x$, $h(y) = y$, and $h(z) = z$, the Fourier-like law, Laplace-like equation and Poisson-like equation are the Fourier law [28], Laplace equation [29] and Poisson equation [30], respectively.

Conclusion

In the work, we proposed the theory of the vector calculus with respect to monotone functions. The Green-like theorem, Stokes-like theorem, Gauss-like theorem and Green-like identities were presented in detail. We proposed the Fourier-like law, Laplace-like equation and Poisson-like equation based on the heat-conduction models arising in the complex phenomenon.

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Nomenclature

T – temperature, [K]
t – time, [s]
x, x, z – space co-ordinates, [m]

Greek symbols

ϖ – heat conductivity, [$Wm^{-1}K^{-1}$]
$\hbar$ – density, [kgm^{-3}]
$\wp$ – specific heat capacity, [$Jkg^{-1}K^{-1}$]

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