ANALYTICAL SOLUTION FOR NON-LINEAR LOCAL FRACTIONAL BRATU-TYPE EQUATION IN A FRACTAL SPACE

by

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In this paper, the non-linear local fractional Bratu-type equation is described by the local fractional derivative in a fractal space, and its variational formulation is successfully established according to semi-inverse transform method. Finally, we find the approximate analytical solution of the local fractional Bratu-type equation by using Adomina decomposition method.

Key words: *local fractional derivative, fractal space, Bratu-type equation, variational principle, Adomina decomposition method*

Introduction

Fractional derivative is an excellent mathematical tool to establish variety of complex mathematical physical models in the fractal space [1, 2]. Yang's local fractional derivative first was proposed by Yang [3-5]. Once proposed, the Yang's local fractional derivative has attracted the attention of many researchers. It has been widely used in the fields of physics and engineering, such as nanoengineering, dynamics system, microelectronics and so on.

In this paper, we consider the local fractional Bratu-type equation:

$$\frac{D^{2\alpha}u}{Dx^{2\alpha}} + \lambda \mathrm{e}^u = 0, \quad x > 0, \quad 0 < \alpha \le 1 \tag{1}$$

where λ is constant, and $D^\alpha u / Dx^\alpha$ is the Yang's local fractional derivative.

When $\alpha = 1$, eq. (1) is the classical Bratu-type equation [6]:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \lambda e^u = 0 \tag{2}$$

which is adopted to elaborate a combustion problem in a numerical slab. The approximate analytical solution of eq. (2) has been researched by many different methods, such as homotopy perturbation method [7-9], variational iteration method [10, 11], reduced differential transform method [12], homotopy analysis method [13-15], and so on.

When the combustion problem in the flat plate occurs in a fractal space, the traditional definitions of the derivatives will be invalid. The Yang's local fractional derivative has to be used to describe this phenomenon.

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In this paper, the variational formulation of the local fractional Bratu-type equation is established by the semi-inverse method [16, 17], and its approximate analytical solution is obtained by Adomina decomposition method.

Mathematical tools

Let $C_\alpha(a,b)$ be the sets of the local fractional continous functions. The local fractional derivative is defined [3-5]:

$$D_x^{(\alpha)}\phi(\mu_0) = \phi^{(\alpha)}(\mu_0) = \left.\frac{\mathrm{d}^\alpha \phi(\mu)}{\mathrm{d}\mu^\alpha}\right|_{\mu=\mu_0} = \lim_{\mu\to\mu_0}\frac{\Delta^\alpha[\phi(\mu)-\phi(\mu_0)]}{(\mu-\mu_0)^\alpha} \tag{3}$$

where $\Delta^\alpha[\phi(\mu)-\phi(\mu_0)] \cong \Gamma(1+\alpha)\Delta[\phi(\mu)-\phi(\mu_0)]$.

Therefore, we can obtain the local fractional partial derivative of the function $\varphi(\mu,t)$ of fractal order α, $(0<\alpha<1)$ at $\mu=\mu_0$, defined by [3]:

$$\left.\frac{\partial^\alpha \varphi(\mu,t)}{\partial\mu^\alpha}\right|_{\mu=\mu_0} = \lim_{\mu\to\mu_0}\frac{\Delta^\alpha[\varphi(\mu,t)-\varphi(\mu_0-t)]}{(\mu-\mu_0)^\alpha} \tag{4}$$

where $\Delta^\alpha[\varphi(\mu,t)-\varphi(\mu_0,t)] \cong \Gamma(1+\alpha)\Delta[\varphi(\mu,t)-\varphi(\mu_0,t)]$.

The local fractional derivative of high order is written [3]:

$$\Psi^{(mv)}(r) = \overbrace{D_r^v \cdots D_r^v \Psi(r)}^{m-timesa} \tag{5}$$

The local fractional partial derivative of high order is [3]:

$$\frac{\partial^{mv} f(r)}{\partial r^{mv}} = \overbrace{\frac{\partial^v}{\partial r^v}\cdots\frac{\partial^v}{\partial r^v} f(r)}^{m-times} \tag{6}$$

In fractal space, the Mittage Leffler function, sine function and cosine function, are respectively defined [3]:

$$E_\alpha(\chi^\beta) = \sum_{k=0}^{\infty}\frac{\chi^{k\beta}}{\Gamma(1+k\beta)}$$

$$\sin_\beta(\mu^\beta) = \sum_{\theta=0}^{\infty}(-1)^\theta\frac{\mu^{(2\theta+1)\beta}}{\Gamma[1+(2\theta+1)\beta]}$$

and

$$\cos_\beta(\mu^\beta) = \sum_{\kappa=0}^{\infty}(-1)^\kappa\frac{\mu^{2k\beta}}{\Gamma(1+2k\beta)}$$

where $\mu\in R$ and $0<\beta<1$.

Let $\varpi_1(\chi_0)$, $\varpi_2(\chi_0)\in C_\alpha(a,b)$.

The local fractional derivatives of the non-differentiable functions have the following properties [18,19]:

$$D^{(\alpha)}\{\varpi_1(\chi_0)+\varpi_2(\chi_0)\} = D^{(\alpha)}\varpi_1(\chi_0) + D^{(\alpha)}\varpi_2(\chi_0)$$

$$D^{(\alpha)}\{\varpi_1(\chi_0)\cdot\varpi_2(\chi_0)\}=\varpi_2(\chi_0)D^{(\alpha)}\varpi_1(\chi_0)+\varpi_1(\chi_0)D^{(\alpha)}\varpi_2(\chi_0)$$

$$D^{(\alpha)}\left\{\frac{\varpi_1(\chi_0)}{\varpi_2(\chi_0)}\right\}=\frac{\varpi_2(\chi_0)D^{(\alpha)}\varpi_1(\chi_0)+\varpi_1(\chi_0)D^{(\alpha)}\varpi_2(\chi_0)}{[\varpi_2(\chi_0)]^2}$$

where $\varpi_2(\chi_0)\neq 0$.

Adomina decomposition method

To illustrate Adomian decomposition method [20, 21], the function $u(x)$ is defined:

$$u(x)=\sum_{n=0}^{\infty}u_n(x) \tag{7}$$

where the components $u_n(x)$ are usually determined recurrently. The non-linear operator $F(u)$ can be decomposed into the following result:

$$F(u)=\sum_{n=0}^{\infty}A_n \tag{8}$$

where A_n are called the Adomian polynomials of $u_0, u_1, u_2 ... u_n$, given:

$$A_n=\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0}, \quad n=0,1,2,3,... \tag{9}$$

or equivalently:

$$D^{(\alpha)}\{\varpi_1(\chi_0)+\varpi_2(\chi_0)\}=D^{(\alpha)}\varpi_1(\chi_0)+D^{(\alpha)}\varpi_2(\chi_0)A_0=F(u_0)$$

$$A_1=u_1F'(u_0)$$

$$A_2=u_2F'(u_0)+\frac{1}{2}u_1^2F''(u_0)$$

$$A_3=u_3F'(u_0)+u_1u_2F''(u_0)+\frac{1}{3}u_1^3F'''(u_0)$$

$$\dots$$

These polynomials can be generated for all classes of non-linearity by using eq. (9). Recently, an alternative algorithm for constructing Adomian polynomials has been proposed by Adomian [22].

Variational principle for the local fractional Bratu-type equation

Consider the local fractional Bratu-type equation:

$$\frac{D^{2\alpha}u}{Dx^{2\alpha}}+\lambda e^u=0, \quad x>0, \quad 0<\alpha\leq 1 \tag{10}$$

where $D^\alpha u/Dx^\alpha$ is the Yang's local fractional derivative.

The variational principle of eq. (10) is established by semi-inverse method [23], which reads:

$$J(u)=\int_0^1\left[-\frac{1}{2}\left(\frac{D^\alpha u}{Dx^\alpha}\right)^2+\lambda e^u\right]\mathrm{d}x^\alpha \tag{11}$$

We obtain the Euler-Lagrange equation, *e. g.*:

$$-\frac{1}{2}\left[-2\frac{D^{\alpha}}{Dx^{\alpha}}\left(\frac{Du^{\alpha}}{Dx^{\alpha}}\right)\right]+\lambda e^{u}=0 \tag{12}$$

Here, eq. (12) is the same as eq. (10).

The approximate analytical solution for local fractional Bratu-type equation

Let us consider the local fractional Bratu-type equation:

$$\frac{D^{2\alpha}u}{Dx^{2\alpha}}-\pi^2 e^{u}=0,\quad x>0,\quad 0<\alpha\le 1 \tag{13}$$

with the initial conditions:

$$u(0)=u(1)=0 \quad \text{and} \quad u^{(\alpha)}(0)=\varepsilon \tag{14a,b}$$

In order to adopt the Adomina decomposition method, eq. (13) can be written into the following form:

$$\ell u=\pi^2 e^{u} \tag{15}$$

and

$$u(0)=u(1)=0 \tag{16}$$

where the operator ℓ is defined:

$$\ell=\frac{D^{2\alpha}}{Dx^{2\alpha}} \tag{17}$$

Hence, the inverse operator of ℓ is given:

$$\ell^{-1}(\cdot)=\int_0^x\int_0^x(\cdot)\,\mathrm{d}x^{\alpha}\mathrm{d}x^{\alpha} \tag{18}$$

Using ℓ^{-1} on both sides of eq. (15), we have:

$$u(x)=\varepsilon x+\ell^{-1}(\pi^2\mathrm{e}^{u}) \tag{19}$$

Substituting eqs. (13) and (14) into eq. (19), we have:

$$\sum_{n=0}^{\infty}u_n(x)=\varepsilon x+\ell^{-1}\left(\pi^2\sum_{n=0}^{\infty}A_n\right) \tag{20}$$

According to Adomian decomposition method, we have the recurrence relation:

$$\begin{cases} u_0(x)=\varepsilon x \\ u_{k+1}(x)=\pi^2\ell^{-1}(A_k) \end{cases} \tag{21}$$

where $k\ge 0$.

The term A_k of eq. (21) can be determined:

$$A_0 = e^{u_0}$$
$$A_1 = u_1 e^{u_0}$$
$$A_2 = \left(u_2 + \frac{1}{2}u_1^2\right)e^{u_0}$$
$$A_3 = \left(u_3 + u_1 u_2 + \frac{1}{6}u_1^3\right)e^{u_0}$$
$$A_4 = \left(u_4 + u_1 u_3 + \frac{1}{2}u_2^2 + \frac{1}{2}u_1^2 u_2 + \frac{1}{24}u_1^4\right)e^{u_0} \tag{22}$$
$$\cdots.$$

Using the eq. (21) and eq. (22), we obtain:

$$u_0(x) = \frac{\varepsilon x^\alpha}{\Gamma(1+\alpha)}$$
$$u_1(x) = -\frac{\pi^2}{\varepsilon^2}\left\{-\exp\left[\frac{\varepsilon x^\alpha}{\Gamma(1+\alpha)}\right] + \frac{\varepsilon x^\alpha}{\Gamma(1+\alpha)} + 1\right\}$$
$$u_2(x) = -\frac{\pi^4}{4\varepsilon^4}\left\{-\exp\left[\frac{\varepsilon x^\alpha}{\Gamma(1+\alpha)}\right] + 4\frac{\varepsilon x^\alpha}{\Gamma(1+\alpha)}\exp\left[\frac{\varepsilon x^\alpha}{\Gamma(1+\alpha)}\right] - 4\exp\left[\frac{\varepsilon x^\alpha}{\Gamma(1+\alpha)}\right] + \frac{2\varepsilon x^\alpha}{\Gamma(1+\alpha)} + 5\right\}$$

and

$$u_3(x) = \frac{\pi^6}{12\varepsilon^6}\left\{\exp\left[\frac{3\varepsilon x^\alpha}{\Gamma(1+\alpha)}\right] + 6\exp\left[\frac{2\varepsilon x^\alpha}{\Gamma(1+\alpha)}\right]\left[1 - \frac{\varepsilon x^\alpha}{\Gamma(1+\alpha)}\right] + \right.$$
$$\left. + 3\exp\left[\frac{\varepsilon x^\alpha}{\Gamma(1+\alpha)}\right]\left[\frac{2\varepsilon^2 x^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{6\varepsilon x^\alpha}{\Gamma(1+\alpha)} + 5\right] - \frac{6\varepsilon x^\alpha}{\Gamma(1+\alpha)} - 22\right\}$$

Therefore, the approximate analytical solution of eq. (13) is the following form:

$$u(x) = \frac{\varepsilon x^\alpha}{\Gamma(1+\alpha)} - \frac{\pi^2}{\varepsilon^2}\left\{-\exp\left[\frac{\varepsilon x^\alpha}{\Gamma(1+\alpha)}\right] + \frac{\varepsilon x^\alpha}{\Gamma(1+\alpha)} + 1\right\} - \frac{\pi^6}{12\varepsilon^6}\left[\frac{6\varepsilon x^\alpha}{\Gamma(1+\alpha)} + 22\right] -$$
$$-\frac{\pi^4}{4\varepsilon^4}\left\{-\exp\left[\frac{\varepsilon x^\alpha}{\Gamma(1+\alpha)}\right] + \frac{4\varepsilon x^\alpha}{\Gamma(1+\alpha)}\exp\left[\frac{\varepsilon x^\alpha}{\Gamma(1+\alpha)}\right] - 4\exp\left[\frac{\varepsilon x^\alpha}{\Gamma(1+\alpha)}\right] + \frac{2\varepsilon x^\alpha}{\Gamma(1+\alpha)} + 5\right\} +$$
$$+\frac{\pi^6}{12\varepsilon^6}[\exp\left[\frac{3\varepsilon x}{\Gamma(1+\alpha)}\right] + 6\exp\left[\frac{2\varepsilon x}{\Gamma(1+\alpha)}\right]\left[1 - \frac{\varepsilon x}{\Gamma(1+\alpha)}\right] +$$
$$+3\exp\left[\frac{\varepsilon x}{\Gamma(1+\alpha)}\right]\left[\frac{2\varepsilon^2 x}{\Gamma(1+2\alpha)} - \frac{6\varepsilon x}{\Gamma(1+\alpha)} + 5\right] + \cdots \tag{23}$$

When $\alpha = 1$, $\varepsilon = \pi$, eq. (13) has exact solution, *e. g.*:

$$u(x) = -\ln[1 - \sin(\pi x)] \tag{24}$$

Here, eq. (24) is very close to eq. (23). It can be concluded that the Adomina decomposition method is a very powerful tool to find the approximate analytical solution for the non-linear local fractional differential equation.

Conclusion

In the present paper, the local fractional Bratu-type equation was described by Yang's local fractional derivative. The variational formulation of the local fractional Bratu-type equation was successfully established according to the semi-inverse transform method in the fractal space and the Adomina decomposition method are adopted to find its approximate analytical solution.

Nomenclature

t – time co-ordinate, [s]
x – space co-ordinate, [m]

Greek symbols

α – a constant, [–]
β – a constant, [1/s]

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