GENERATION AND SOLUTIONS TO THE TIME-SPACE FRACTIONAL COUPLED NAVIER-STOKES EQUATIONS

by

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In this paper, a Lagrangian of the coupled Navier-Stokes equations is proposed based on the semi-inverse method. The fractional derivatives in the sense of Riemann-Liouville definition are used to replace the classical derivatives in the Lagrangian. Then the fractional Euler-Lagrange equation can be derived with the help of the fractional variational principles. The Agrawal's method is devoted to lead to the time-space fractional coupled Navier-Stokes equations from the above Euler-Lagrange equation. The solution of the time-space fractional coupled Navier-Stokes equations is obtained by means of RPS algorithm. The numerical results are presented by using exact solutions.

Key words: *time-space fractional coupled Navier-Stokes equations, Agrawal's method, residual power series method, fractional derivatives*

Introduction

Fractional PDE can describe natural and physical phenomena more realistically and accurately. For example, it was widely used in rheology, electrostatics, fluid flow, biology, reaction diffusion and so on [1-5]. Many problems can be described by partial differential equations, such as the KP equation, the mKdV equation, the Schrodinger equation, and the Boussinesq equation [6]. In this paper, we use the semi-inverse method , the Euler-Lagrange equation and Agrawal's method [7] to introduce the (3+1)-dimensional time-space fractional coupled Navier-Stokes equations, which have certain development significance for the study of fractional equation.

In the study of the fractional equation, the solution is important [8-10]. The solving mothod is also important for fractional PDE. Recently, there are some importent methods to abtain the solution of PDE, such as the Hirota method [11], the optimal homotopyasmptotic, the homotopy analysis method [12], and so on [13]. In this paper, the residual power series (RPS) method is used to obtain the analytical solution of the Navier-Stokes equations. Unlike the classical power series method, the RPS method does not need to compare the corresponding coefficients and recursive relations, and does not require linearization, discretization, and per-

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turbation. The main advantage of this approach is that it is easier and more accurate to derive solutions than integration.

Derivation of time-space fractional coupled Navier-Stokes equations

The Navier-Stokes equations is derived from the semi-inverse method, the Euler-Lagrange equations and Agrawal's method. And the equation is considered to describe the motion of fluid in models related to ocean currents, water flow in pipes, weather and so forth.

The unsteady (3+1)-dimensional incompressible Navier-Stokes equations in the Cartesian coordinate system is:

$$\begin{cases} u_t + Ruu_x + Rvu_y + Rwu_z + P_x - u_{xx} - u_{yy} - u_{zz} = 0 \\ v_t + Ruv_x + Rvv_y + Rwv_z + P_y - v_{xx} - v_{yy} - v_{zz} = 0 \\ w_t + Ruw_x + Rvw_y + Rww_z + P_z - w_{xx} - w_{yy} - w_{zz} = 0 \end{cases} \tag{1}$$

with the incompressibility and boundary and initial conditions:

üụiĭüụiĭüạị $= 0, \quad (\; , \; , \; , \;) = \;_b, \quad (\; , \; , \;) \in \Gamma, \quad (\; , \; , \; ,0) = \;_i(\; , \; , \;)$

where $U(u,v,w) = [u(x,y,z), v(x,y,z), w(x,y,z)]$ is the fluid velocity vector field with the components $u(x,y,z,t)$, $v(x,y,z,t)$, and $w(x,y,z,t)$ at the point (x,y,z) and time t, $(x,y,t) \in \Omega \subseteq R^3$, Γ – the boundary of Ω, $(i = 1,2)$, P – the pressure, V – the maximum velocity of the object, L – the characteristic linear dimension, μ – the dynamic viscosity, ν – the kinematic viscosity, ρ – the density of the fluid, and R – the Reynolds number, given as $R = \rho VL/\mu = VL/\nu$.

Let $u(x,y,z,t) = A_x(x,y,z,t)$, $v(x,y,z,t) = B_x(x,y,z,t)$, and $w(x,y,z,t) = C_x(x,y,z,t)$. The functional of eq. (1) can be represented:

$$\begin{aligned} J = \int_R \mathrm{d}x \int_Y \mathrm{d}y \int_Z \mathrm{d}z \int_T \mathrm{d}t \times \\ \times\Big[A\big(A_1A_{xt} + A_2RA_xA_{xx} + A_3RB_xA_{xy} + A_4RC_xA_{xz} + A_5P_x - A_6A_{xxx} - A_7A_{xyy} - A_8A_{xzz}\big) + \\ + B\big(B_1B_{xt} + B_2RA_xB_{xx} + B_3RB_xB_{xy} + B_4RC_xB_{xz} + B_5P_y - B_6B_{xxx} - B_7B_{xyy} - B_8B_{xzz}\big) + \\ + C\big(C_1C_{xt} + C_2RA_xC_{xx} + C_3RB_xC_{xy} + C_4RC_xC_{xz} + C_5P_z - C_6C_{xxx} - C_7C_{xyy} - C_8C_{xzz}\big)\Big] \end{aligned} \tag{2}$$

where A_i, B_i, and C_i $(i = 1,\cdots,8)$ are the Lagrange multipliers.

From eq. (2) we obtain:

$$A_x|_R = A_x|_T = A_x|_Y = A_x|_Z = B_x|_R = B_x|_T = B_x|_Y = B_x|_Z = C_x|_R = C_x|_T = C_x|_Y = C_x|_Z = 0$$

Using the variation optimum conditions and $\delta J(A,B,C) = 0$, we have:

$$\begin{aligned} 2A_1A_{xt} + 3A_2RA_xA_{xx} + 3A_3RB_xA_{xy} + 3A_4RC_xA_{xz} + A_5P_x - 2A_6A_{xxx} - 2A_7A_{xyy} - 2A_8A_{xzz} + \\ +2B_1B_{xt} + 3B_2RA_xB_{xx} + 3B_3RB_xB_{xy} + 3B_4RC_xB_{xz} + B_5P_y - 2B_6B_{xxx} - 2B_7B_{xyy} - 2B_8B_{xzz} + \\ +2C_1C_{xt} + 3C_2RA_xC_{xx} + 3C_3RB_xC_{xy} + 3C_4RC_xC_{xz} + C_5P_z - 2C_6C_{xxx} - 2C_7C_{xyy} - 2C_8C_{xzz} = 0 \end{aligned} \tag{3}$$

Comparing eq. (2) with eq. (3), we get:

$$\theta_{1,6,7,8} = \frac{1}{2}, \quad \theta_{2,3,4} = \frac{1}{3}, \quad \theta_5 = 1, \quad (\theta = A,B,C)$$

Substituting the value of $\theta_i (i=1,\cdots,8)$ into eq. (3), the Lagrangian form of the Navier--Stokes equations can be written:

$$\begin{aligned} L = & -\frac{1}{2}A_x A_t - \frac{1}{3}R\left(A_x^2 + AA_{xx}\right)A_x - \frac{1}{3}R(A_y B_x + AB_{xy})A_x - \\ & -\frac{1}{3}R(A_z C_x + AC_{xz})A_x + P_x A + \frac{1}{2}A_{xx}A_x + \frac{1}{2}A_{xy}A_y + \frac{1}{2}A_{xz}A_z - \\ & -\frac{1}{2}B_x B_t - \frac{1}{3}R\left(B_x A_x + BA_{xx}\right)B_x - \frac{1}{3}R(B_y B_x + BB_{xy})B_x - \\ & -\frac{1}{3}R(B_z C_x + BC_{xz})B_x + P_y B + \frac{1}{2}B_{xx}B_x + \frac{1}{2}B_{xy}B_y + \frac{1}{2}B_{xz}B_z - \\ & -\frac{1}{2}C_x C_t - \frac{1}{3}R\left(C_x A_x + CA_{xx}\right)C_x - \frac{1}{3}R(C_y B_x + CB_{xy})C_x - \\ & -\frac{1}{3}R(C_z C_x + CC_{xz})C_x + P_z C + \frac{1}{2}C_{xx}C_x + \frac{1}{2}C_{xy}C_y + \frac{1}{2}C_{xz}C_z \end{aligned} \tag{4}$$

In the same way, we have:

$$\begin{aligned} F = & \left[-\frac{1}{2}D_{\mathrm{x}}^{\beta} A D_t^{\alpha} A - \frac{1}{3}R\left[(D_x^{\beta} A)^2 + A D_x^{2\beta} A\right] D_x^{\beta} A - \frac{1}{3}R(D_y^{\gamma} A D_x^{\beta} B + A D_x^{\beta} D_y^{\gamma} B) D_x^{\beta} A - \right. \\ & \left. -\frac{1}{3}R(D_z^{\xi} A D_x^{\beta} C + A D_x^{\beta} D_z^{\xi} C) D_x^{\beta} A + P_x A + \frac{1}{2}D_x^{2\beta} A D_x^{\beta} A + \frac{1}{2}D_x^{\beta} D_y^{\gamma} A D_y^{\gamma} A + \frac{1}{2}D_x^{\beta} D_z^{\xi} A D_z^{\xi} A\right] + \\ & + \left[-\frac{1}{2}D_x^{\beta} B D_t^{\alpha} B - \frac{1}{3}R(D_x^{\beta} B D_x^{\beta} A + B D_x^{2\beta} A) D_x^{\beta} B - \frac{1}{3}R(D_y^{\gamma} B D_x^{\beta} B + B D_x^{\beta} D_y^{\gamma} B) D_x^{\beta} B - \frac{1}{3}R \times \right. \\ & \left. \times (D_z^{\xi} B D_x^{\beta} C + B D_x^{\beta} D_z^{\xi} C) D_x^{\beta} D_z^{\xi} B + P_y B + \frac{1}{2}D_x^{2\beta} B D_x^{\beta} B + \frac{1}{2}D_x^{\beta} D_y^{\gamma} B D_y^{\gamma} B + \frac{1}{2}D_x^{\beta} D_z^{\xi} B D_z^{\xi} B\right] + \\ & + \left[-\frac{1}{2}D_x^{\beta} C D_t^{\alpha} C - \frac{1}{3}R(D_x^{\beta} C D_x^{\beta} A + C D_x^{2\beta} A) D_x^{\beta} C - \frac{1}{3}R(D_y^{\gamma} C D_x^{\beta} B + C D_x^{\beta} D_y^{\gamma} B) D_x^{\beta} C - \frac{1}{3}R \times \right. \\ & \left. \times (D_z^{\xi} C D_x^{\beta} C + C D_x^{\beta} D_z^{\xi} C) D_x^{\beta} C + P_z C + \frac{1}{2}D_x^{2\beta} C D_x^{\beta} C + \frac{1}{2}D_x^{\beta} D_y^{\gamma} C D_y^{\gamma} C + \frac{1}{2}D_x^{\beta} D_z^{\xi} C D_z^{\xi} C\right] \end{aligned} \tag{5}$$

where $D_x^{\beta} f(x)$ is the modified Riemann-Liouville (mRL) fractional derivative.

The time-space fractional Navier-Stokes equations is:

$$J_f = \int_R \left(\mathrm{d}x\right)^{\beta} \int_Y \left(\mathrm{d}y\right)^{\gamma} \int_Z \left(\mathrm{d}z\right)^{\xi} \int_T \left(\mathrm{d}t\right)^{\alpha} F \tag{6}$$

As $\delta J_f = 0$, the Euler-Lagrangian equation of the time-space fractional Navier-Stokes equations is obtained. Using the fractional potential function, denoted by:

$$D_x^{\beta}\theta(x,y,z,t) = \Delta T \quad [\theta = A, B, C, \Delta T = (u, v, w)]$$

we get the time-space fractional Navier-Stokes equation:

$$D_t^{\alpha}\Delta T + R\Delta T D_1 \Delta T + \Delta P' - D_2 \Delta T = 0 \tag{7}$$

where

$$D_1 = (D_x^\beta, D_y^\gamma, D_z^\xi)^T, \quad D_2 = (D_x^{2\beta}, D_y^{2\gamma}, D_z^{2\xi})^T, \quad \Delta P' = (P_x, P_y, P_z)$$

and $D_t^\alpha f$ is the mRL fractional derivative of function *f*.

Solving time-space fractional Navier-Stokes equations by RPS algorithm

From eq. (7), the RPS method [14] implies the solution of the equations as a fractional power series about the initial point $t = 0$ in the following forms:

$$\Delta T = \sum_{n=0}^{\infty} \frac{\tilde{\tau}_n t^{n\alpha}}{\Gamma(n\alpha+1)}, \quad \tilde{\tau}_n = (f_n, g_n, h_n), \quad \text{and} \quad P = \sum_{n=0}^{\infty} \frac{h_n(x,y,z) t^{n\alpha}}{\Gamma(n\alpha+1)} \tag{8}$$

where $0 < \alpha < 1$, $(x, y, z) \in \Omega$, and $0 \le t < R$.

For $t = 0$, the initial conditions can be written:

$$u(x,y,z,0) = f(x,y,z), \quad v(x,y,z,0) = g(x,y,z), \quad w(x,y,z,0) = m(x,y,z) \tag{9}$$

The initial approximation of $u(x,y,z,t)$, $v(x,y,z,t)$, and $w(x,y,z,t)$ can be expressed:

$$\begin{aligned} u_0(x,y,z,0) &= f_0(x,y,z) = f(x,y,z) \\ v_0(x,y,z,0) &= g_0(x,y,z) = g(x,y,z) \\ w_0(x,y,z,0) &= m_0(x,y,z) = m(x,y,z) \end{aligned} \tag{10}$$

We make a shift of the index *n* from 0 to 1 as follows:

$$P(x,y,z,t) = \sum_{n=1}^{\infty} \frac{h_{n-1}(x,y,z) t^{(n-1)\alpha}}{\Gamma[(n-1)\alpha+1]} \tag{11}$$

The series of u, v, w, and P, denoted as u_k, v_k, w_k, and P_k, can be expressed:

$$\Delta T_k\big|_{t=0} = \tilde{\tau} + \sum_{n=1}^{k} \frac{\tilde{\tau}_n t^{n\alpha}}{\Gamma(n\alpha+1)}, \;\; \Delta T_k = (u_k, v_k, w_k) \;\; \tilde{\tau} = (f, g, m), \;\; P_k\big|_{t=0} = \sum_{n=1}^{k} \frac{h_{n-1} t^{(n-1)\alpha}}{\Gamma[(n-1)\alpha+1]} \tag{12}$$

where $k = 1, 2, 3$.

The residual functions for eq. (7), denoted as $\mathrm{Re}s_u$, $\mathrm{Re}s_v$, and $\mathrm{Re}s_w$, are defined:

$$\begin{aligned} \mathrm{Re}s_u &= D_t^\alpha u + RuD_x^\beta u + RvD_y^\gamma u + RwD_z^\xi u + P_x - D_x^{2\beta} u - D_y^{2\gamma} u - D_z^{2\xi} u \\ \mathrm{Re}s_v &= D_t^\alpha v + RuD_x^\beta v + RvD_y^\gamma v + RwD_z^\xi v + P_y - D_x^{2\beta} v - D_y^{2\gamma} v - D_z^{2\xi} v \\ \mathrm{Re}s_w &= D_t^\alpha w + RuD_x^\beta w + RvD_y^\gamma w + RwD_z^\xi w + P_z - D_x^{2\beta} w - D_y^{2\gamma} w - D_z^{2\xi} w \end{aligned} \tag{13}$$

From eq. (13), the k^{th} truncation error functions can be given:

$$\begin{aligned} \mathrm{Re}s_{uk} &= D_t^\alpha u_k + Ru_k D_x^\beta u_k + Rv_k D_y^\gamma u_k + Rw_k D_z^\xi u_k + P_{kx} - D_x^{2\beta} u_k - D_y^{2\gamma} u_k - D_z^{2\xi} u_k \\ \mathrm{Re}s_{vk} &= D_t^\alpha v_k + Ru_k D_x^\beta v_k + Rv_k D_y^\gamma v_k + Rw_k D_z^\xi v_k + P_{ky} - D_x^{2\beta} v_k - D_y^{2\gamma} v_k - D_z^{2\xi} v_k \\ \mathrm{Re}s_{wk} &= D_t^\alpha w_k + Ru_k D_x^\beta w_k + Rv_k D_y^\gamma w_k + Rw_k D_z^\xi w_k + P_{kz} - D_x^{2\beta} w_k - D_y^{2\gamma} w_k - D_z^{2\xi} w_k \end{aligned} \tag{14}$$

Substituting eq. (8) into eq. (14), the new forms of $\mathrm{Re}s_{uk}(x,y,z,t)$, $\mathrm{Re}s_{vk}(x,y,z,t)$, and $\mathrm{Re}s_{wk}(x,y,z,t)$ are obtained. It is known that $\mathrm{Re}s(x,y,z,t)=0$, $\lim_{k\to\infty}\mathrm{Re}s_k(x,y,z,t)=0$, $t\in[t_0,t_0+R]$, where R is a non-negative real number and represents the radius of convergence. So $D_t^{r\alpha}\mathrm{Re}s(x,y,z,t)=0$, the Caputo fractional derivative of a constant is zero, the fractional derivative $D_t^{r\alpha}$ of $\mathrm{Re}s(x,y,z,t)$ and $\mathrm{Re}s_k(x,y,z,t)$ are matching at $t=t_0$ for each $r=0,1,2\cdots$. Then, let $t_0=0, r=k-1$, we have:

$$D_t^{(k-1)\alpha}\mathrm{Re}s_{uk}(x,y,z,0)=0,\ D_t^{(k-1)\alpha}\mathrm{Re}s_{vk}(x,y,z,0)=0,\ D_t^{(k-1)\alpha}\mathrm{Re}s_{wk}(x,y,z,0)=0 \quad (15)$$

In the following step, we can calculate the coefficients $f_n(x,y,z)$, $\mathrm{g}_n(x,y,z)$, $m_n(x,y,z)$, and $h_{n-1}(x,y,z)$, where $n=1,\cdots,k$. Finally, we solve the algebraic system of eq. (15).

Approximate RPS solutions

For $k=1$, we have:

$$\Delta T_1=\tilde{\tau}+\tilde{\tau}_1 t^{\alpha}/\Gamma(\alpha+1) \quad \text{and} \quad P_1(x,y,z,t)=h_0(x,y,z) \quad (16)$$

Substituting eq. (16) into $\mathrm{Re}s_{uk}, \mathrm{Re}s_{vk}$, $\mathrm{Re}s_{wk}$ at $t=0$, we have the first approximate RPS solutions:

$$\begin{gathered}
u_1(x,y,z,t)=f(x,y,z)+[-RfD_x^{\beta}f-RgD_y^{\gamma}f-RmD_z^{\xi}f-\\
-\varphi_1(x)+D_x^{2\beta}f+D_y^{2\gamma}f+D_z^{2\xi}f]\frac{t^{2\alpha}}{\Gamma(\alpha+1)}\\
v_1(x,y,z,t)=g(x,y,z)+\\
+[-RfD_x^{\beta}g-RgD_y^{\gamma}g-RmD_z^{\xi}g-\phi_1(y)+D_x^{2\beta}g+D_y^{2\gamma}g+D_z^{2\xi}g]\frac{t^{2\alpha}}{\Gamma(\alpha+1)}\\
w_1(x,y,z,t)=m(x,y,z)+\\
+[-RfD_x^{\beta}m-RgD_y^{\gamma}m-RmD_z^{\xi}m-\psi_1(z)+D_x^{2\beta}m+D_y^{2\gamma}m+D_z^{2\xi}m]\frac{t^{2\alpha}}{\Gamma(\alpha+1)}\\
P_1(x,y,z,t)=h_0(x,y,z)=\int\varphi_1(x)\mathrm{d}x+\int\phi_1(y)\mathrm{d}y+\int\psi_1(z)\mathrm{d}z
\end{gathered} \quad (17)$$

For $k=2$, we have:

$$\Delta T_1=\tilde{\tau}+\frac{\tilde{\tau}_1 t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\tilde{\tau}_2 t^{2\alpha}}{\Gamma(2\alpha+1)} \quad \text{and} \quad P_1(x,y,z,t)=h_0(x,y,z)+\frac{h_1(x,y,z)t^{\alpha}}{\Gamma(\alpha+1)} \quad (18)$$

Substituting eq. (18) into $\mathrm{Re}s_{uk}$, $\mathrm{Re}s_{vk}, \mathrm{Re}s_{wk}$ at $t=0$, we obtain the second approximate RPS solutions:

$$\begin{gathered}
u_2=f+[-RfD_x^{\beta}f-RgD_y^{\gamma}f-RmD_z^{\xi}f-\varphi_1(x)+D_x^{2\beta}f+D_y^{2\gamma}f+D_z^{2\xi}f]\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\\
+[-\varphi_2(x)-Rf(x,y_b,z)D_x^{\beta}f_1(x,y_b,z)-Rf_1(x,y_b,z)D_x^{\beta}f(x,y_b,z)+D_x^{2\beta}f_1(x,y_b,z)+\\
+D_y^{2\gamma}f_1(x,y_b,z)+D_z^{2\xi}f_1(x,y_b,z)-Rg(x,y_b,z)D_y^{\gamma}f_1(x,y_b,z)-Rg_1(x,y_b,z)\times\\
\times D_y^{\gamma}f(x,y_b,z)-Rm(x,y_b,z)D_z^{\xi}f_1(x,y_b,z)-Rm_1(x,y_b,z)D_z^{\xi}f(x,y_b,z)]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}
\end{gathered} \quad (19)$$

$$v_2 = g + [-RfD_x^\beta g - RgD_y^\gamma g - RmD_z^\xi g - \phi_1(y) + D_x^{2\beta} g + D_y^{2\gamma} g + D_z^{2\xi} g]\frac{t^\alpha}{\Gamma(\alpha+1)} +$$
$$+[-\phi_2(y) - Rf(x,y,z_b)D_x^\beta g_1(x,y,z_b) - Rf_1(x,y,z_b)D_x^\beta g(x,y,z_b) + D_x^{2\beta} g_1(x,y,z_b) +$$
$$+D_y^{2\gamma} g_1(x,y,z_b) + D_z^{2\xi} g_1(x,y,z_b) - Rg(x,y,z_b)D_y^\gamma g_1(x,y,z_b) - Rg_1(x,y,z_b)\times$$
$$\times D_y^\gamma g(x,y,z_b) - Rm(x,y,z_b)D_z^\xi g_1(x,y,z_b) - Rm_1(x,y,z_b)D_z^\xi g(x,y,z_b)]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \quad (20)$$

$$w_2 = m + [-RfD_x^\beta m - RgD_y^\gamma m - RmD_z^\xi m - \psi_1(z) + D_x^{2\beta} m + D_y^{2\gamma} m + D_z^{2\xi} m]\frac{t^\alpha}{\Gamma(\alpha+1)} +$$
$$+[-\psi_2(z) - Rf(x_b,y,z)D_x^\beta m_1(x_b,y,z) - Rf_1(x_b,y,z)D_x^\beta m(x_b,y,z) + D_x^{2\beta} m_1(x_b,y,z) +$$
$$+D_y^{2\gamma} m_1(x_b,y,z) + D_z^{2\xi} m_1(x_b,y,z) - Rg(x_b,y,z)D_y^\gamma m_1(x_b,y,z) - Rg_1(x_b,y,z)\times$$
$$\times D_y^\gamma m(x_b,y,z) - Rm(x_b,y,z)D_z^\xi m_1(x_b,y,z) - Rm_1(x_b,y,z)D_z^\xi m(x_b,y,z)]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$\mathrm{P}_2(x,y,z,t) = h_1(x,y,z) = \int \varphi_2(x)dx + \int \phi_2(y)dy + \int \psi_2(z)dz \quad (21)$$

Table 1. The errors of *u*(*x*, *y*, *z*)

$y = z = \pi/2$ $\|\|u_2 - u_{\text{Exact}}\|\|$				
x	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
π/6	$4.03\cdot10^{-2}$	$1.27\cdot10^{-1}$	$2.62\cdot10^{-1}$	$4.43\cdot10^{-1}$
π/5	$3.87\cdot10^{-2}$	$1.17\cdot10^{-1}$	$2.36\cdot10^{-1}$	$3.95\cdot10^{-1}$
π/4	$3.50\cdot10^{-2}$	$9.82\cdot10^{-2}$	$1.90\cdot10^{-1}$	$3.10\cdot10^{-1}$
π/3	$2.69\cdot10^{-2}$	$6.34\cdot10^{-2}$	$1.10\cdot10^{-1}$	$1.65\cdot10^{-1}$
π/2	$1.47\cdot10^{-2}$	$2.95\cdot10^{-2}$	$4.42\cdot10^{-2}$	$5.89\cdot10^{-2}$

Table 2. The errors of *v*(*x*, *y*, *z*)

$x = z = \pi/2$ $\|\|v_2 - v_{\text{Exact}}\|\|$				
x	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
π/6	$7.19\cdot10^{-1}$	$9.49\cdot10^{-1}$	$1.12\cdot10^{0}$	$1.25\cdot10^{0}$
π/5	$7.02\cdot10^{-1}$	$9.27\cdot10^{-1}$	$1.09\cdot10^{0}$	$1.22\cdot10^{0}$
π/4	$6.45\cdot10^{-1}$	$8.52\cdot10^{-1}$	$1.00\cdot10^{0}$	$1.12\cdot10^{0}$
π/3	$5.00\cdot10^{-1}$	$6.60\cdot10^{-1}$	$7.76\cdot10^{-1}$	$8.71\cdot10^{-1}$
π/2	$1.26\cdot10^{-1}$	$1.67\cdot10^{-1}$	$1.96\cdot10^{-1}$	$2.20\cdot10^{-1}$

Table 3. The errors of *w*(*x*, *y*, *z*)

$x = z = \pi/2$ $\|\|w_2 - w_{\text{Exact}}\|\|$				
x	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
π/6	$7.82\cdot10^{-1}$	$1.03\cdot10^{0}$	$1.21\cdot10^{0}$	$1.36\cdot10^{0}$
π/5	$7.90\cdot10^{-1}$	$1.04\cdot10^{0}$	$1.23\cdot10^{0}$	$1.38\cdot10^{0}$
π/4	$7.67\cdot10^{-1}$	$1.01\cdot10^{0}$	$1.19\cdot10^{0}$	$1.34\cdot10^{0}$
π/3	$6.19\cdot10^{-1}$	$8.17\cdot10^{-1}$	$9.61\cdot10^{-1}$	$1.08\cdot10^{0}$
π/2	$2.28\cdot10^{-16}$	$3.01\cdot10^{-16}$	$3.54\cdot10^{-16}$	$3.98\cdot10^{-16}$

Numerical results

In this section, in order to verify the accuracy of the RPS method, we study the numerical solution of the time-space fractional Navier-Stokes equations. The errors between the exact solutions and the third order RPS approximate solutions when $\alpha = 1$ at different time t are shown in tabs. 1-3.

Conclusion

In the present work, the time-space fractional coupled Navier-Stokes equations was constructed using the semi-inverse method and the Agrawal's method. The analytical solutions of the three-dimensional time-space fractional Navier-Stokes equations were obtained by using the RPS method, and the numerical results were in good agreement with the exact solution. It is shown that the RPS method is a very simple and effective tool for solving linear and nonlinear fractional partial differential equations.

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