A NEW GENERAL FRACTIONAL DERIVATIVE GOLDSTEIN-KAC-TYPE TELEGRAPH EQUATION

by

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In this paper, we consider the Riemann-Liouville-type general fractional derivatives of the non-singular kernel of the one-parametric Lorenzo-Hartley function. A new general fractional-order-derivative Goldstein-Kac-type telegraph equation is proposed for the first time. The analytical solution of the considered model with the graphs is obtained with the aid of the Laplace transform. The general fractional-order-derivative formula is as a new mathematical tool proposed to model the anomalous behaviors in complex and power-law phenomena.

Key words: *general fractional-order derivative, Lorenzo-Hartley function, telegraph equation, Laplace transform*

Introduction

The telegraph equation, also known as the transmission line equations, was firstly proposed by Kelvin in [1]. It has long been the focus of the attention of mathematicians and engineers [2, 3]. As early as the Atlantic cable was communicated, Kelvin discovered a long-line effect. The reflection and transmission of the telegraph signals are very different from the low frequencies. Subsequently, Kelvin further proposed the theory of the signal propagation based on *telegraph equation*, which was used to explain how the signal propagates in the cable and to solve the major theoretical problems of laying cables on the seabed. The telegraph equation can be widely used to characterize both the wave and conduction processes.

In recent years, the fractional telegraph equation has received the great attention in theory and applications [4-7]. At present, the fractional telegraph equation has been applied in many scientific fields to describe the various phenomena, such as the fluid-flow of the porous materials, reaction diffusion phenomena, acoustic wave propagation in viscoelastic materials, self-similar structural dynamics, signals in bioelectric systems and processing and so on [8-10]. Due to the characteristics of the theory of the fractional calculus, it compensates for the defects of the classical differential equations in the scoring objective reality.

The general fractional calculus, as the kernels of the special functions, for example, the exponential, Lorenzo-Hartley, Miller-Ross, Mittag-Leffler and Prabhakar functions, were considered in [11-13]. The general fractional-order-derivative (GFD) relaxation via exponential kernel was discussed in [14]. The rheological model in the general fractional-order via

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Prabhakar kernel was investigated in [15]. The general fractional-order Burgers model via Mittag-Leffler was reported in [16].

In 1998, Lorenzo and Hartley applied the one-parametric Lorenzo-Hartley (LH) function [17-19] to describe the relaxation process in [20-24]. The Goldstein-Kac-type telegraph model within the GFD with one-parametric LH functions has not been proposed. Motivated by this idea, the main purpose of the paper is to propose the Goldstein-Kac-type telegraph equation within the GFD with one-parametric LH function in the sense of Riemann-Liouville-type and to investigate the analytical solution of the proposed model with the help of the Laplace transform.

General fractional derivative with one-parametric LH function

Let $\mathbb{R}, \mathbb{R}_0^+, \mathbb{C}, \mathbb{N}$, and $\mathbb{N}_0$ be the sets of real numbers, non-negative real numbers, complex numbers, positive integers and $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, respectively.

Special functions

The one-parametric LH function is defined [17-19]:

$$\Omega_\alpha(\mu x^\alpha) = \sum_{i=0}^{\infty} \frac{\mu^i x^{(i+1)\alpha-1}}{\Gamma\left[(i+1)\alpha\right]} \tag{1}$$

where $x \in \mathbb{R},\ \alpha \in \mathbb{R}_0^+,\ \mu \in \mathbb{R}_0^+$, and $\kappa \in \mathbb{N}_0$.

The Laplace transform of eq. (1) is given [12]:

$$\mathfrak{A}\{\Omega_\alpha(\mu x^\alpha)\} = \tau^{-\alpha}(1 - \mu\tau^{-\alpha})^{-1} \tag{2}$$

where $\mathfrak{A}\{f(x)\} = f(\tau)$ is the Laplace transform operator.

The Prabhakar function is defined:

$$E_{\alpha,\nu}^{\theta}(x^\alpha) = \sum_{\kappa=0}^{\infty} \frac{\Gamma(\theta+\kappa)}{\Gamma(\kappa+1)\Gamma(\theta)} \frac{x^{\kappa\alpha}}{\Gamma(\kappa\alpha+\nu)} \tag{3}$$

The Laplace transform of eq. (3) is:

$$\mathfrak{A}\{x^{\nu-1}E_{\alpha,\nu}^{\theta}(\lambda x^\alpha)\} = t^{-\nu}(1 - \lambda t^{-\alpha})^{-\theta} \tag{4}$$

Riemann-Liouville-type GFD within LH kernal

Let $\alpha \in \mathbb{C},\ 1 > \mathrm{Re}(\alpha) > 0,\ \kappa = [\mathrm{Re}(\alpha)] + 1,\ -\infty < a < b < \infty,\ \kappa \in \mathbb{N}$, and $\gamma \in \mathbb{C}$.

The left-sided Riemann-Liouville-type GFD of one-parametric LH function is defined [11]:

$${}_{A}^{RL}D_{a+}^{\alpha,\kappa,\gamma}\Xi(x) = \frac{\mathrm{d}^\kappa}{\mathrm{d}x^\kappa}\left[{}_{A}I_{a+}^{\alpha,\gamma}\Xi(x)\right] = \frac{\mathrm{d}^\kappa}{\mathrm{d}x^\kappa}\int_a^x \Omega_\alpha\left[-\gamma(x-\tau)^\alpha\right]\Xi(\tau)\,\mathrm{d}\tau \tag{5}$$

and the right-sided Riemann-Liouville-type GFD with one-parametric LH function by [11]:

$${}_{A}^{RL}D_{b-}^{\alpha,\kappa,\gamma}\Xi(x) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^\kappa\left[{}_{A}I_{b-}^{\alpha,\gamma}\Xi(x)\right] = (-1)^\kappa\frac{\mathrm{d}^\kappa}{\mathrm{d}x^\kappa}\int_x^b \Omega_\alpha\left[-\gamma(\tau-x)^\alpha\right]\Xi(\tau)\,\mathrm{d}\tau \tag{6}$$

In particular, when $\kappa = 1$, eqs. (4) and (5) become [11]:

$${}_{A}^{RL}D_{a+}^{\alpha,\gamma}\Xi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[{}_{A}I_{a+}^{\alpha,\gamma}\Xi(x)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \Omega_\alpha\left[-\gamma(x-\tau)^\alpha\right]\Xi(\tau)\,\mathrm{d}\tau \tag{7}$$

and

$$ {}_{A}^{RL}D_{b-}^{\alpha,\gamma}\Xi(x)=-\frac{\mathrm{d}}{\mathrm{d}x}\left[{}_{A}I_{b-}^{\alpha,\gamma}\Xi(x)\right]=-\frac{\mathrm{d}}{\mathrm{d}x}\int_{x}^{b}\Omega_{\alpha}\left[-\gamma(\tau-t)^{\alpha}\right]\Xi(\tau)\,\mathrm{d}\tau \tag{8} $$

where

$$ {}_{A}I_{a+}^{\alpha,\gamma}\Xi(x)=\int_{a}^{x}\Omega_{\alpha}\left[-\gamma(x-\tau)^{\alpha}\right]\Xi(\tau)\,\mathrm{d}\tau \tag{9} $$

and

$$ {}_{A}I_{b-}^{\alpha,\gamma}\Xi(x)=\int_{x}^{b}\Omega_{\alpha}\left[-\gamma(\tau-x)^{\alpha}\right]\Xi(\tau)\,\mathrm{d}\tau \tag{10} $$

For $a=0$ eqs. (4) and (5) can be written [11]:

$$ {}_{A}^{RL}D_{0+}^{\alpha,\kappa,\gamma}\Xi(x)=\frac{\mathrm{d}^{\kappa}}{\mathrm{d}x^{\kappa}}\left[{}_{A}I_{0+}^{\alpha,\gamma}\Xi(x)\right]=\frac{\mathrm{d}^{\kappa}}{\mathrm{d}x^{\kappa}}\int_{0}^{x}\Omega_{\alpha}\left[-\gamma(x-\tau)^{\alpha}\right]\Xi(\tau)\,\mathrm{d}\tau \tag{11} $$

and

$$ {}_{A}^{RL}D_{0+}^{\alpha,\gamma}\Xi(x)=\frac{\mathrm{d}}{\mathrm{d}x}\left[{}_{A}I_{0+}^{\alpha,\gamma}\Xi(x)\right]=\frac{\mathrm{d}}{\mathrm{d}x}\int_{0}^{x}\Omega_{\alpha}\left[-\gamma(x-\tau)^{\alpha}\right]\Xi(\tau)\,\mathrm{d}\tau \tag{12} $$

respectively.

Let $\alpha\in\mathbb{C}, 1>\mathrm{Re}(\alpha)>0$, $-\infty<a<b<\infty$, and $\gamma\in\mathbb{C}$. Then we have [11, 25]:

$$ \mathfrak{A}\{{}_{A}^{RL}D_{0}^{\alpha,\gamma}\Xi(x)\}=t^{1-\alpha}(1+\gamma t^{-\alpha})^{-1}\overline{\Xi}(t)-{}_{A}I_{0+}^{\alpha,\gamma}\Xi(0) \tag{13} $$

where

$$ {}_{A}I_{0+}^{\alpha,\gamma}\Xi(x)=\int_{0}^{x}\Omega_{\alpha}\left[-\gamma(x-\tau)^{\alpha}\right]\Xi(\tau)\,\mathrm{d}\tau \tag{14} $$

A GFD Goldstein-Kac-type telegraph model

We now consider a new model of the GFD Goldstein-Kac-type telegraph equation:

$$ {}_{A}^{RL}\partial_{0+}^{\alpha,2,\gamma}u(x,t)-c^{2}\frac{\partial^{2}u(x,t)}{\partial x^{2}}+2a\,{}_{A}^{RL}\partial_{0+}^{\alpha,1,\gamma}u(x,t)=0\quad\left(0<x<L,\ t>0\right) \tag{15} $$

with the initial and boundary conditions:

$$ u(0,t)=0 \tag{16} $$

$$ u(x,0)=0 \tag{17} $$

$$ {}_{A}I_{0+}^{\alpha,\gamma}u(x,0)=0 \tag{18} $$

$$ \frac{\partial}{\partial t}\left[{}_{A}I_{0+}^{\alpha,\gamma}u(x,0)\right]=\cos(\pi x) \tag{19} $$

where

$$ {}_{A}^{RL}\partial_{0+}^{\alpha,2,\gamma} u(x,t) = \frac{\partial^2}{\partial t^2}\left[{}_{A}I_{0+}^{\alpha;\gamma} u(x,t) \right] = \frac{\partial^2}{\partial t^2}\int_0^t \Omega_\alpha\left[-\gamma(t-\tau)^\alpha \right] u(x,\tau)\,\mathrm{d}\tau \tag{20} $$

and

$$ {}_{A}^{RL}\partial_{0+}^{\alpha,1,\gamma} u(x,t) = \frac{\partial}{\partial t}\left[{}_{A}I_{0+}^{\alpha;\gamma} u(x,t) \right] = \frac{\partial}{\partial t}\int_0^t \Omega_\alpha\left[-\gamma[t-\tau]^\alpha \right] u(x,\tau)\,\mathrm{d}\tau \tag{21} $$

With the use of eqs. (13) and (19), we have that:

$$ \frac{\partial^2}{\partial x^2} u(x,s) = \frac{1}{c^2}\left[s^{2-\alpha}\left(1+\gamma s^{-\alpha}\right)^{-1} u(x,s) - \cos(\pi x) \right] + \frac{2a}{c^2} s^{1-\alpha}\left(1+\gamma s^{-\alpha}\right)^{-1} u(x,s) = $$
$$ = \frac{1}{c^2} s^{1-\alpha}(s+2a)\left(1+\gamma s^{-\alpha}\right)^{-1} u(x,s) - \frac{1}{c^2}\cos(\pi x) \tag{22} $$

Setting the general solution of eq. (15), we have that:

$$ u(x,s) = A_1 \sin(\pi x) + A_2 \cos(\pi x) \tag{23} $$

where A_1 and A_2 are the constants.

Then, we have:

$$ \frac{\cos(\pi x)}{c^2} = \left[\pi^2 + \frac{1}{c^2} s^{1-\alpha}(s+2a)(1+\gamma s^{-\alpha})^{-1} \right]\left[A_1 \sin(\pi x) + A_2 \cos(\pi x) \right] \tag{24} $$

which leads to:

$$ A_1 = 0 \tag{25} $$

and

$$ A_2 = \frac{1}{c^2\pi^2 + s^{1-\alpha}(s+2a)(1+\gamma s^{-\alpha})^{-1}} \tag{26} $$

Thus, we have:

$$ u(x,s) = \frac{\cos(\pi x)}{c^2\pi^2}\sum_{k=0}^{n}\left(-\frac{1}{c^2\pi^2} \right)^k \sum_{r=0}^{k} C_k^r s^{(1-\alpha)k+r}(2a)^{k-r}(1+\gamma s^{-\alpha})^{-k} \tag{27} $$

where

$$ n \to \infty, \quad \left| \frac{s^{1-\alpha}}{c^2\pi^2}(s+2a)\left(1+\gamma s^{-\alpha}\right)^{-1} \right| < 1 \tag{28} $$

By considering the inverse Laplace transform, we can obtain the solution of eq. (27):

$$ u(x,t) = \frac{\cos(\pi x)}{c^2\pi^2}\sum_{k=0}^{n}\left(-\frac{1}{c^2\pi^2} \right)^k \sum_{r=0}^{k} C_k^r (2a)^{k-r} t^{(\alpha-1)k-r-1} E_{\alpha,(\alpha-1)k-r}^{k}(-\gamma t^\alpha) \tag{29} $$

In particular, for $a = 0$ we have:

$$u(x,s)=\frac{\cos(\pi x)}{c^2\pi^2}\sum_{k=0}^{n}\left(-\frac{1}{c^2\pi^2}\right)^k s^{(2-\alpha)k}(1+\gamma s^{-\alpha})^{-k} \tag{30}$$

Thus, we have:

$$u(x,t)=\frac{\cos(\pi x)}{c^2\pi^2}\sum_{k=0}^{n}\left(-\frac{1}{c^2\pi^2}\right)^k t^{(\alpha-2)k-1}E_{\alpha,(\alpha-2)k}^{k}(-\gamma t^{\alpha}) \tag{31}$$

and the graph of the solution of (27) is showed in fig. 1.

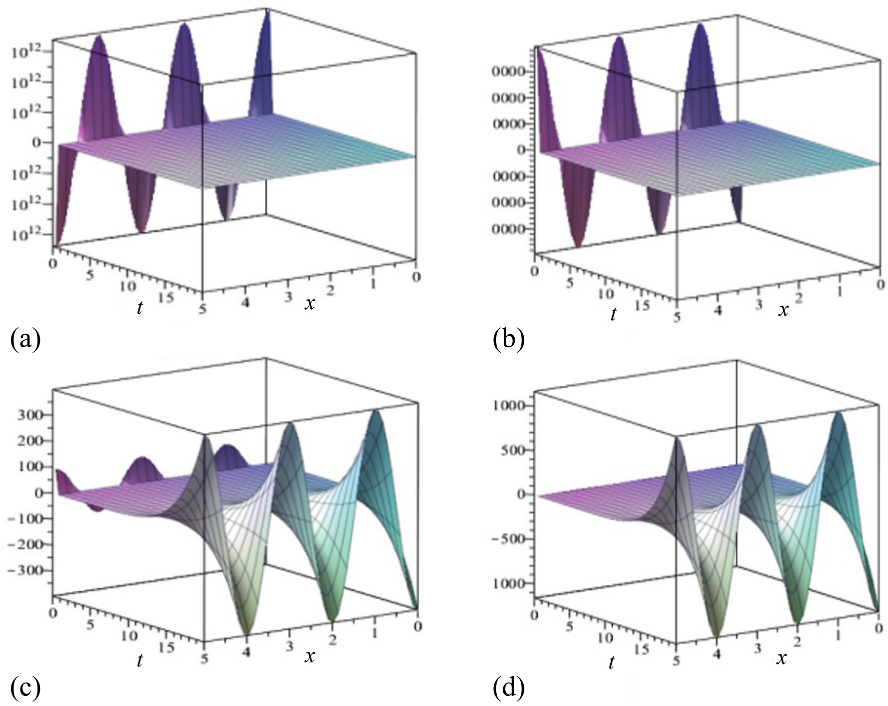

Figure 1. The solution of (27) with the different values; (a) $\alpha = 0.2$, (b) $\alpha = 0.6$, (c) $\alpha = 0.8$, and (d) $\alpha = 0.9$

Conclusion

In our work, based on the GFD with one-parametric LH function, the GFD Goldstein-Kac-type telegraph equation was proposed for the first time. The analytical solution of the mathematical model was discussed by using the Laplace transform scheme. The GFD formula can be used to express other nonlinear phenomena appear in complex and power-law phenomena.

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