ON TRAVELING-WAVE SOLUTIONS FOR THE SCALING-LAW TELEGRAPH EQUATIONS

by

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The aim of the study is to address the scaling-law telegraph equations with the Mandelbrot-scaling-law derivative. The traveling-wave solutions with use of the Kohlrausch-Williams-Watts function are considered in detail. The works are proposed to describe the physical models in complex topology.

Key words: *scaling-law telegraph equation, Leibniz derivative, Mandelbrot-scaling-law derivative, traveling-wave solution, Kohlrausch-Williams-Watts function*

Introduction

The telegraph equations, which are the original tasks proposed by Heaviside in 1876 [1], have been developed to describe the transmission line models with the classical derivative operator [2], which leads the diffusion problem. The mathematical models for the fractional telegraph equations based on the Riemann-Liouville sense [3] leads to the power-law behaviors of the Brownian time [4], and contacted the fractional diffusion process in [5].

The calculus with respect to monotone function (so-called the general calculus) goes back to many mathematicians, *e. g.*, Leibniz and Stieltjes [6-10]. The Leibniz derivative (so-called the derivative with respect to monotone function) was proposed by Leibniz in 1676 [11]. Based on the Riemann's work [12], the Stieltjes-Riemann integral (so-called the integral with respect to monotone function) was proposed by Stieltjes in 1894 [13]. The scaling-law derivative (so-called the derivative with respect to scaling-law function) was proposed in [7] and further reported in [9]. The scaling-law integral (so-called the integral with respect to scaling-law function) was proposed in [9]. Their geometric interpretations and the topology calculus were discussed in [6, 9].

The scaling theory can be applied to the complex circuit, which is experimentally confirmed in [14,15]. Considering the Brownian time and Mandelbrot-scaling-law behaviors [16] in the circuit, the aim of the paper is to derive the electrical phenomena, which involve the Leibniz derivative and Mandelbrot-scaling-law derivative, to propose the scaling-law telegraph equations and to present the traveling wave solutions via Kohlrausch-Williams-Watts function [17, 18].

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The Leibniz derivative, Stieltjes integral and scaling-law calculus

The general calculus

Let $\psi_h(t) = (\psi \circ h)(t) = \psi[h(t)]$, where $h^{(1)}(t) > 0$, and let $\mathbb{N}$ be the set of the integer numbers.

The Leibniz derivative of the function $\psi_h(t)$ is defined [6, 11]:

$$D_{t,h(\cdot)}^{(1)}\psi_h(t) = \frac{1}{h^{(1)}(t)}\frac{\mathrm{d}\psi_h(t)}{\mathrm{d}t} \tag{1}$$

The Leibniz derivative of high-order is defined [6, 11]:

$$D_{t,h(\cdot)}^{(n)}\psi_h(t) = \left[\frac{1}{h^{(1)}(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right]^n \psi_h(t) \tag{2}$$

where $n \in \mathbb{N}$.

The Riemann-Stieltjes integral of the function $\psi_h(t)$ is defined:

$$_aI_{b,h(\cdot)}^{(1)}\psi_h(t) = \int_a^b \psi_h(t)h^{(1)}(t)\mathrm{d}t \tag{3}$$

Here, eqs. (1) and (3) are called the calculus with respect to monotone function [6].

The scaling-law calculus

Let $\psi(t) = [\psi \circ (\kappa t^\beta)](t) = \psi(\kappa t^\beta)$, where κ is the normalization constant and β is the scaling exponent [7, 9].

The scaling-law derivative of the function $\psi(t)$ is defined [7]:

$$^{SL}D_t^{(1)}\psi(t) = \frac{1}{\beta\kappa t^{\beta-1}}\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} \tag{4}$$

The scaling-law derivative of high-order is defined [6, 11]:

$$^{SL}D_t^{(n)}\psi(t) = \left(\frac{1}{\beta\kappa t^{\beta-1}}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \psi(t) \tag{5}$$

where $n \in \mathbb{N}$.

The scaling-law integral of the function $\psi(t)$ is defined [9]:

$$^{SL}_{\;\;a}I_t^{(1)}\psi(t) = \beta\kappa\int_a^t \psi(t)t^{\beta-1}\mathrm{d}t \tag{6}$$

Here, eqs. (4) and (5) are called the scaling-law calculus [6, 7, 9].

Mandelbrot-scaling-law calculus

Let $\psi(t) = \left[\psi \circ (\kappa t^{1-D})\right](t) = \psi(\kappa t^{1-D})$, where κ is the normalization constant and D is the scaling exponent.

The Mandelbrot scaling-law function $\psi(t)$ is defined as [16]:

$$g(t) = \kappa t^{1-D} \tag{7}$$

where $\kappa \in \mathbb{R}_+$, $t \in \mathbb{R}_+$, and D is the fractal dimension with $D \in \mathbb{R}_+$.

The Mandelbrot-scaling-law derivative of the function $\psi(t)$ is defined:

$$ {}^{MSL}D_t^{(1)}\psi(t) = \frac{t^D}{(1-D)\kappa}\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} \tag{8} $$

The Mandelbrot-scaling-law derivative of high-order is defined:

$$ {}^{MSL}D_t^{(n)}\psi(t) = \left[\frac{t^D}{(1-D)\kappa}\frac{\mathrm{d}}{\mathrm{d}t}\right]^n \psi(t) \tag{9} $$

where $n \in \mathbb{N}$.

The Mandelbrot-scaling-law integral of the function $\psi(t)$ is defined:

$$ {}^{MSL}_{a}I_t^{(1)}\psi(t) = (1-D)\kappa\int_a^t \psi(t)t^D\mathrm{d}t \tag{10} $$

Here, eqs. (8) and (10) are called the Mandelbrot-scaling-law calculus.

The Mandelbrot-scaling-law partial derivative of the function $\psi(x,t)$ respect to the variable x is defined:

$$ {}^{MSL}\partial_x^{(1)}\psi(x,t) = \frac{t^D}{(1-D)\kappa}\frac{\partial\psi(x,t)}{\partial x} \tag{11} $$

and the Mandelbrot-scaling-law partial derivative of the function $\psi(x,t)$ respect to the variable t as:

$$ {}^{MSL}\partial_t^{(1)}\psi(x,t) = \frac{t^D}{(1-D)\kappa}\frac{\partial\psi(x,t)}{\partial t} \tag{12} $$

The indefinite Mandelbrot-scaling-law integral, which is called the antidifferentiation with respect to Mandelbrot-scaling-law function, is defined:

$$ {}^{MSL}I_t^{(1)}\psi(t) = (1-D)\kappa\int \psi(t)t^D\mathrm{d}t = \Phi(t) + \vartheta \tag{13} $$

which leads to:

$$ {}^{MSL}D_t^{(1)}\left[{}^{MSL}I_t^{(1)}\psi(t)\right] = {}^{MSL}D_t^{(1)}\left[\Phi(t)+\vartheta\right] = {}^{MSL}D_t^{(1)}\Phi(t) = \psi(t) \tag{14} $$

and

$$ {}^{MSL}I_t^{(1)}\left[{}^{MSL}D_t^{(1)}\Phi(t)\right] = \Phi(t) + \vartheta \tag{15} $$

where ϑ is the constant.

Let $\psi(t) = \Lambda(\kappa t^{D-1})$, $\psi_1(t) = \Lambda_1(\kappa t^{D-1})$, and $\psi_2(t) = \Lambda_2(\kappa t^{D-1})$ be continuous functions.

The properties of the Mandelbrot-scaling-law calculus can be given:

– (S1) The sum and difference rules for the Mandelbrot-scaling-law derivative:

$$ {}^{MSL}D_t^{(1)}\left[\psi_1(t) \pm \psi_2(t)\right] = {}^{MSL}D_t^{(1)}\psi_1(t) \pm {}^{MSL}D_t^{(1)}\psi_2(t) \tag{16} $$

where ${}^{MSL}D_t^{(1)}\psi_1(t)$ and ${}^{MSL}D_t^{(1)}\psi_2(t)$ exist.

– (S2) The constant multiple rule for the Mandelbrot-scaling-law derivative:

$$^{MSL}D_t^{(1)}[\mu\psi(t)] = \mu\, ^{MSL}D_t^{(1)}\psi(t) \tag{17}$$

where $^{SL}D_t^{(1)}\psi(t)$ exists and μ is a constant.

– (S3) The product rule for the Mandelbrot-scaling-law derivative:

$$^{MSL}D_t^{(1)}\left[\psi_1(t)\psi_2(t)\right] = \psi_2(t)\, ^{MSL}D_t^{(1)}\psi_1(t) + \psi_1(t)\, ^{MSL}D_t^{(1)}\psi_2(t) \tag{18}$$

where $^{MSL}D_t^{(1)}\psi_1(t)$ and $^{MSL}D_t^{(1)}\psi_2(t)$ exist.

– (S4) The quotient rule for the Mandelbrot-scaling-law derivative:

$$^{MSL}D_t^{(1)}\left[\frac{\psi_1(t)}{\psi_2(t)}\right] = \frac{\psi_2(t)\, ^{MSL}D_t^{(1)}\psi_1(t) - \psi_1(t)\, ^{MSL}D_t^{(1)}\psi_2(t)}{\psi_2(t)\psi_2(t)} \tag{19}$$

where $^{MSL}D_t^{(1)}\psi_1(t)$ and $^{MSL}D_t^{(1)}\psi_2(t)$ exist, and $\varphi_2(t) \neq 0$.

– (S5) The chain rule for the Mandelbrot-scaling-law derivative:

$$^{MSL}D_t^{(1)}\left\{w\left[\psi(t)\right]\right\} = \frac{\mathrm{d}w(\psi)}{\mathrm{d}\psi}\, ^{MSL}D_t^{(1)}\psi(t) \tag{20}$$

where $\mathrm{d}w(\psi)/\mathrm{d}\psi$ and $^{MSL}D_t^{(1)}\psi(t)$ exist.

– (S6) The sum and difference rules for the Mandelbrot-scaling-law integral:

$$^{MSL}_{\ \ a}I_t^{(1)}\left[\psi_1(t) \pm \psi_2(t)\right] = {}^{MSL}_{\ \ a}I_t^{(1)}\psi_1(t) \pm {}^{MSL}_{\ \ a}I_t^{(1)}\psi_2(t) \tag{21}$$

where $^{MSL}_{\ \ a}I_t^{(1)}\psi_1(t)$ and $^{MSL}_{\ \ a}I_t^{(1)}\psi_2(t)$ exist.

– (S7) The integration by parts for the Mandelbrot-scaling-law integral:

$$^{MSL}_{\ \ a}I_t^{(1)}\left[\psi_2(t)\, ^{MSL}D_t^{(1)}\psi_1(t)\right] = \left[\psi_1(t)\psi_2(t)\right]_a^t - {}^{MSL}_{\ \ a}I_t^{(1)}\left[\psi_1(t)\, ^{MSL}D_t^{(1)}\psi_2(t)\right] \tag{22}$$

where $[\psi_1(t)\psi_2(t)]_a^t = \psi_1(t)\psi_2(t) - \psi_1(a)\psi_2(a)$.

– (S8) The sum and difference rules for the indefinite Mandelbrot-scaling-law integral:

$$^{MSL}I_t^{(1)}\left[\psi_1(t) \pm \psi_2(t)\right] = {}^{MSL}I_t^{(1)}\psi_1(t) \pm {}^{MSL}I_t^{(1)}\psi_2(t) \tag{23}$$

where $^{MSL}I_t^{(1)}\psi_1(t)$ and $^{MSL}I_t^{(1)}\psi_2(t)$ exist.

– (S9) The integration by parts for the indefinite Mandelbrot-scaling-law integral:

$$^{MSL}I_t^{(1)}\left[\psi_2(t)\, ^{MSL}D_t^{(1)}\psi_1(t)\right] = \psi_2(t)\psi_1(t) - {}^{MSL}I_t^{(1)}\left[\psi_1(t)\, ^{MSL}D_t^{(1)}\psi_2(t)\right] \tag{24}$$

The $n-$order Mandelbrot-scaling-law partial derivative of the function $\psi(x,t)$ respect to the variable x is defined:

$$^{MSL}\partial_x^{(n)}\psi(x,t) = \left[\frac{t^D}{(1-D)\kappa}\frac{\partial}{\partial x}\right]^n \psi(x,t) \tag{25}$$

and the $n-$order Mandelbrot-scaling-law partial derivative of the function $\psi(x,t)$ respect to the variable t as:

$$^{MSL}\partial_t^{(n)}\psi(x,t) = \left[\frac{t^D}{(1-D)\kappa}\frac{\partial}{\partial t}\right]^n \psi(x,t) \tag{26}$$

The Mandelbrot-scaling-law-telegraph models with the traveling-wave solution

The Mandelbrot-scaling-law-telegraph equation

Let us derive the Mandelbrot-scaling-law telegraph equation based on the Mandelbrot-scaling-law calculus.

The voltage across the resistor, which is derived from the well-known Ohm's law, is given:

$$u(x,t)=Ri(x,t) \tag{27}$$

where R is the resistance of the cable, $i(x,t)$ is the current on the cable at any point x and any time t, and $u(x,t)$ is the voltage on the cable at any point x and any time t.

The voltage drop across the inductor element via Mandelbrot-scaling-law derivative is given:

$$u(x,t)=L\,{}^{MSL}\partial_t^{(1)}i(x,t)=\frac{Lt^D}{(1-D)\kappa}\frac{\mathrm{d}i(x,t)}{\mathrm{d}t} \tag{28}$$

which leads to:

$$u(x,t)=\frac{1}{L}\,{}^{MSL}I_t^{(1)}i(x,t)=\frac{(1-D)\kappa}{L}\int i(x,t)t^D\mathrm{d}t \tag{29}$$

where L is denotes the inductance of the cable, κ is the normalization constant, and $D(0\le D\le 1)$ is the fractal dimension.

The voltage drop across the capacitor element via Mandelbrot-scaling-law integral is given:

$$u(x,t)=\frac{1}{C}\,{}^{MSL}I_t^{(1)}\psi(t)=\frac{(1-D)\kappa}{C}\int i(x,t)t^D\mathrm{d}t \tag{30}$$

which reduces to:

$$i(x,t)=C\partial_t^{(1)}u(x,t)=\frac{Ct^D}{(1-D)\kappa}\frac{\mathrm{d}u(x,t)}{\mathrm{d}t} \tag{31}$$

where C denotes the inductance of the cable, κ is the normalization constant, and D is the fractal dimension.

In the telegraphic transmission line with leakage, we present:

$$\frac{\partial u(x,t)}{\partial x}=-Ri(x,t)-L\,{}^{MSL}\partial_t^{(1)}i(x,t) \tag{32}$$

In the current through leakage to the ground, we may give:

$$\frac{\partial i(x,t)}{\partial x}=-Gu(x,t)-C\,{}^{MSL}\partial_t^{(1)}u(x,t) \tag{33}$$

where G denotes the conductance to the ground.

Finding the Mandelbrot-scaling-law derivative of (32), we have:

$$\frac{\partial^2 u(x,t)}{\partial x^2}=-R\frac{\partial}{\partial x}i(x,t)-L\,{}^{MSL}\partial_t^{(1)}\left[\frac{\partial}{\partial x}i(x,t)\right] \tag{34}$$

Substituting eq. (33) into eq. (34), we have:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = RGu(x,t) + \left(RC + LG\right) {}^{MSL}\partial_t^{(1)} u(x,t) + LC\, {}^{MSL}\partial_t^{(2)} u(x,t) \tag{35}$$

From eq. (35) we have:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = RGu(x,t) + \left(RC + LG\right) {}^{MSL}\partial_t^{(1)} u(x,t) + LC\, {}^{MSL}\partial_t^{(2)} u(x,t) \tag{36}$$

which can be written as:

$$\frac{1}{LC}\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{RG}{LC} u(x,t) + \frac{[RC + LG]}{LC} {}^{MSL}\partial_t^{(1)} u(x,t) + {}^{MSL}\partial_t^{(2)} u(x,t) \tag{37}$$

Thus, the Mandelbrot-scaling-law-telegraph model reads:

$$\theta^2 \frac{\partial^2 u(x,t)}{\partial x^2} = {}^{MSL}\partial_t^{(2)} u(x,t) + \gamma\, {}^{MSL}\partial_t^{(1)} u(x,t) + \rho u(x,t) \tag{38}$$

or

$$\theta^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \left[\frac{t^D}{(1-D)\kappa}\frac{\partial}{\partial t}\right]^2 u(x,t) + \frac{\gamma t^D}{(1-D)\kappa}\frac{\partial u(x,t)}{\partial t} + \rho u(x,t) \tag{39}$$

where $\theta^2 = 1/(LC)$, $\rho = RG/(LC)$, and $\gamma = (RC + LG)/(LC)$.

The Mandelbrot-scaling-law-telegraph model can be repeated as:

$$\theta^2 \frac{\partial^2 u(x,t)}{\partial x^2} = a(\kappa) t^{2D} \frac{\partial^2}{\partial t^2} u(x,t) + b(\kappa) t^D \frac{\partial u(x,t)}{\partial t} + \rho u(x,t) \tag{40}$$

where $a(\kappa) = 1/[(1-D)^2 \kappa^2]$ and $b(\kappa) = \gamma/[(1-D)\kappa]$.

The traveling-wave solution

Let us consider the similar variable of Mandelbrot-scaling-law type, given:

$$z = x + \kappa t^{1-D} \tag{41}$$

where κ is the normalization constant, and D is the fractal dimension.

Taking $u(x,t) = u(z)$, we have from eq. (20) that:

$$\frac{\partial^2 u(z)}{\partial z^2} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad \frac{\partial u(z)}{\partial z} = {}^{MSL}\partial_t^{(1)} u(x,t), \text{ and } \frac{\partial^2 u(z)}{\partial z^2} = {}^{MSL}\partial_t^{(2)} u(x,t), \tag{42a, b, c}$$

we have from eq. (38) that:

$$(\theta^2 - 1)\frac{\mathrm{d}^2 u(z)}{\mathrm{d}z^2} - \gamma \frac{\mathrm{d}u(z)}{\mathrm{d}z} - \rho u(z) = 0 \tag{43}$$

with the well-known solution ($\vartheta = [\gamma^2 + 4(\theta^2 - 1)\rho]^{1/2}$), given:

$$u(z) = \lambda_1 \mathrm{e}^{\frac{\gamma + \vartheta z}{2(\theta^2 - 1)}} + \lambda_2 \mathrm{e}^{\frac{\gamma - \vartheta z}{2(\theta^2 - 1)}} \tag{44}$$

where λ_1 and λ_2 are two constants.

Thus, the traveling-wave solution for eq. (38) can be given:

$$u(x,t) = \lambda_1 e^{\frac{\gamma+\vartheta x}{2(\theta^2-1)}} e^{\zeta t^{1-D}} + \lambda_2 e^{\frac{\gamma-\vartheta x}{2(\theta^2-1)}} e^{-\zeta t^{1-D}} \tag{45}$$

where $\zeta = \vartheta\kappa/[2(\theta^2-1)]$ and the Kohlrausch-Williams-Watts function is denoted as [6, 17, 18] $e^{-\zeta t^{1-D}} = \sum_{n=0}^{\infty}(-\zeta)^n t^{n(1-D)}/n!$ (For the more details for the functions related to the Kohlrausch-Williams-Watts function, see [6, 10]).

Conclusion

In this work, we addressed the Mandelbrot-scaling-law calculus to model the scaling-law telegraph models. The traveling-wave solution for the scaling-law telegraph equations was presented with the aid of the Kohlrausch-Williams-Watts function. The work may be used to describe the Brownian time and Mandelbrot-scaling-law behaviors in the circuit.

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Nomenclature

$i(x,t)$ – current on the cable, [A]
t – time, [s]
$u(x,t)$ – voltage on the cable, [V]
x – space co-ordinate, [m]

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