NEW INSIGHT INTO THE FOURIER-LIKE AND DARCY-LIKE MODELS IN POROUS MEDIUM

by

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In this study, we propose the general calculus operators based on the Richardson scaling law and Korcak scaling law. The Richardson-scaling-law calculus is considered to investigate the Fourier-like law for the scaling-law flow of the heat in the heat-transfer process. The Korcak-scaling-law calculus is used to model the Darcy-like law for describing the scaling-law flow of the fluid in porous medium. The formulas are as the special cases of the topology calculus proposed for descriptions of the fractal scaling-law behaviors in nature phenomena.

Key words: *scaling law, Richardson-scaling-law calculus, Fourier-like law, Korcak-scaling-law calculus, Darcy-like law, topology calculus*

Introduction

The scaling law is a mathematical relationship, which is used to describe the complex behaviors in the nature phenomena, for instance, anomalous Hall effect [1], slow earthquakes [2], grey matter and white matter of cerebral cortex [3], nano-structured materials [4], turbulent shear flows [5], and human behavioral organization [6].

Let us recall the scaling laws as follows. The Mandelbrot scaling law, proposed by Mandelbrot in 1967, is presented as follows [7]:

$$\phi(t) = \kappa t^{1-D} \tag{1}$$

where $\kappa \in (0,+\infty)$, $t \in (0,+\infty)$, and $D \in (0,+\infty)$ is the fractal dimension. The Richardson scaling law, coined by Richardson in 1926 [8], is given:

$$\psi(t) = \kappa t^{D} \tag{2}$$

where $\kappa \in (0,+\infty)$, $t \in (0,+\infty)$, and $D \in (0,+\infty)$ is the scaling exponent. The Korcak scaling law, suggested by Korcak in 1938 [9]:

$$\omega(t) = \kappa t^{-D} \tag{3}$$

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where $\kappa \in (0,+\infty)$, $t \in (0,+\infty)$, and $D \in (0,+\infty)$ is the scaling exponent. The scaling law in life, presented by West *et al.* in 1999 [10], reads:

$$g(t) = \kappa t^D \tag{4}$$

where $\kappa \in (0,+\infty)$ is the normalization constant, $t \in (0,+\infty)$, and $D \in (-\infty,+\infty)$ is the scaling exponent. In the mathematics, the complex topology can be expressed by the scaling-law function, *e. g.*:

$$\varphi(t) = \kappa t^\beta + c \tag{5}$$

where $\kappa \in (0,+\infty)$ is the normalization constant, $\beta \in (-\infty,+\infty)$ is the scaling exponent, $c \in (-\infty,+\infty)$ is the constant, and $t \in (-\infty,+\infty)$ is the radius. The topology calculus was proposed in [11] based on the Leibniz derivative [12], Stieltjes integral (or Stieltjes-Riemann integral) [13] and Riemann integral [14] (for more details, see [15]). The topology calculus was proposed in [15].

Due to the scaling-law behaviors in the temperature scaling law [16] and in the porous media [17], the main targets of the present paper are to propose the general calculus operators containing the Richardson scaling law and Korcak scaling law, and to consider the Fourier-like law for the scaling-law flow of the heat in the heat-transfer process and Darcy-like law for the scaling-law flow of the fluid in porous medium.

The general calculus operators involving the Richardson scaling law and Korcak scaling law

In this section, we propose the Richardson-scaling-law calculus and the Korcak-scaling-law calculus and discuss their properties based on the topology calculus.

Let $\aleph(\Phi)$ be the set of the continuous functions $\Phi(\varphi)$ in the domain *A* and let $\Im(\varphi)$ be the set of the continuous derivatives of the functions $\varphi(t)$ in the domain *B*.

Let $\Phi_\varphi(t) = (\Phi \circ \varphi)(t) = \Phi[\varphi(t)]$.

Let us consider the sets of the composite functions, given:

$$\Re(\Phi_\varphi) = \left\{\Phi_\varphi(t) : \Phi_\varphi(t) = (\Phi \circ \varphi)(t),\ \Phi \in \aleph(\varphi),\ \Phi \in \Im(\Phi),\ \varphi \in \Im(\varphi)\right\} \tag{6}$$

The topology calculus

Let $\Phi_\varphi \in \Re(\Phi_\varphi)$, where $\varphi(t) = \kappa t^\beta + c$.

The topology derivative of the function $\Phi_\varphi(t)$ is defined as [15]:

$${}^{T}D_t^{(1)}\Phi_\varphi(t) = \frac{1}{\left(\kappa t^\beta + c\right)^{(1)}} \frac{\mathrm{d}\Phi_\varphi(t)}{\mathrm{d}t} \tag{7}$$

where κ is the normalization constant, β is the scaling exponent, t is the radius, and c is the moving term.

The topology partical derivatives of the function $\Phi_\omega = \Phi_\omega(x,y,z)$ are defined:

$${}^{T}\partial_x^{(1)}\Phi_\varphi = \frac{1}{(\kappa x^\beta + c)^{(1)}} \frac{\partial \Phi_\varphi}{\partial x},\quad {}^{T}\partial_y^{(1)}\Phi_\varphi = \frac{1}{(\kappa y^\beta + c)^{(1)}} \frac{\partial \Phi_\varphi}{\partial z},\quad {}^{T}\partial_x^{(1)}\Phi_\varphi = \frac{1}{(\kappa z^\beta + c)^{(1)}} \frac{\partial \Phi_\varphi}{\partial z}$$

$${}^{T}\partial_x^{(1)}\left({}^{T}\partial_x^{(1)}\Phi_\varphi\right) = {}^{T}\partial_x^{(2)}\Phi_\varphi,\quad {}^{T}\partial_x^{(1)}\left({}^{T}\partial_y^{(1)}\Phi_\varphi\right) = {}^{T}\partial_{y,x}^{(2)}\Phi_\varphi,\quad {}^{T}\partial_x^{(1)}\left({}^{T}\partial_z^{(1)}\Phi_\varphi\right) = {}^{T}\partial_{z,x}^{(2)}\Phi_\varphi$$

$${}^{T}\partial_{y}^{(1)}({}^{T}\partial_{y}^{(1)}\Phi_{\varphi}) = {}^{T}\partial_{y}^{(2)}\Phi_{\varphi}, \quad {}^{T}\partial_{y}^{(1)}({}^{T}\partial_{x}^{(1)}\Phi_{\varphi}) = {}^{T}\partial_{x,y}^{(2)}\Phi_{\varphi}, \quad {}^{T}\partial_{y}^{(1)}({}^{T}\partial_{z}^{(1)}\Phi_{\varphi}) = {}^{T}\partial_{z,y}^{(2)}\Phi_{\varphi}$$

$${}^{T}\partial_{z}^{(1)}({}^{T}\partial_{z}^{(1)}\Phi_{\varphi}) = {}^{T}\partial_{z}^{(2)}\Phi_{\varphi}, \quad {}^{T}\partial_{z}^{(1)}({}^{T}\partial_{x}^{(1)}\Phi_{\varphi}) = {}^{T}\partial_{x,z}^{(2)}\Phi_{\varphi} \text{ and } {}^{T}\partial_{z}^{(1)}({}^{T}\partial_{y}^{(1)}\Phi_{\varphi}) = {}^{T}\partial_{y,z}^{(2)}\Phi_{\varphi}$$

The topology differential of the function $\Phi_{\varphi}(t)$, denoted by $\mathrm{d}\Phi_{\varphi}(t)$, is given:

$$\mathrm{d}\Phi_{\varphi}(t) = (\kappa t^{\beta} + c)^{(1)}\, {}^{T}D_{t}^{(1)}\Phi_{\varphi}(t)\mathrm{d}t \tag{8}$$

Let $\Theta_{\varphi} \in \Re(\Theta_{\varphi})$, where $\varphi(t) = \kappa t^{\beta} + c$.

The topology integral of the function $\Theta_{\varphi}(t)$ is defined [15]:

$${}_{a}^{T}I_{t}^{(1)}\Theta_{\varphi}(t) = \int_{a}^{t} \Theta_{\varphi}(t)(\kappa t^{\beta} + c)^{(1)}\,\mathrm{d}t \tag{9}$$

where κ is the normalization constant, β is the scaling exponent, t is the radius, and c is the moving term.

The indefinite topology integral of the function $\Theta_{\varphi}(t)$ is defined [15]:

$${}^{T}I_{t}^{(1)}\Theta_{\varphi}(t) = \int \Theta_{\varphi}(t)(\kappa t^{\beta} + c)^{(1)}\,\mathrm{d}t \tag{10}$$

where κ is the normalization constant, β is the scaling exponent, t is the radius, and c is the moving term.

Let $\Theta_{\varphi} \in \Re(\Phi_{\varphi})$ and $\Pi_{\varphi} \in \Re(\Phi_{\varphi})$.

The properties of the topology calculus can be given:

(A1) The sum and difference rules for the topology derivative:

$${}^{T}D_{t}^{(1)}\left[\Theta_{\varphi}(t) \pm \Pi_{\varphi}(t)\right] = {}^{T}D_{t}^{(1)}\Theta_{\varphi}(t) \pm {}^{T}D_{t}^{(1)}\Pi_{\varphi}(t) \tag{11}$$

(A2) The constant multiple rule for the topology derivative:

$${}^{T}D_{t}^{(1)}\left[C\Theta_{\varphi}(t)\right] = C\, {}^{T}D_{t}^{(1)}\Theta_{\varphi}(t) \tag{12}$$

where C is a constant;

(A3) The product rule for the topology derivative [15]:

$${}^{T}D_{t}^{(1)}\left[\Theta_{\varphi}(t) \cdot \Pi_{\varphi}(t)\right] = \Pi_{\varphi}(t)\, {}^{T}D_{t}^{(1)}\,\Theta_{\varphi}(t) + \Theta_{\varphi}(t)\, {}^{T}D_{t}^{(1)}\,\Pi_{\varphi}(t) \tag{13}$$

(A4) The quotient rule for the topology derivative [15]:

$${}^{T}D_{t}^{(1)}\left[\frac{\Theta_{\varphi}(t)}{\Pi_{\varphi}(t)}\right] = \frac{\Pi_{\varphi}(t)\, {}^{T}D_{t}^{(1)}\,\Theta_{\varphi}(t) - \Theta_{\varphi}(t)\, {}^{T}D_{t}^{(1)}\,\Pi_{\varphi}(t)}{\Pi_{\varphi}(t) \cdot \Pi_{\varphi}(t)} \tag{14}$$

where $\Pi_{\varphi}\left(t\right) \neq 0$.

(A5) The chain rule for the topology derivative:

$${}^{T}D_{t}^{(1)}\left\{w\left[\Theta_{\varphi}(t)\right]\right\} = w^{(1)}(\Theta_{\varphi}) \cdot {}^{T}D_{t}^{(1)}\,\Theta_{\varphi}(t) \tag{15}$$

where $w^{(1)}(\Theta_{\varphi}) = \mathrm{d}w(\Theta_{\varphi})/\mathrm{d}\Theta_{\varphi}$ exists.

(A6) The first fundamental theorem of the topology integral:

$$\Theta_{\varphi}(t) - \Theta_{\varphi}(a) = {}_{a}^{T}I_{t}^{(1)}\left[{}^{T}D_{t}^{(1)}\,\Theta_{\varphi}(t)\right] \tag{16}$$

(A7) The mean value theorem for the topology integral:

$$ {}_{a}^{T}I_{t}^{(1)}\Theta_{\varphi}(t)=\Theta_{\varphi}(l)\left[\varphi(t)-\varphi(a)\right] \tag{17}$$

where $a<l<t$.

(A8) The second fundamental theorem of the topology integral:

$$\Theta_{\varphi}(t)={}^{T}D_{t}^{(1)}\left[{}_{a}^{T}I_{t}^{(1)}\Theta_{\varphi}(t)\right] \tag{18}$$

(A9) The net change theorem for the topology integral:

$$\Theta_{\varphi}(b)-\Theta_{\varphi}(a)={}_{a}^{T}I_{t}^{(1)}\left[{}^{T}D_{t}^{(1)}\,\Theta_{\varphi}(t)\right] \tag{19}$$

(A10) The integration by parts for the topology integral [15]:

$$ {}_{a}^{T}I_{t}^{(1)}\left[\Theta_{\varphi}(t)\,{}^{T}D_{t}^{(1)}\,\Pi_{\varphi}(t)\right]=\Theta_{\varphi}(t)\cdot\Pi_{\varphi}(t)-\Theta_{\varphi}(a)\cdot\Pi_{\varphi}(a)-{}_{a}^{T}I_{t}^{(1)}\left[\Theta_{\varphi}(t)\,{}^{T}D_{t}^{(1)}\,\Pi_{\varphi}(t)\right] \tag{20}$$

(A11) The topology integral for the composite function:

$$\int_{a}^{b}{}^{T}D_{t}^{(1)}\left\{w\left[\Theta_{\varphi}(t)\right]\right\}\mathrm{d}t=\int_{a}^{b}w^{(1)}(\Theta_{\varphi})\cdot{}^{T}D_{t}^{(1)}\,\Theta_{\varphi}(x)\mathrm{d}t \tag{21}$$

(A12) The second fundamental theorem of the topology integral:

$$\Theta_{\varphi}(t)={}^{T}D_{t}^{(1)}\left[{}^{T}I_{t}^{(1)}\Theta_{\varphi}(t)\right] \tag{22}$$

(A13) The net change theorem for the topology integral:

$$ {}^{T}I_{t}^{(1)}\left[{}^{T}D_{t}^{(1)}\,\Theta_{\varphi}(t)\right]=\Theta_{\varphi}(t)+C \tag{23}$$

(A14) The integration by parts for the topology integral [15]:

$$ {}^{T}I_{t}^{(1)}\left[\Theta_{\varphi}(t)\,{}^{T}D_{t}^{(1)}\,\Pi_{\varphi}(t)\right]=\Theta_{\varphi}(t)\cdot\Pi_{\varphi}(t)-{}^{T}I_{t}^{(1)}\left[\Theta_{\varphi}(t)\,{}^{T}D_{t}^{(1)}\,\Pi_{\varphi}(t)\right] \tag{24}$$

(A15) The topology integral for the composite function:

$$\int w^{(1)}(\Theta_{\varphi})\cdot{}^{T}D_{t}^{(1)}\,\Theta_{\varphi}(t)\,\mathrm{d}t=w\left[\Theta_{\varphi}(t)\right]+C \tag{25}$$

where C is the constant.

The Richardson-scaling-law calculus

Let $\Phi_{\psi}\in\Re(\Phi_{\psi})$, where $\psi(t)=\kappa t^{D}$.

The Richardson-scaling-law derivative of the function $\Phi_{\psi}(t)$ is defined:

$$ {}^{RSL}D_{t}^{(1)}\Phi_{\psi}(t)=\frac{t^{1-D}}{D\kappa}\frac{\mathrm{d}\Phi_{\psi}(t)}{\mathrm{d}t} \tag{26}$$

where κ is the normalization constant, D is the scaling exponent, and t is the radius.

The Richardson-scaling-law partical derivatives of the function $\Phi_{\psi}=\Phi_{\psi}(x,y,z)$ are defined:

$$ {}^{RSL}\partial_{x}^{(1)}\Phi_{\psi}=\frac{t^{1-D}}{D\kappa}\frac{\partial\Phi_{\psi}}{\partial x},\quad {}^{RSL}\partial_{y}^{(1)}\Phi_{\psi}=\frac{t^{1-D}}{D\kappa}\frac{\partial\Phi_{\psi}}{\partial y},\quad {}^{RSL}\partial_{z}^{(1)}\Phi_{\psi}=\frac{t^{1-D}}{D\kappa}\frac{\partial\Phi_{\psi}}{\partial z}$$

$$ {}^{RSL}\partial_x^{(1)}\left[{}^{RSL}\partial_x^{(1)}\Phi_\psi\right] = {}^{RSL}\partial_x^{(2)}\Phi_\psi, \quad {}^{RSL}\partial_x^{(1)}\left[{}^{RSL}\partial_y^{(1)}\Phi_\psi\right] = {}^{RSL}\partial_{y,x}^{(2)}\Phi_\psi $$

$$ {}^{RSL}\partial_x^{(1)}\left[{}^{RSL}\partial_z^{(1)}\Phi_\psi\right] = {}^{RSL}\partial_{z,x}^{(2)}\Phi_\psi $$

$$ {}^{RSL}\partial_y^{(1)}\left[{}^{RSL}\partial_y^{(1)}\Phi_\psi\right] = {}^{RSL}\partial_y^{(2)}\Phi_\psi, \quad {}^{RSL}\partial_y^{(1)}\left[{}^{RSL}\partial_x^{(1)}\Phi_\psi\right] = {}^{RSL}\partial_{x,y}^{(2)}\Phi_\psi $$

$$ {}^{RSL}\partial_y^{(1)}\left[{}^{RSL}\partial_z^{(1)}\Phi_\psi\right] = {}^{RSL}\partial_{z,y}^{(2)}\Phi_\psi $$

$$ {}^{RSL}\partial_z^{(1)}\left[{}^{RSL}\partial_z^{(1)}\Phi_\psi\right] = {}^{RSL}\partial_z^{(2)}\Phi_\psi, \quad {}^{RSL}\partial_z^{(1)}\left[{}^{RSL}\partial_x^{(1)}\Phi_\psi\right] = {}^{RSL}\partial_{x,z}^{(2)}\Phi_\psi $$

and

$$ {}^{RSL}\partial_z^{(1)}\left[{}^{RSL}\partial_y^{(1)}\Phi_\psi\right] = {}^{RSL}\partial_{y,z}^{(2)}\Phi_\psi $$

The Richardson-scaling-law differential of the function $\Phi_\psi(t)$, denoted by $\mathrm{d}\Phi_\psi(t)$, is given:

$$ \mathrm{d}\Phi_\psi(t) = D\kappa t^{D-1}\, {}^{RSL}D_t^{(1)}\Phi_\psi(t)\,\mathrm{d}t \tag{27} $$

Let $\Theta_\psi \in \Re(\Theta_\psi)$, where $\psi(t) = \kappa t^D$.

The Richardson-scaling-law integral of the function $\Theta_\psi(t)$ is defined:

$$ {}^{RSL}_{\ a}I_t^{(1)}\Theta_\psi(t) = \int_a^t \Theta_\psi(t) D\kappa t^{D-1}\mathrm{d}t \tag{28} $$

where κ is the normalization constant, D is the scaling exponent, and t is the radius.

The indefinite Richardson-scaling-law integral of the function $\Theta_\varphi(t)$ is defined:

$$ {}^{RSL}I_t^{(1)}\Theta_\psi(t) = \int \Theta_\psi(t) D\kappa t^{D-1}\mathrm{d}t \tag{29} $$

where κ is the normalization constant, D is the scaling exponent, and t is the radius.

Let $\Theta_\psi \in \Re(\Phi_\psi)$ and $\Pi_\psi \in \Re(\Phi_\psi)$.

The properties of the Richardson-scaling-law calculus can be given:

(B1) The sum and difference rules for the Richardson-scaling-law derivative:

$$ {}^{RSL}D_t^{(1)}\left[\Theta_\psi(t) \pm \Pi_\psi(t)\right] = {}^{RSL}D_t^{(1)}\Theta_\psi(t) \pm {}^{RSL}D_t^{(1)}\Pi_\psi(t) \tag{30} $$

(B2) The constant multiple rule for the Richardson-scaling-law derivative:

$$ {}^{RSL}D_t^{(1)}\left[C\Theta_\psi(t)\right] = C\, {}^{RSL}D_t^{(1)}\Theta_\psi(t) \tag{31} $$

where C is a constant;

(B3) The product rule for the Richardson-scaling-law derivative [15]:

$$ {}^{RSL}D_t^{(1)}\left[\Theta_\psi(t)\cdot\Pi_\psi(t)\right] = \Pi_\psi(t)\, {}^{RSL}D_t^{(1)}\Theta_\psi(t) + \Theta_\psi(t)\, {}^{RSL}D_t^{(1)}\Pi_\psi(t) \tag{32} $$

(B4) The quotient rule for the Richardson-scaling-law derivative:

$$ {}^{RSL}D_t^{(1)}\left[\frac{\Theta_\psi(t)}{\Pi_\psi(t)}\right] = \frac{\Pi_\psi(t)\, {}^{RSL}D_t^{(1)}\Theta_\psi(t) - \Theta_\psi(t)\, {}^{RSL}D_t^{(1)}\Pi_\psi(t)}{\Pi_\psi(t)\cdot\Pi_\psi(t)} \tag{14} $$

where $\Pi_\psi(t) \neq 0$.

(B5) The chain rule for the Richardson-scaling-law derivative:

$$ {}^{RSL}D_t^{(1)}\left\{w\left[\Theta_\psi(t)\right]\right\} = w^{(1)}(\Theta_\psi)\cdot {}^{RSL}D_t^{(1)}\,\Theta_\psi(t) \tag{33}$$

where $w^{(1)}(\Theta_\psi) = \mathrm{d}w(\Theta_\psi)/\mathrm{d}\Theta_\psi$ exists.

(B6) The first fundamental theorem of the Richardson-scaling-law integral:

$$\Theta_\psi(t)-\Theta_\psi(a) = {}^{RSL}_{\ a}I_t^{(1)}\left[{}^{RSL}D_t^{(1)}\,\Theta_\psi(t)\right] \tag{34}$$

(B7) The mean value theorem for the Richardson-scaling-law integral:

$${}^{RSL}_{\ a}I_t^{(1)}\Theta_\psi(t) = \Theta_\psi(l)\left[\psi(t)-\psi(a)\right] \tag{35}$$

where $a<l<t$.

(B8) The second fundamental theorem of the Richardson-scaling-law integral:

$$\Theta_\psi(t) = {}^{RSL}D_t^{(1)}\left[{}^{RSL}_{\ a}I_t^{(1)}\Theta_\psi(t)\right] \tag{36}$$

(B9) The net change theorem for the Richardson-scaling-law integral:

$$\Theta_\psi(b)-\Theta_\psi(a) = {}^{RSL}_{\ a}I_t^{(1)}\left[{}^{RSL}D_t^{(1)}\,\Theta_\psi(t)\right] \tag{37}$$

(B10) The integration by parts for the Richardson-scaling-law integral:

$${}^{RSL}_{\ a}I_t^{(1)}\left[\Theta_\psi(t)\,{}^{RSL}D_t^{(1)}\,\Pi_\psi(t)\right]=\Theta_\psi(t)\cdot\Pi_\psi(t)-\Theta_\psi(a)\cdot\Pi_\psi(a)-{}^{RSL}_{\ a}I_t^{(1)}\left[\Theta_\psi(t)\,{}^{RSL}D_t^{(1)}\,\Pi_\psi(t)\right] \tag{38}$$

(B11) The Richardson-scaling-law integral for the composite function:

$$\int_a^b {}^{RSL}D_t^{(1)}\left\{w\left[\Theta_\psi(t)\right]\right\}\mathrm{d}t = \int_a^b w^{(1)}(\Theta_\psi)\cdot {}^{RSL}D_t^{(1)}\,\Theta_\psi(t)\mathrm{d}t \tag{39}$$

(B12) The second fundamental theorem of the Richardson-scaling-law integral:

$$\Theta_\psi(t) = {}^{RSL}D_t^{(1)}\left[{}^{RSL}I_t^{(1)}\Theta_\psi(t)\right] \tag{40}$$

(B13) The net change theorem for the Richardson-scaling-law integral:

$${}^{RSL}I_t^{(1)}\left[{}^{RSL}D_t^{(1)}\,\Theta_\psi(t)\right] = \Theta_\psi(t)+C \tag{41}$$

(B14) The integration by parts for the Richardson-scaling-law integral:

$${}^{RSL}I_t^{(1)}\left[\Theta_\psi(t)\,{}^{RSL}D_t^{(1)}\,\Pi_\psi(t)\right]=\Theta_\psi(t)\cdot\Pi_\psi(t)-{}^{RSL}I_t^{(1)}\left[\Theta_\psi(t)\,{}^{RSL}D_t^{(1)}\,\Pi_\psi(t)\right] \tag{42}$$

(B15) The Richardson-scaling-law integral for the composite function:

$$\int w^{(1)}(\Theta_\psi)\cdot {}^{RSL}D_t^{(1)}\,\Theta_\psi(t)\,\mathrm{d}t = w\left[\Theta_\psi(t)\right]+C \tag{43}$$

where C is the constant.

The basic formulas for the Richardson-scaling-law calculus can be given:

$${}^{RSL}D_t^{(1)}1=0,\quad {}^{RSL}D_t^{(1)}(\kappa t^D)=1,\quad {}^{RSL}D_t^{(1)}(\kappa t^D)^n = n(\kappa t^D)^{n-1} \tag{44a,b,c}$$

$${}^{RSL}D_t^{(1)}\mathrm{e}^{\kappa t^D} = \mathrm{e}^{\kappa t^D},\quad {}^{RSL}D_t^{(1)}\ln(\kappa t^D) = \frac{1}{\kappa t^D},\quad {}^{RSL}D_t^{(1)}s^{\kappa t^D} = (\ln s)s^{\kappa t^D} \tag{45a,b,c}$$

$$ {}^{RSL}D_t^{(1)} \log_s(\kappa t^D) = \frac{1}{\kappa t^D \ln s}, \quad {}^{RSL}D_t^{(1)} \mathrm{e}^{\Theta_\psi(t)} = \mathrm{e}^{\Theta_\psi(t)}\, {}^{RSL}D_t^{(1)} \Theta_\psi(t) \tag{46a,b} $$

$$ {}^{RSL}D_t^{(1)} \ln \Theta_\psi(t) = \frac{{}^{RSL}D_t^{(1)}\Theta_\psi(t)}{\Theta_\psi(t)}, \quad {}^{RSL}D_t^{(1)} \log_s \Theta_\psi(t) = \frac{{}^{RSL}D_t^{(1)}\Theta_\psi(t)}{(\ln s)\Theta_\psi(t)} \tag{47a,b} $$

$$ {}^{RSL}D_t^{(1)} s^{\Theta_\psi(t)} = \left[(\ln s) s^{\Theta_\psi(t)}\right] \cdot {}^{RSL}D_t^{(1)}\Theta_\psi(t), \quad {}^{RSL}I_t^{(1)} 1 = \kappa t^D + C \tag{48a,b} $$

$$ {}^{RSL}I_t^{(1)}\left[n(\kappa t^D)^{n-1}\right] = [\kappa t^D]^n + C, \quad {}^{RSL}I_t^{(1)}\left[\frac{{}^{T}D_t^{(1)}\Theta_\psi(t)}{(\ln s)\Theta_\psi(t)}\right] = \log_s \Theta_\psi(t) + C \tag{49a,b} $$

$$ {}^{RSL}I_t^{(1)}\left(\frac{1}{\kappa t^D}\right) = \ln(\kappa t^D) + C, \quad {}^{RSL}I_t^{(1)}\left[\frac{1}{(\ln s)} \cdot \frac{1}{\kappa t^D}\right] = \log_s(\kappa t^D) + C \tag{50a,b} $$

$$ {}^{RSL}I_t^{(1)}\left[(\ln s) s^{\kappa t^D}\right] = s^{\kappa t^D} + C, \quad {}^{RSL}I_t^{(1)}\left[\mathrm{e}^{\Theta_\psi(t)}\, {}^{RSL}D_t^{(1)}\Theta_\psi(t)\right] = e^{\Theta_\psi(t)} + C \tag{51a,b} $$

$$ {}^{RSL}I_t^{(1)}\left[\frac{\Theta_\psi(t)}{\left|\Theta_\psi(t)\right|}\, {}^{RSL}D_t^{(1)}\Theta_\psi(t)\right] = \left|\Theta_\psi(t)\right| + C, \quad {}^{RSL}I_t^{(1)}\left[\frac{{}^{RSL}D_t^{(1)}\Theta_\psi(t)}{\Theta_\psi(t)}\right] = \ln \Theta_\psi(t) + C \tag{52a,b} $$

$$ {}^{RSL}I_t^{(1)}(\mathrm{e}^{\kappa t^D}) = e^{\kappa t^D} + C, \quad {}^{RSL}I_t^{(1)}\left\{\left[(\ln s) s^{\Theta_\psi(t)}\right] \cdot {}^{RSL}D_t^{(1)}\Theta_\psi(t)\right\} = s^{\Theta_\psi(t)} + C \tag{53a,b} $$

where C is the constant and $\mathrm{e}^{\kappa t^D}$ is the Kohlrausch-Williams-Watts function [11,15].

The Korcak-scaling-law calculus

Let $\Phi_\omega \in \Re(\Phi_\omega)$, where $\omega(t) = \kappa t^{-D}$.

The Korcak-scaling-law derivative of the function $\Phi_\omega(t)$ is defined:

$$ {}^{KSL}D_t^{(1)}\Phi_\omega(t) = -\frac{t^{1+D}}{D\kappa}\frac{\mathrm{d}\Phi_\omega(t)}{\mathrm{d}t} \tag{54} $$

where κ is the normalization constant, D is the scaling exponent, and t is the radius.

The Korcak-scaling-law partical derivatives of the function $\Phi_\omega = \Phi_\omega(x,y,z)$ are defined:

$$ {}^{KSL}\partial_x^{(1)}\Phi_\omega = -\frac{t^{1+D}}{D\kappa}\frac{\partial\Phi_\omega}{\partial x}, \quad {}^{KSL}\partial_y^{(1)}\Phi_\omega = -\frac{t^{1+D}}{D\kappa}\frac{\partial\Phi_\omega}{\partial y}, \quad {}^{KSL}\partial_z^{(1)}\Phi_\omega = -\frac{t^{1+D}}{D\kappa}\frac{\partial\Phi_\omega}{\partial z} $$

$$ {}^{KSL}\partial_x^{(1)}\left[{}^{KSL}\partial_x^{(1)}\Phi_\omega\right] = {}^{KSL}\partial_x^{(2)}\Phi_\omega, \quad {}^{KSL}\partial_x^{(1)}\left[{}^{KSL}\partial_y^{(1)}\Phi_\omega\right] = {}^{KSL}\partial_{y,x}^{(2)}\Phi_\omega $$

$$ {}^{KSL}\partial_x^{(1)}\left[{}^{KSL}\partial_z^{(1)}\Phi_\omega\right] = {}^{KSL}\partial_{z,x}^{(2)}\Phi_\omega $$

$$ {}^{KSL}\partial_y^{(1)}\left[{}^{KSL}\partial_y^{(1)}\Phi_\omega\right] = {}^{KSL}\partial_y^{(2)}\Phi_\omega, \quad {}^{KSL}\partial_y^{(1)}\left[{}^{KSL}\partial_x^{(1)}\Phi_\omega\right] = {}^{KSL}\partial_{x,y}^{(2)}\Phi_\omega $$

$$ {}^{KSL}\partial_y^{(1)}\left[{}^{KSL}\partial_z^{(1)}\Phi_\omega\right] = {}^{KSL}\partial_{z,y}^{(2)}\Phi_\omega $$

$$ {}^{KSL}\partial_z^{(1)}\left[{}^{KSL}\partial_z^{(1)}\Phi_\omega\right] = {}^{KSL}\partial_z^{(2)}\Phi_\omega, \quad {}^{KSL}\partial_z^{(1)}\left[{}^{KSL}\partial_x^{(1)}\Phi_\omega\right] = {}^{KSL}\partial_{x,z}^{(2)}\Phi_\omega $$

and

$$ {}^{KSL}\partial_z^{(1)}\left[{}^{KSL}\partial_y^{(1)}\Phi_\omega\right] = {}^{KSL}\partial_{y,z}^{(2)}\Phi_\omega $$

The Korcak-scaling-law differential of the function $\Phi_\omega(t)$, denoted by $\mathrm{d}\Phi_\omega(t)$, is given:

$$\mathrm{d}\Phi_\omega(t) = -D\kappa t^{-(D+1)}\, {}^{KSL}D_t^{(1)}\Phi_\omega(t)\,\mathrm{d}t \tag{55}$$

Let $\Theta_\omega \in \Re(\Theta_\omega)$, where $\omega(t) = \kappa t^{-D}$.

The Korcak-scaling-law integral of the function $\Theta_\omega(t)$ is defined:

$${}^{KSL}_{\ a}I_t^{(1)}\Theta_\omega(t) = -\int_a^t \Theta_\omega(t) D\kappa t^{-(D+1)}\mathrm{d}t \tag{56}$$

where κ is the normalization constant, D is the scaling exponent, and t is the radius.

The indefinite Korcak-scaling-law integral of the function $\Theta_\varphi(t)$ is defined:

$${}^{KSL}I_t^{(1)}\Theta_\omega(t) = -\int \Theta_\omega(t) D\kappa t^{-(D+1)}\mathrm{d}t \tag{57}$$

where κ is the normalization constant, D is the scaling exponent, and t is the radius.

Let $\Theta_\omega \in \Re(\Phi_\omega)$ and $\Pi_\omega \in \Re(\Phi_\omega)$.

The properties of the Korcak-scaling-law calculus can be given:

(C1) The sum and difference rules for the Korcak-scaling-law derivative:

$${}^{KSL}D_t^{(1)}\left[\Theta_\omega(t) \pm \Pi_\omega(t)\right] = {}^{KSL}D_t^{(1)}\Theta_\omega(t) \pm {}^{KSL}D_t^{(1)}\Pi_\omega(t) \tag{58}$$

(C2) The constant multiple rule for the Korcak-scaling-law derivative:

$${}^{KSL}D_t^{(1)}\left[C\Theta_\omega(t)\right] = C\, {}^{KSL}D_t^{(1)}\Theta_\omega(t) \tag{59}$$

where C is a constant;

(C3) The product rule for the Korcak-scaling-law derivative [15]:

$${}^{KSL}D_t^{(1)}\left[\Theta_\omega(t)\cdot\Pi_\omega(t)\right] = \Pi_\omega(t)\, {}^{KSL}D_t^{(1)}\,\Theta_\omega(t) + \Theta_\omega(t)\, {}^{KSL}D_t^{(1)}\,\Pi_\omega(t) \tag{60}$$

(C4) The quotient rule for the Korcak-scaling-law derivative:

$${}^{KSL}D_t^{(1)}\left[\frac{\Theta_\omega(t)}{\Pi_\omega(t)}\right] = \frac{\Pi_\omega(t)\, {}^{KSL}D_t^{(1)}\,\Theta_\omega(t) - \Theta_\omega(t)\, {}^{KSL}D_t^{(1)}\,\Pi_\omega(t)}{\Pi_\omega(t)\cdot\Pi_\omega(t)} \tag{61}$$

where $\Pi_\omega(t) \neq 0$.

(C5) The chain rule for the Korcak-scaling-law derivative:

$${}^{KSL}D_t^{(1)}\left\{w\left[\Theta_\omega(t)\right]\right\} = w^{(1)}(\Theta_\omega)\cdot {}^{KSL}D_t^{(1)}\,\Theta_\omega(t) \tag{62}$$

where $w^{(1)}(\Theta_\omega) = \mathrm{d}w(\Theta_\omega)/\mathrm{d}\Theta_\omega$ exists.

(C6) The first fundamental theorem of the Korcak-scaling-law integral:

$$\Theta_\omega(t) - \Theta_\omega(a) = {}^{KSL}_{\ a}I_t^{(1)}\left[{}^{KSL}D_t^{(1)}\,\Theta_\omega(t)\right] \tag{63}$$

(C7) The mean value theorem for the Korcak-scaling-law integral:

$${}^{KSL}_{\ a}I_t^{(1)}\Theta_\omega(t) = \Theta_\omega(l)\left[\omega(t) - \omega(a)\right] \tag{64}$$

where $a < l < t$.

(C8) The second fundamental theorem of the Korcak-scaling-law integral:

$$\Theta_\omega(t) = {}^{KSL}D_t^{(1)}\left[{}^{KSL}_{\ a}I_t^{(1)}\Theta_\omega(t)\right] \tag{65}$$

(C9) The net change theorem for the Korcak-scaling-law integral:

$$\Theta_\omega(b)-\Theta_\omega(a)={}_{a}^{KSL}I_t^{(1)}\left[{}^{KSL}D_t^{(1)}\,\Theta_\omega(t)\right] \tag{66}$$

(C10) The integration by parts for the Korcak-scaling-law integral:

$${}_{a}^{KSL}I_t^{(1)}\left[\Theta_\omega(t)\,{}^{KSL}D_t^{(1)}\,\Pi_\omega(t)\right]=\Theta_\omega(t)\cdot\Pi_\omega(t)-\Theta_\omega(a)\cdot\Pi_\omega(a)-{}_{a}^{KSL}I_t^{(1)}\left[\Theta_\omega(t)\,{}^{KSL}D_t^{(1)}\,\Pi_\omega(t)\right] \tag{67}$$

(C11) The Korcak-scaling-law integral for the composite function:

$$\int_a^b {}^{KSL}D_t^{(1)}\left\{w\left[\Theta_\omega(t)\right]\right\}\mathrm{d}t=\int_a^b w^{(1)}(\Theta_\omega)\cdot{}^{KSL}D_t^{(1)}\,\Theta_\omega(t)\mathrm{d}t \tag{68}$$

(C12) The second fundamental theorem of the Korcak-scaling-law integral:

$$\Theta_\omega(t)={}^{KSL}D_t^{(1)}\left[{}^{KSL}I_t^{(1)}\Theta_\omega(t)\right] \tag{69}$$

(C13) The net change theorem for the Korcak-scaling-law integral:

$${}^{KSL}I_t^{(1)}\left[{}^{KSL}D_t^{(1)}\,\Theta_\omega(t)\right]=\Theta_\omega(t)+C \tag{70}$$

(C14) The integration by parts for the Korcak-scaling-law integral:

$${}^{KSL}I_t^{(1)}\left[\Theta_\omega(t)\,{}^{KSL}D_t^{(1)}\,\Pi_\omega(t)\right]=\Theta_\omega(t)\cdot\Pi_\omega(t)-{}^{KSL}I_t^{(1)}\left[\Theta_\omega(t)\,{}^{KSL}D_t^{(1)}\,\Pi_\omega(t)\right] \tag{71}$$

(C15) The Korcak-scaling-law integral for the composite function:

$$\int w^{(1)}(\Theta_\omega)\cdot{}^{KSL}D_t^{(1)}\,\Theta_\omega(t)\mathrm{d}t=w\left[\Theta_\omega(t)\right]+C \tag{72}$$

where C is the constant.

The basic formulas for the Korcak-scaling-law calculus can be presented as follows:

$${}^{KSL}D_t^{(1)}1=0,\quad {}^{KSL}D_t^{(1)}(\kappa t^{-D})=1,\quad {}^{KSL}D_t^{(1)}(\kappa t^{-D})^n=n(\kappa t^{-D})^{n-1} \tag{73a,b,c}$$

$${}^{KSL}D_t^{(1)}\mathrm{e}^{\kappa t^{-D}}=\mathrm{e}^{\kappa t^{-D}},\quad {}^{KSL}D_t^{(1)}\ln(\kappa t^{-D})=\frac{1}{\kappa t^{-D}},\quad {}^{KSL}D_t^{(1)}s^{\kappa t^{-D}}=(\ln s)s^{\kappa t^{-D}} \tag{74a,b,c}$$

$${}^{KSL}D_t^{(1)}\log_s(\kappa t^{-D})=\frac{1}{\kappa t^{-D}\ln s},\quad {}^{KSL}D_t^{(1)}\mathrm{e}^{\Theta_\omega(t)}=\mathrm{e}^{\Theta_\omega(t)}\,{}^{KSL}D_t^{(1)}\Theta_\omega(t) \tag{75a,b}$$

$${}^{KSL}D_t^{(1)}\ln\Theta_\omega(t)=\frac{{}^{KSL}D_t^{(1)}\Theta_\omega(t)}{\Theta_\omega(t)},\quad {}^{KSL}D_t^{(1)}\log_s\Theta_\omega(t)=\frac{{}^{KSL}D_t^{(1)}\Theta_\omega(t)}{(\ln s)\Theta_\omega(t)} \tag{76a,b}$$

$${}^{KSL}D_t^{(1)}s^{\Theta_\omega(t)}=\left[(\ln s)s^{\Theta_\omega(t)}\right]\cdot{}^{KSL}D_t^{(1)}\Theta_\omega(t),\quad {}^{KSL}I_t^{(1)}1=\kappa t^{-D}+C \tag{77a,b}$$

$${}^{KSL}I_t^{(1)}\left[n(\kappa t^{-D})^{n-1}\right]=(\kappa t^{-D})^n+C,\quad {}^{KSL}I_t^{(1)}\left[\frac{{}^{T}D_t^{(1)}\Theta_\omega(t)}{(\ln s)\Theta_\omega(t)}\right]=\log_s\Theta_\omega(t)+C \tag{78a,b}$$

$${}^{KSL}I_t^{(1)}\left(\frac{1}{\kappa t^{-D}}\right)=\ln(\kappa t^{-D})+C,\quad {}^{KSL}I_t^{(1)}\left[\frac{1}{(\ln s)}\cdot\frac{1}{\kappa t^{-D}}\right]=\log_s(\kappa t^{-D})+C \tag{79a,b}$$

$${}^{KSL}I_t^{(1)}\left[(\ln s)s^{\kappa t^{-D}}\right]=s^{\kappa t^{-D}}+C,\quad {}^{KSL}I_t^{(1)}\left[\mathrm{e}^{\Theta_\omega(t)}\,{}^{KSL}D_t^{(1)}\Theta_\omega(t)\right]=\mathrm{e}^{\Theta_\omega(t)}+C \tag{80a,b}$$

$$ {}^{KSL}I_t^{(1)}\left[\frac{\Theta_\omega(t)}{|\Theta_\omega(t)|}\,{}^{KSL}D_t^{(1)}\Theta_\omega(t)\right]=|\Theta_\omega(t)|+C\,,\quad {}^{KSL}I_t^{(1)}\left[\frac{{}^{KSL}D_t^{(1)}\Theta_\omega(t)}{\Theta_\omega(t)}\right]=\ln\Theta_\omega(t)+C \quad (81a,b) $$

$$ {}^{KSL}I_t^{(1)}(\mathrm{e}^{\kappa t^{-D}})=\mathrm{e}^{\kappa t^{-D}}+C,\quad {}^{KSL}I_t^{(1)}\left\{\left[(\ln s)s^{\Theta_\omega(t)}\right]\cdot{}^{KSL}D_t^{(1)}\Theta_\omega(t)\right\}=s^{\Theta_\omega(t)}+C \quad (82a,b) $$

where C is the constant and $\mathrm{e}^{t^{\kappa/D}}$ is the Kohlrausch-Williams-Watts function [11, 15].

Applications

In this section, we propose the Fourier-like law for the scaling-law flow in the heat-transfer process and the Darcy-like law for the scaling-law flow of the fluid in porous medium.

The Fourier-like law for the scaling-law flow

The Fourier-like law for the scaling-law flow in the heat-transfer process can be defined:

$$ \boldsymbol{q}(x,y,z,t)=-\alpha\,{}^{RSL}\nabla_D T(x,y,z,t) $$
$$ =-i\alpha(\kappa Dx^{D-1})\frac{\partial T(x,y,z,t)}{\partial x}-j\alpha(\kappa Dy^{D-1})\frac{\partial T(x,y,z,t)}{\partial y}-k\alpha(\kappa Dz^{D-1})\frac{\partial T(x,y,z,t)}{\partial z} \quad (83) $$

where $T(x,y,z,t)$ is the temperature field in the unit volume at the Cartesian co-ordinates x, y and z and at the time t, $\boldsymbol{q}(x,y,z,t)$ is the vector of the local heat flux density, i, j, and k denote the unit vectors in the Cartesian co-ordinate system, κ is the normalization constant, D is the scaling exponent, α is the material conductivity, and the Richardson-scaling-law gradient in a Cartesian co-ordinate system is defined:

$$ {}^{RSL}\nabla_D=i(\kappa Dx^{D-1})\frac{\partial}{\partial x}+j(\kappa Dy^{D-1})\frac{\partial}{\partial y}+k(\kappa Dz^{D-1})\frac{\partial}{\partial z} \quad (84) $$

which is connected with the Laplace-like operator, represented:

$$ {}^{RSL}\Delta_D={}^{RSL}\nabla_D^2={}^{RSL}\nabla_D\bullet{}^{RSL}\nabla_D=\left(\kappa Dx^{D-1}\frac{\partial}{\partial x}\right)^2+\left(\kappa Dy^{D-1}\frac{\partial}{\partial y}\right)^2+\left(\kappa Dz^{D-1}\frac{\partial}{\partial z}\right)^2 \quad (85) $$

which is connected the Laplace operator [18] when $D=1$.

In 1-D case, the Fourier-like law for the scaling-law flow in the heat-transfer process reads:

$$ q(x,t)=-\alpha(\kappa Dx^{D-1})\frac{\partial T(x,t)}{\partial x} \quad (86) $$

where $T(x,t)$ is the temperature field, $q(x,t)$ is the local heat flux density and α is the material conductivity.

When $D=1$, the Fourier-like law for the scaling-law flow of the heat is the Fourier law for the flow of the heat [19].

The Darcy-like law for the scaling-law flow of the fluid

The Darcy-like law for the scaling-law flow of the fluid in porous medium can be defined:

$$ \boldsymbol{\Theta}(x,y,z,t)=-\lambda\,{}^{KSL}\nabla_D\Xi(x,y,z,t)=i\lambda(\kappa Dx^{-(D+1)})\frac{\partial\Xi(x,y,z,t)}{\partial x} $$
$$ +j\lambda(\kappa Dy^{-(D+1)})\frac{\partial\Xi(x,y,z,t)}{\partial y}+k\lambda(\kappa Dz^{-(D+1)})\frac{\partial\Xi(x,y,z,t)}{\partial z} \quad (87) $$

where $\boldsymbol{\Theta}(x,y,z,t)$ is the specific discharge, $\Xi(x,y,z,t)$ is the hydraulic head, κ is the normalization constant, D is the scaling exponent, λ is the hydraulic conductivity, the Korcak-scaling-law gradient in a Cartesian co-ordinate system is defined:

$$^{KSL}\nabla_D = i\left[-\kappa D x^{-(D+1)}\right]\frac{\partial}{\partial x} + j\left[-\kappa D y^{-(D+1)}\right]\frac{\partial}{\partial y} + k\left[-\kappa D z^{-(D+1)}\right]\frac{\partial}{\partial z} \tag{88}$$

which is connected with the Laplace-like operator, given:

$$^{KSL}\Delta_D = {}^{KSL}\nabla_D^2 = {}^{KSL}\nabla_D \bullet {}^{KSL}\nabla_D$$

$$= \left[\kappa D x^{-(D+1)}\frac{\partial}{\partial x}\right]^2 + \left[\kappa D y^{-(D+1)}\frac{\partial}{\partial y}\right]^2 + \left[\kappa D z^{-(D+1)}\frac{\partial}{\partial z}\right]^2 \tag{89}$$

which is connected the Laplace operator [18] when $D = -1$.

In 1-D case, the Darcy-like law for the scaling-law flow of the fluid in porous medium can be expressed:

$$\Theta(x,t) = \lambda\kappa D x^{-(D+1)}\frac{\partial \Xi(x,t)}{\partial x} \tag{90}$$

where $\Xi(x,t)$ is the hydraulic head, $\Theta(x,t)$ is the specific discharge, λ is the hydraulic conductivity, κ is the normalization constant, and D is the scaling exponent.

When $D = 1$, the Darcy-like law for the scaling-law flow of the fluid is the Darcy law for the flow of the fluid [20].

Conclusion

In the present work, we proposed the Richardson-scaling-law calculus and Korcak-scaling-law calculus for the first time. Based on the results for the Richardson-scaling-law gradient and the Korcak-scaling-law gradient, we considered the Fourier-like law for the scaling-law flow of the heat and the Darcy-like law for describing the scaling-law flow of the fluid, respectively. The obtained results are as mathematical tools proposed for decriptions of the fractal scaling-law phenomena in applied sciences.

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Nomenclature

t – time, [s]
x, y, z – co-ordinates, [m]
$\boldsymbol{q}(x, y, z, t)$ – local heat flux density, [W]
$T(x, y, z, t)$ – temperature field, [K]

Greek symbols

λ – hydraulic conductivity, [ms^{-1}]
$\Theta(x,t)$ – specific discharge, [ms^{-1}]
$\Xi(x,t)$ – hydraulic head, [m]

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