A NEW INSIGHT INTO VECTOR CALCULUS WITH RESPECT TO MONOTONE FUNCTIONS FOR THE COMPLEX FLUID-FLOWS

by

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In the paper, the Navier-Stokes-type equations of the complex fluid-flows, the equations of the complex turbulent flows, and Euler-type equations of the complex fluid-flows based on the theory of the new vector calculus with respect to monotone functions are investigated for the first time.

Key words: *Navier-Stokes-type equation, complex turbulent flow, Euler-type equation, vector calculus with respect to monotone function*

Introduction

The theory of calculus, proposed by Newton in 1665 [1, 2] and by Leibniz in 1684 [3] and 1686 [4], respectively, has been the important contributions in the fields of science, socialty, and applied sciences. The calculus with respect to monotone functions, which consists of the Leibniz derivative (the differential calculus with respect to monotone function), proposed by Leibniz in 1667 [5-7], and Riemann-Stieltjes integral (the integral calculus with respect to monotone function), proposed by Stieltjes in 1894 [8] based on Riemann's task [9], and further developed by Lebesgue in 1902 [10] and Stoll in 2001 [11]. The double Stieltjes Riemann-Stieltjes integral in Euclidean space was proposed by Horst in 1986 [12] and further developed by Carter and Brunt in [13]. The vector calculus operators with respect to monotone function were proposed to consider the heat-condution problem in [14] and the special cases of the operators were discussed in [15-18].

The main target of the paper is to present the special case of the theoy of the vector calculus with respect to monotone functions in [14] and to propose the Navier-Stokes-type equations of the complex fluid-flows, the equations of the complex turbulent flows, and Euler-type equations of the complex fluid-flows.

The calculus with respect to monotone function

In this section, we introduce the Leibniz derivative and Riemann-Stieltjes integral, which are calld the calculus with respect to monotone function.

Suppose that $\psi_{\varpi}(t)=(\psi\circ\varpi)(t)=\psi[\varpi(t)]$, where $\varpi(t)$ is the monotone function, *e. g.*, $\varpi^{(1)}(t)=\mathrm{d}\varpi(t)/\mathrm{d}t>0$. Let $\Re(\psi)$ be the set of the countinous functions $\psi(\varpi)$ in the do-

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main $\wp$. Let $\Im(\varpi)$ be the set of the continuous derivatives of the functions $\varpi(t)$ in the domain $\aleph$. Let us consider the sets of the composite functions, given as:

$$\hbar(\psi_{\varpi}) = \left\{\psi_{\varpi}(t) : \psi_{\varpi}(t) \in \Re(\psi), \quad \varpi \in \Im(\varpi)\right\}$$

The Leibniz derivative

Let $\psi_{\varpi} \in \hbar(\psi_{\varpi})$. The Leibniz derivative of the function $\psi_u(t)$ is defined as [5-7, 14-17]:

$$D_{t,\varpi}^{(1)}\psi_{\varpi}(\mathrm{t}) = \frac{1}{\varpi^{(1)}(\mathrm{t})}\frac{\mathrm{d}\psi_{\varpi}(\mathrm{t})}{\mathrm{d}t} \tag{1}$$

The geometric interpretations of the Leibniz derivative is the rate of change of the functional $\psi_{\varpi}(t)$ with the function $\varpi(t)$ in the independent variable t [15, 17].

The total Leibniz-type differential with respect to monotone function $\varpi(t)$ of the function $\psi_{\varpi}(t)$, denoted as $\mathrm{d}\psi_{\varpi}(t) = \mathrm{d}\left[(\psi \circ \varpi)(t)\right] = \mathrm{d}\psi[\varpi(t)]$, can be defined:

$$\mathrm{d}\psi_{\varpi}(t) = \left[\varpi^{(1)}(t)D_{t,\varpi}^{(1)}\psi_{\varpi}(t)\right]\mathrm{d}t \tag{2}$$

The Riemann-Stieltjes integral

Let $\psi_{\varpi} \in \hbar(\psi_{\varpi})$. The Riemann-Stieltjes integral of the function $\psi_{\varpi}(t)$ is defined as [8, 14-17]:

$${}_aI_{b,\varpi}^{(1)}\psi_u(t) = \int_a^b \psi_{\varpi}(t)\varpi^{(1)}(t)\mathrm{d}t \tag{3}$$

Similarly, the geometric interpretations of the Stieltjes-Riemann integral is the area enclosed by the integrand function $\Phi_{\varpi}(t)$ and the function $\varpi(t)$ in the independent variable $t \in [a,b]$ [15, 17].

Their properties are given as follows [14, 15, 17]:

($U1$) The chain rule for the Leibniz derivative:

$$D_{t,\varpi}^{(1)}w\left\{\psi_{\omega}(t)\right\} = w^{(1)}(\psi)\cdot D_{t,\varpi}^{(1)}\psi_{\omega}(t) \tag{4}$$

where $w^{(1)}(\psi) = \mathrm{d}w(\psi)/\mathrm{d}\psi$ exists.

($U2$) The change-of-variable theorem for the Riemann-Stieltjes integral:

$$\int_a^t w^{(1)}(\psi)\cdot D_{t,\varpi}^{(1)}\psi_{\omega}(t)\varpi^{(1)}(t)\mathrm{d}t = w\left\{\psi_{\omega}(t)\right\} - w\left\{\psi_{\omega}(a)\right\} \tag{5}$$

The Leibniz-type partial derivatives

Let $\mathrm{T} = \mathrm{T}_{\varpi}(x,y,z) = \mathrm{T}[\alpha(x),\beta(y),\gamma(z)]$, $\alpha^{(1)}(x) > 0$, $\beta^{(1)}(y) > 0$, and $\gamma^{(1)}(z) > 0$.

The Leibniz-type partial derivatives of the scalar field T are defined [14]:

$$\partial_{x,\alpha}^{(1)}\mathrm{T} = \frac{1}{\alpha^{(1)}(x)}\frac{\partial\mathrm{T}}{\partial x}, \quad \partial_{y,\beta}^{(1)}\mathrm{T} = \frac{1}{\beta^{(1)}(y)}\frac{\partial\mathrm{T}}{\partial y}, \quad \text{and} \quad \partial_{z,\gamma}^{(1)}\mathrm{T} = \frac{1}{\gamma^{(1)}(z)}\frac{\partial\mathrm{T}}{\partial z} \tag{6a,b,c}$$

respectively.

The total Leibniz-type differential of the scalar field T is defined [14]:

$$\mathrm{dT} = \left[\alpha^{(1)}(x)\partial_{x,\alpha}^{(1)}\mathrm{T}\right]\mathrm{d}x + \left[\beta^{(1)}(x)\partial_{y,\beta}^{(1)}\mathrm{T}\right]\mathrm{d}y + \left[\gamma^{(1)}(z)\partial_{z,\gamma}^{(1)}\mathrm{T}\right]\mathrm{d}z \tag{7}$$

Thus, we show that:

$$\frac{\mathrm{dT}}{\mathrm{d}t}=\left[\alpha^{(1)}(x)\partial_{x,\alpha}^{(1)}\mathrm{T}\right]\frac{\mathrm{d}x}{\mathrm{d}t}+\left[\beta^{(1)}(y)\partial_{y,\beta}^{(1)}\mathrm{T}\right]\frac{\mathrm{d}y}{\mathrm{d}t}+\left[\gamma^{(1)}(z)\partial_{z,\gamma}^{(1)}\mathrm{T}\right]\frac{\mathrm{d}z}{\mathrm{d}t} \tag{8}$$

The gradient with respect to monotone functions

In the Cartesian co-ordinate system, the gradient with respect to monotone functions is defined [14]:

$$\nabla^{(\alpha,\beta,\gamma)}=i\alpha^{(1)}(x)\partial_{x,\alpha}^{(1)}+j\beta^{(1)}(y)\partial_{y,\beta}^{(1)}+k\gamma^{(1)}(z)\partial_{z,\gamma}^{(1)} \tag{9}$$

where i, j, and k are the unit vector in the Cartesian co-ordinate system.

The gradient with respect to monotone functions of the scalar field T is given [14]:

$$\nabla^{(\alpha,\beta,\gamma)}\mathrm{T}=i\alpha^{(1)}(x)\partial_{x,\alpha}^{(1)}\mathrm{T}+j\beta^{(1)}(y)\partial_{y,\beta}^{(1)}\mathrm{T}+k\gamma^{(1)}(z)\partial_{z,\gamma}^{(1)}\mathrm{T} \tag{10}$$

From eqs. (9) and (10) we show that [14]:

$$\mathrm{dT}=\left[\alpha^{(1)}(x)\partial_{x,\alpha}^{(1)}\mathrm{T}\right]\mathrm{d}x+\left[\beta^{(1)}\left(y\right)\partial_{y,\beta}^{(1)}\mathrm{T}\right]\mathrm{d}y+\left[\gamma^{(1)}(z)\partial_{z,\gamma}^{(1)}\mathrm{T}\right]\mathrm{d}z=\nabla^{(\alpha,\beta,\gamma)}\mathrm{T}\bullet\boldsymbol{n}\mathrm{d}r \tag{11}$$

where $\boldsymbol{n}$ is the unit normal to the surface, dr is the distance measured along the normal $\boldsymbol{n}$, and $\mathbf{d}\boldsymbol{r}=\boldsymbol{n}\mathrm{d}r=i\mathrm{d}x+j\mathrm{d}y+k\mathrm{d}z$.

The direction derivative with respect to monotone functions of the scalar field T along the normal $\boldsymbol{n}$ is defined [14]:

$$\frac{\mathrm{dT}}{\mathrm{d}r}=\nabla^{(\alpha,\beta,\gamma)}\mathrm{T}\bullet\boldsymbol{n}=\partial_n^{(\alpha,\beta,\gamma)}\mathrm{T} \tag{12}$$

The Laplace-like operator with respect to monotone functions, denoted as $\nabla^{(2\alpha,2\beta,2\gamma)}=\nabla^{(\alpha,\beta,\gamma)}\bullet\nabla^{(\alpha,\beta,\gamma)}$, of the scalar field T is defined [14]:

$$\nabla^{(2\alpha,2\beta,2\gamma)}\mathrm{T}=\left[\alpha^{(1)}(x)\partial_{x,\alpha}^{(1)}\right]^2\mathrm{T}+\left[\beta^{(1)}(y)\partial_{y,\beta}^{(1)}\right]^2\mathrm{T}+\left[\gamma^{(1)}(z)\partial_{z,\gamma}^{(1)}\right]^2\mathrm{T} \tag{13}$$

Let $\boldsymbol{O}=\boldsymbol{O}[\alpha(x),\beta(y),\gamma(z)]=\mathrm{O}_x i+\mathrm{O}_y j+\mathrm{O}_z k$.

The divergence with respect to monotone functions of the vector field $\boldsymbol{O}$ is defined [14]:

$$\nabla^{(\alpha,\beta,\gamma)}\bullet\boldsymbol{O}=\alpha^{(1)}(x)\partial_{x,\alpha}^{(1)}\mathrm{O}_x+\beta^{(1)}(y)\partial_{y,\beta}^{(1)}\mathrm{O}_y+\gamma^{(1)}(z)\partial_{z,\gamma}^{(1)}\mathrm{O}_z \tag{14}$$

The curl respect to monotone functions of the vector field $\boldsymbol{O}$ is defined [14]:

$$\nabla^{(\alpha,\beta,\gamma)}\times\boldsymbol{O}=\begin{pmatrix} i & j & k \\ \alpha^{(1)}(x)\partial_{x,\alpha}^{(1)} & \beta^{(1)}(y)\partial_{y,\beta}^{(1)} & \gamma^{(1)}(z)\partial_{z,\gamma}^{(1)} \\ \mathrm{O}_x & \mathrm{O}_y & \mathrm{O}_z \end{pmatrix} \tag{15}$$

The properties for the gradient with respect to monotone functions can be given:

$$\nabla^{(\alpha,\beta,\gamma)}\times\left[\nabla^{(\alpha,\beta,\gamma)}\times\boldsymbol{O}\right]=\nabla^{(\alpha,\beta,\gamma)}\left[\nabla^{(\alpha,\beta,\gamma)}\bullet\boldsymbol{O}\right]-\nabla^{(2\alpha,2\beta,2\gamma)}\boldsymbol{O} \tag{16}$$

$$\nabla^{(\alpha,\beta,\gamma)}\bullet\left[\nabla^{(\alpha,\beta,\gamma)}\times\boldsymbol{O}\right]=0 \tag{17}$$

$$\nabla^{(\alpha,\beta,\gamma)}\times\left(\nabla^{(\alpha,\beta,\gamma)}\mathrm{T}\right)=0 \tag{18}$$

and

$$\nabla^{(\alpha,\beta,\gamma)}\left(\mathrm{T}\Xi\right)=\mathrm{T}\nabla^{(\alpha,\beta,\gamma)}\left(\Xi\right)+\Xi\nabla^{(\alpha,\beta,\gamma)}\mathrm{T} \tag{19}$$

where $\Xi=\Xi_{\varpi}(x,y,z)=\Xi[\alpha(x),\beta(y),\gamma(z)]$.

The vector calculus with respect to monotone functions

Let $\boldsymbol{l}=\ell_{\varpi}(x,y,z)=[\alpha(x),\beta(y),\gamma(z)]$ is the vector line.

The arc length is given:

$$\ell=\int_0^{\ell}\mathrm{d}\ell=\int_a^b\sqrt{\left[\alpha^{(1)}(x)\right]^2\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left[\beta^{(1)}(y)\right]^2\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2+\left[\gamma^{(1)}(z)\right]^2\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \tag{20}$$

The line Riemann-Stieltjes-type integral of the vector field $\boldsymbol{O}$ along the subcurve $\boldsymbol{l}$, denoted by Θ, is defined:

$$\Theta=\int_{\ell}\boldsymbol{O}\bullet\mathbf{d}\boldsymbol{l}=\int_{\ell}\boldsymbol{O}\bullet\mathbf{n}\mathrm{d}\ell \tag{21}$$

which can be rewritten as:

$$\Theta=\int_{\ell}\boldsymbol{O}\bullet\mathbf{d}\boldsymbol{l}=\int_{\ell}\boldsymbol{O}\bullet\mathbf{n}\mathrm{d}\ell=\int_{\ell}\alpha^{(1)}(x)\mathrm{O}_x\mathrm{d}x+\beta^{(1)}(y)\mathrm{O}_y\mathrm{d}y+\gamma^{(1)}(z)\mathrm{O}_z\mathrm{d}z \tag{22}$$

where the element of the line is:

$$\mathbf{d}\boldsymbol{l}=\mathbf{n}\mathrm{d}\ell=i\alpha^{(1)}(x)\mathrm{d}x+j\beta^{(1)}(y)\mathrm{d}y+k\gamma^{(1)}(z)\mathrm{d}z \tag{23}$$

with the unit vector $\boldsymbol{n}$ tangent to vector line $\boldsymbol{l}$.

From eq. (19) we show that:

$$\Theta=\sqrt{\mathrm{O}_x\alpha^{(1)}(x)\frac{\mathrm{d}x}{\mathrm{d}t}+\mathrm{O}_y\beta^{(1)}(y)\frac{\mathrm{d}y}{\mathrm{d}t}+\mathrm{O}_z\gamma^{(1)}(z)\frac{\mathrm{d}z}{\mathrm{d}t}}\,\mathrm{d}t \tag{24}$$

The result is the special case of the work in [14].

Let $S=S_{\varpi}(x,y)=S[\alpha(x),\beta(y)]$.

The Riemann-Stieltjes-type double integral with respect to monotone functions of the scalar field T on the region S, denoted by $\mathrm{N(T)}$, is defined as [12, 13]:

$$\mathrm{N}\left(\mathrm{T}\right)=\iint_S\mathrm{T}\mathrm{d}S=\iint_S\mathrm{T}\alpha^{(1)}(x)\beta^{(1)}(y)\mathrm{d}x\mathrm{d}y=\iint_S\mathrm{T}\mathrm{d}\alpha(x)\mathrm{d}\beta(y) \tag{25}$$

where $\mathrm{d}S=\alpha^{(x)}(x)\beta^{(1)}(y)\mathrm{d}x\mathrm{d}y=\mathrm{d}\alpha(x)\mathrm{d}\beta(y)$ is the element of area.

Thus, we have that [12, 13]:

$$\iint_S\mathrm{T}\mathrm{d}S=\int_c^d\left[\int_a^b\Phi\alpha^{(1)}(x)\mathrm{d}x\right]\beta^{(1)}(y)\mathrm{d}y=\int_a^b\left[\int_c^d\Phi\beta^{(1)}(y)\mathrm{d}y\right]\alpha^{(1)}(x)\mathrm{d}x \tag{26}$$

where $x\in[a,b]$ and $y\in[c,d]$.

The Riemann-Stieltjes-type volume integral with respect to monotone functions of the scalar field Φ in the domain Ω is defined:

$$V\left(\mathrm{T}\right)=\iiint_{\Omega}\mathrm{T}\mathrm{d}V=\iiint_{\Omega}\mathrm{T}\alpha^{(1)}(x)\beta^{(1)}(y)\gamma^{(1)}(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint_{\Omega}\mathrm{T}\mathrm{d}\alpha(x)\mathrm{d}\beta(y)\mathrm{d}\gamma(z) \tag{27}$$

where $\mathrm{d}V = \alpha^{(1)}(x)\beta^{(1)}(y)\gamma^{(1)}(z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \mathrm{d}\alpha(x)\,\mathrm{d}\beta(y)\,\mathrm{d}\gamma(z)$ is the element of volume.

Thus, we have that:

$$\begin{aligned}\iiint_{\Omega} \mathrm{T}\mathrm{d}V &= \int_f^g \left[\int_c^d \left(\int_a^b \mathrm{T}\alpha^{(1)}(x)\mathrm{d}x\right)\beta^{(1)}(y)\mathrm{d}y\right]\gamma^{(1)}(z)\mathrm{d}z \\ &= \int_f^g \left[\int_a^b \left(\int_c^d \mathrm{T}\beta^{(1)}(y)\mathrm{d}y\right)\alpha^{(1)}(x)\mathrm{d}x\right]\gamma^{(1)}(z)\,\mathrm{d}z \\ &= \int_a^b \left[\int_c^d \left(\int_e^f \mathrm{T}\gamma^{(1)}(z)\mathrm{d}z\right)\beta^{(1)}(y)\mathrm{d}y\right]\alpha^{(1)}(x)\,\mathrm{d}x \end{aligned} \tag{28}$$

where $x \in [a,b]$, $y \in [c,d]$, and $z \in [f,g]$.

Let $\boldsymbol{S} = \boldsymbol{S}_{\varpi}(x,y,z) = \boldsymbol{S}[\alpha(x),\beta(y),\gamma(z)]$.

The Riemann-Stieltjes-type surface integral with respect to monotone functions of the vector field $\boldsymbol{O}$ on the surface $\boldsymbol{S}$ is defined:

$$\iint_{\mathbf{S}} \boldsymbol{O}\bullet\mathbf{d}\boldsymbol{S} = \iint_{\mathbf{S}} \boldsymbol{O}\bullet\mathbf{u}\mathrm{d}S \tag{29}$$

where $\boldsymbol{u} = \mathbf{d}\boldsymbol{S}/|\mathbf{d}\boldsymbol{S}| = \mathbf{d}\boldsymbol{S}/\mathrm{d}S$ is the unit normal vector to the surface $\boldsymbol{S}$ with $\mathrm{d}S = |\mathbf{d}\boldsymbol{S}|$, and:

$$\begin{aligned}\mathbf{d}\boldsymbol{S} &= \mathrm{d}\beta(y)\mathrm{d}\gamma(z)i + \mathrm{d}\alpha(x)\mathrm{d}\gamma(z)j + \mathrm{d}\alpha(x)\mathrm{d}\beta(y)k \\ &= i\beta^{(1)}(y)\gamma^{(1)}(z)\,\mathrm{d}y\,\mathrm{d}z + j\alpha^{(1)}(x)\gamma^{(1)}(z)\,\mathrm{d}x\,\mathrm{d}z + k\alpha^{(1)}(x)\beta^{(1)}(y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned} \tag{30}$$

is the element of the surface.

From eqs. (29) and (30) we have that:

$$\iint_{\mathbf{S}} \boldsymbol{O}\bullet\mathbf{d}\boldsymbol{S} = \iint_{S} \mathrm{O}_x\beta^{(1)}(y)\gamma^{(1)}(z)\,\mathrm{d}y\,\mathrm{d}z + \mathrm{O}_y\alpha^{(1)}(x)\gamma^{(1)}(z)\,\mathrm{d}x\,\mathrm{d}z + \mathrm{O}_z\alpha^{(1)}(x)\beta^{(1)}(y)\,\mathrm{d}x\,\mathrm{d}y \tag{31}$$

The flux of the vector field $\boldsymbol{O}$ across the surface $\mathbf{d}\boldsymbol{S}$, denoted by $J(\boldsymbol{O})$, is defined:

$$J(\boldsymbol{O}) = \oiint_{\mathbf{S}} \boldsymbol{O}\bullet\mathbf{d}\boldsymbol{S} \tag{32}$$

The divergence with respect to monotone functions of the vector field $\boldsymbol{O}$ is defined:

$$\nabla^{(\alpha,\beta,\gamma)}\bullet\boldsymbol{O} = \lim_{\Delta V_m \to 0}\frac{1}{\Delta V_m}\oiint_{\Delta\mathbf{S}_m} \boldsymbol{O}\bullet\mathbf{d}\boldsymbol{S} \tag{33}$$

where the volume V is divided into a large number of small subvolumes ΔV_m with surfaces $\Delta\mathbf{S}_m$, and $\mathbf{d}\boldsymbol{S}$ is the element of the surface $\boldsymbol{S}$ bounding the solid Ω.

It is noted that eq. (33) in the Cartesian co-ordinate system is the alternative form of eq. (14).

The curl with respect to monotone functions of the vector field $\boldsymbol{O}$ is defined:

$$\nabla^{(\alpha,\beta,\gamma)}\times\boldsymbol{O} = \lim_{\Delta S_m \to 0}\frac{1}{\Delta S_m}\oint_{\Delta\ell_m} \boldsymbol{O}\bullet\mathbf{d}\boldsymbol{l} \tag{34}$$

where $\mathbf{d}\boldsymbol{l}$ is the element of the vector line, ΔS_m is a small surface element perpendicular to $\boldsymbol{n}$, $\Delta\ell_m$ is the closed curve of the boundary of ΔS_m, and $\boldsymbol{n}$ is oriented in a postive sense.

Here, eq. (34) in the Cartesian co-ordinate system is the alternative form of eq. (15).

From eq. (33) we obtain the Gauss-like theorem as follows.

Let us present that:

$$\oiint_{\mathbf{S}} \boldsymbol{O}\bullet\boldsymbol{n}dS = \oiint_{\mathbf{S}} \boldsymbol{O}\bullet\mathbf{d}\boldsymbol{S}$$

$$= \oiint_{\boldsymbol{S}(x,y,z)} \mathrm{O}_x\beta^{(1)}(y)\gamma^{(1)}(z)\,\mathrm{d}y\,\mathrm{d}z + \mathrm{O}_y\alpha^{(1)}(x)\gamma^{(1)}(z)\,\mathrm{d}x\,\mathrm{d}z + \mathrm{O}_z\alpha^{(1)}(x)\beta^{(1)}(y)\,\mathrm{d}x\,\mathrm{d}y$$

The Gauss-Ostrogradsky-like theorem states that:

$$\iiint_{\Omega} \nabla^{(\alpha,\beta,\gamma)}\bullet\boldsymbol{O}\,\mathrm{d}V = \oiint_{\mathbf{S}} \boldsymbol{O}\bullet\boldsymbol{u}\,\mathrm{d}S \quad \text{or} \quad \iiint_{\Omega} \nabla^{(\alpha,\beta,\gamma)}\bullet\boldsymbol{O}\,\mathrm{d}V = \oiint_{\mathbf{S}} \boldsymbol{O}\bullet\mathbf{d}\boldsymbol{S} \tag{35}$$

When $\alpha(x) = x$, $\beta(y) = y$, and $\gamma(z) = z$, eq. (35) becomes the Gauss-Ostrogradsky theorem, proposed by Gauss in 1813 [19] and by Ostrogradsky in 1828 [20].

With use of eq. (34), we give the Stokes-like theorem as follows.

Let us show that:

$$\oint_{\ell} \boldsymbol{O}\bullet\mathbf{d}\boldsymbol{l} = \oint_{\ell} \mathrm{O}_x\alpha^{(1)}(x)\,\mathrm{d}x + \mathrm{O}_y\beta^{(1)}(y)\,\mathrm{d}y + \mathrm{O}_z\gamma^{(1)}(z)\,\mathrm{d}z$$

The Stokes-like theorem states that:

$$\oiint_{S} \left(\nabla^{(\alpha,\beta,\gamma)} \times \boldsymbol{O}\right)\bullet\boldsymbol{u}\,\mathrm{d}S = \oint_{\ell} \boldsymbol{O}\bullet\mathbf{d}\boldsymbol{l} \quad \text{or} \quad \oiint_{S} \left[\nabla^{(\alpha,\beta,\gamma)} \times \boldsymbol{O}\right]\bullet\mathbf{d}\boldsymbol{S} = \oint_{\ell} \boldsymbol{O}\bullet\mathbf{d}\boldsymbol{l} \tag{36}$$

Here, when $\alpha(x) = x$, $\beta(y) = y$, and $\gamma(z) = z$, eq. (36) becomes the Stokes theorem, proposed by Stokes in 1845 [21].

From eq. (36) we derive the Green-like theorem as follows.

The Green-like theorem states:

$$\oint_{\ell} \mathrm{O}_x\alpha^{(1)}(x)\,\mathrm{d}x + \mathrm{O}_y\beta^{(1)}(y)\,\mathrm{d}y$$

$$= \iint_{S} \left[\alpha^{(1)}(x)\,{}^{PL}\partial_x^{(1)}\mathrm{O}_y - \beta^{(1)}(y)\,{}^{PL}\partial_y^{(1)}\mathrm{O}_x\right]\alpha^{(1)}(x)\beta^{(1)}(y)\,\mathrm{d}x\,\mathrm{d}y \tag{37}$$

where S is the domain bounded by the contour ℓ.

The Green-like identity of first type states that:

$$\iiint_{\Omega} \nabla^{(\alpha,\beta,\gamma)}\bullet\left(\mathrm{T}\nabla^{(2\alpha,2\beta,2\gamma)}\Xi + \nabla^{(\alpha,\beta,\gamma)}\Xi\bullet\nabla^{(\alpha,\beta,\gamma)}\mathrm{T}\right)dV = \oiint_{\mathbf{S}(x,y,z)} \mathrm{T}\partial_u^{(\alpha,\beta,\gamma)}\Xi dS \tag{38}$$

The Green-like identity of second type states that:

$$\iiint_{\Omega} \nabla^{(\alpha,\beta,\gamma)}\bullet\left[\mathrm{T}\nabla^{(2\alpha,2\beta,2\gamma)}\Phi - \Phi\nabla^{(2\alpha,2\beta,2\gamma)}\mathrm{T}\right]\mathrm{d}V = \oiint_{\mathbf{S}} \left[\mathrm{T}\partial_u^{(\alpha,\beta,\gamma)}\Phi - \Phi\partial_u^{(\alpha,\beta,\gamma)}\mathrm{T}\right]\mathrm{d}S \tag{39}$$

Here, when $\alpha(x) = x$, $\beta(y) = y$, and $\gamma(z) = z$, Green-like theorem and identities are the Green theorem and identities, proposed by Green in 1828 [22].

Applied to describe the complex fliud flow

Let us consider the co-ordinate system, given as $\left[\alpha(x),\beta(y),\gamma(z),\eta(t)\right] = \eta(t) + i\alpha(x) + j\beta(y) + k\gamma(z)$, where i, j, and k are the unit vector in the Cartesian co-ordinate system.

Let $\mathrm{H} = \mathrm{H}_{\varpi}(x, y, z, t) = \mathrm{H}\left[\alpha(x), \beta(y), \gamma(z), \eta(t)\right]$ be the scalar complex fluid field.

The material Leibniz-type derivative of the complex fluid-flows

The total Leibniz-type differential of the complex scalar field is presented as follows:

$$\mathrm{dH} = \left[\alpha^{(1)}(x)\partial_{x,\alpha}^{(1)}\mathrm{H}\right]\mathrm{d}x + \left[\beta^{(1)}(y)\partial_{y,\beta}^{(1)}\mathrm{H}\right]\mathrm{d}y + \left[\gamma^{(1)}(z)\partial_{z,\gamma}^{(1)}\mathrm{H}\right]\mathrm{d}z + \left[\eta^{(1)}(t)\partial_{t,\eta}^{(1)}\mathrm{H}\right]\mathrm{d}t \quad (40)$$

which yields that:

$$\frac{\mathrm{dH}}{\mathrm{d}t} = \left[\alpha^{(1)}(x)\partial_{x,\alpha}^{(1)}\mathrm{H}\right]\frac{\mathrm{d}x}{\mathrm{d}t} + \left[\beta^{(1)}(y)\partial_{y,\beta}^{(1)}\mathrm{H}\right]\frac{\mathrm{d}y}{\mathrm{d}t} + \left[\gamma^{(1)}(z)\partial_{z,\gamma}^{(1)}\mathrm{H}\right]\frac{\mathrm{d}z}{\mathrm{d}t} + \left[\eta^{(1)}(t)\partial_{t,\eta}^{(1)}\mathrm{H}\right] \quad (41)$$

From eq. (41) the material Leibniz-type derivative of the complex fluid density Ξ is defined as follows:

$$\frac{D\Xi}{Dt} = \eta^{(1)}(t)\partial_{t,\eta}^{(1)}\Xi + \boldsymbol{\upsilon}\cdot\nabla^{(\alpha,\beta,\gamma)}\Xi \quad (42)$$

where $\boldsymbol{\upsilon} = \left(\partial x/\partial t, \partial y/\partial t, \partial z/\partial t\right) = i\upsilon_x + j\upsilon_y + k\upsilon_z$ is the velocity vector.

The Stokes material derivative, proposed by Stokes in 1845 [23] and 1851 [24], is the special case of the material Leibniz-type derivative when $\alpha(x) = x$, $\beta(y) = y$, and $\gamma(z) = z$.

The transport theorem for the complex fluid-flows

From eq. (42) the transport theorem for the complex fluid-flow can be given:

$$\frac{D}{Dt}\iiint_{\Omega(t)} \mathrm{T}\mathrm{d}V = \iiint_{\Omega(t)} \left[\eta^{(1)}(t)\partial_{t,\eta}^{(1)}\mathrm{T} + \boldsymbol{\upsilon}\cdot\nabla^{(\alpha,\beta,\gamma)}\mathrm{T}\right]\mathrm{d}V \quad (43)$$

which, by using eq. (42), leads to:

$$\frac{D}{Dt}\iiint_{\Omega(t)} \mathrm{T}\mathrm{d}V = \iiint_{\Omega(t)} \eta^{(1)}(t)\partial_{t,\eta}^{(1)}\mathrm{T}\mathrm{d}V + \oiint_{\mathbf{S}(t)} \mathrm{T}\boldsymbol{\upsilon}\bullet\mathbf{d}\boldsymbol{S} \quad (44)$$

since

$$\iiint_{\Omega(t)} \boldsymbol{\upsilon}\cdot\nabla^{(\alpha,\beta,\gamma)}\mathrm{T}\mathrm{d}V = \oiint_{\boldsymbol{S}(t)} \mathrm{T}\left(\boldsymbol{\upsilon}\bullet\boldsymbol{u}\right)\mathrm{d}S = \oiint_{\mathbf{S}(t)} \mathrm{T}\boldsymbol{\upsilon}\bullet\mathbf{d}\boldsymbol{S} \quad (45)$$

where $\boldsymbol{S}(t)$ is the surface of $\Omega(t)$, $\boldsymbol{u}$ is the unit normal to the surface, and $\boldsymbol{\upsilon}$ is the velocity vector.

The Reynolds transport theorem for the fluid-flow, proposed by Reynolds in 1903 [25], is the special case of the transport theorem for the complex fluid-flow when $\alpha(x) = x$, $\beta(y) = y$, and $\gamma(z) = z$.

The conservation of the mass of the complex fluid-flows

The conservation of the mass of the complex fluid-flow is given:

$$\eta^{(1)}(t)\partial_{t,\eta}^{(1)}\rho + \boldsymbol{\upsilon}\cdot\nabla^{(\alpha,\beta,\gamma)}\rho = 0 \quad \text{and} \quad \eta^{(1)}(t)\partial_{t,\eta}^{(1)}\rho + \nabla^{(\alpha,\beta,\gamma)}\cdot(\boldsymbol{\upsilon}\rho) = 0 \quad (46\text{a,b})$$

since

$$\frac{D}{Dt}\iiint_{\Omega(t)} \rho\mathrm{d}V = \iiint_{\Omega(t)} \left[\eta^{(1)}(t)\partial_{t,\eta}^{(1)}\rho + \boldsymbol{\upsilon}\cdot\nabla^{(\alpha,\beta,\gamma)}\rho\right]\mathrm{d}V = 0 \quad (47)$$

which is derived from the mass of the complex fluid-flow, defined:

$$\mathrm{M} = \iiint\limits_{\Omega(t)} \rho \mathrm{d}V$$

where ρ and M are the density and mass of the complex fluid-flow, respectively.

The conservation of the mass, proposed by Euler in 1757 [26], is the special case of the conservation of the mass of the complex fluid-flow when $\alpha(x) = x$, $\beta(y) = y$, and $\gamma(z) = z$.

The Cauchy-type strain tensor, Stokes-type strain tensor, and Stokes-type velocity gradient tensor for the complex fluid-flows

The Cauchy-type strain tensor for the complex fluid-flow, denoted by ϑ, is defined as:

$$\vartheta = \frac{1}{2}\left[\nabla^{(\alpha,\beta,\gamma)} \cdot \boldsymbol{\upsilon} + \boldsymbol{\upsilon} \cdot \nabla^{(\alpha,\beta,\gamma)}\right] \tag{48}$$

The Cauchy strain tensor, proposed by Cauchy in 1823 [27], is the special case of the Cauchy-type strain tensor for the complex fluid-flow when $\alpha(x) = x$, $\beta(y) = y$, and $\gamma(z) = z$.

The Stokes-type strain tensor for the complex fluid-flow, denoted by Λ, is defined as:

$$\Lambda = \frac{1}{2}\left[\nabla^{(\alpha,\beta,\gamma)} \cdot \boldsymbol{\upsilon} - \boldsymbol{\upsilon} \cdot \nabla^{(\alpha,\beta,\gamma)}\right] \tag{49}$$

The Stokes-type velocity gradient tensor for the complex fluid-flow, denoted by $\nabla^{(\alpha,\beta,\gamma)} \cdot \boldsymbol{\upsilon}$, is given:

$$\nabla^{(\alpha,\beta,\gamma)} \cdot \boldsymbol{\upsilon} = \vartheta + \Lambda = \frac{1}{2}\left[\nabla^{(\alpha,\beta,\gamma)} \cdot \boldsymbol{\upsilon} + \boldsymbol{\upsilon} \cdot \nabla^{(\alpha,\beta,\gamma)}\right] + \frac{1}{2}\left[\nabla^{(\alpha,\beta,\gamma)} \cdot \boldsymbol{\upsilon} - \boldsymbol{\upsilon} \cdot \nabla^{(\alpha,\beta,\gamma)}\right] \tag{50}$$

The stress tensor for the complex fluid-flow, denoted by $\boldsymbol{X}$, is defined:

$$\boldsymbol{X} = -p\boldsymbol{I} + 2\lambda\boldsymbol{h} \tag{51}$$

where λ is the shear moduli of the viscosity, and $\boldsymbol{I}$ is the unit tensor.

The Stokes-type strain tensor and Stokes-type velocity gradient tensor, proposed by Stokes in 1845 [23], are the special cases of the Stokes strain tensor and Stokes velocity gradient tensor for the complex fluid-flow when $\alpha(x) = x$, $\beta(y) = y$, and $\gamma(z) = z$.

The conservation of the momentums for the complex fluid-flows

The conservation of the momentums for the complex fluid-flow is:

$$\frac{D}{Dt}\iiint\limits_{\Omega(t)} \rho\boldsymbol{\upsilon} \mathrm{d}V = \iiint\limits_{\Omega(t)} \boldsymbol{U} \mathrm{d}V + \oiint\limits_{\mathbf{S}(t)} \boldsymbol{X} \bullet \mathbf{d}\boldsymbol{S} \tag{52}$$

where $\boldsymbol{U}$ is the specific body force.

Thus, we have:

$$\eta^{(1)}(t)\partial_{t,\eta}^{(1)}(\rho\boldsymbol{\upsilon}) + \boldsymbol{\upsilon} \cdot \nabla^{(\alpha,\beta,\gamma)}(\rho\boldsymbol{\upsilon}) = \nabla^{(\alpha,\beta,\gamma)} \bullet \boldsymbol{X} + \boldsymbol{U} \tag{53}$$

since

$$\iiint\limits_{\Omega(t)} \left[\eta^{(1)}(t)\partial_{t,\eta}^{(1)}(\rho\boldsymbol{\upsilon}) + \boldsymbol{\upsilon} \cdot \nabla^{(\alpha,\beta,\gamma)}(\rho\boldsymbol{\upsilon}) - \boldsymbol{U} - \nabla^{(\alpha,\beta,\gamma)} \bullet \boldsymbol{X}\right] \mathrm{d}V = 0 \tag{54}$$

where

$$\frac{D}{Dt}\iiint\limits_{\Omega(t)}\rho\boldsymbol{\upsilon}\mathrm{d}V=\iiint\limits_{\Omega(t)}\left[\eta^{(1)}(t)\partial_{t,\eta}^{(1)}(\rho\boldsymbol{\upsilon})+\boldsymbol{\upsilon}\cdot\nabla^{(\alpha,\beta,\gamma)}(\rho\boldsymbol{\upsilon})\right]\mathrm{d}V \tag{55}$$

and

$$\oiint\limits_{\mathbf{S}(t)}\boldsymbol{X}\bullet\mathbf{d}\boldsymbol{S}=\iiint\limits_{\Omega(t)}\nabla^{(\alpha,\beta,\gamma)}\bullet\boldsymbol{X}\mathrm{d}V \tag{56}$$

The Navier-Stokes-type equations of the complex fluid-flows

From eq. (51) we have:

$$\nabla^{(\alpha,\beta,\gamma)}\bullet\boldsymbol{X}=-\nabla^{(\alpha,\beta,\gamma)}p+\beta\nabla^{(2\alpha,2\beta,2\gamma)}\boldsymbol{\upsilon} \tag{57}$$

such that:

$$\eta^{(1)}(t)\partial_{t,\eta}^{(1)}(\rho\boldsymbol{\upsilon})+\boldsymbol{\upsilon}\cdot\nabla^{(\alpha,\beta,\gamma)}(\rho\boldsymbol{\upsilon})=-\nabla^{(\alpha,\beta,\gamma)}p+\beta\nabla^{(2\alpha,2\beta,2\gamma)}\boldsymbol{\upsilon}+\boldsymbol{U} \tag{58}$$

Thus, with aid of eq. (58), we have:

$$\rho\left[\eta^{(1)}(t)\partial_{t,\eta}^{(1)}\boldsymbol{\upsilon}+\boldsymbol{\upsilon}\cdot\nabla^{(\alpha,\beta,\gamma)}\boldsymbol{\upsilon}\right]=-\nabla^{(\alpha,\beta,\gamma)}p+\beta\nabla^{(2\alpha,2\beta,2\gamma)}\boldsymbol{\upsilon}+\boldsymbol{U} \tag{59}$$

The Navier-Stokes equations of the complex fluid-flow, proposed by Stokes in 1845 [23] and by Navier in 1822 [28], are the special case of the Navier-Stokes-type equations of the complex fluid-flow when $\alpha(x)=x$, $\beta(y)=y$, and $\gamma(z)=z$.

The equations of the complex turbulent flows

From eqs. (50) and (58) we consider that:

$$\nabla^{(\alpha,\beta,\gamma)}\cdot\boldsymbol{\upsilon}=0 \tag{60}$$

and

$$\eta^{(1)}(t)\partial_{t,\eta}^{(1)}(\rho\boldsymbol{\upsilon})+\boldsymbol{\upsilon}\cdot\nabla^{(\alpha,\beta,\gamma)}(\rho\boldsymbol{\upsilon})=-\nabla^{(\alpha,\beta,\gamma)}p+\beta\nabla^{(2\alpha,2\beta,2\gamma)}\boldsymbol{\upsilon} \tag{61}$$

which lead to the equations of the complex turbulent flows:

$$\nabla^{(\alpha,\beta,\gamma)}\cdot\boldsymbol{\upsilon}=0 \tag{62}$$

and

$$\eta^{(1)}(t)\partial_{t,\eta}^{(1)}\boldsymbol{\upsilon}+\boldsymbol{\upsilon}\cdot\nabla^{(\alpha,\beta,\gamma)}\boldsymbol{\upsilon}=-\frac{1}{\rho}\nabla^{(\alpha,\beta,\gamma)}p+\frac{\beta}{\rho}\nabla^{(2\alpha,2\beta,2\gamma)}\boldsymbol{\upsilon} \tag{63}$$

The equations of the turbulent flows, pointed out by Frisch in 1995 [29], are the special cases of the equations of the complex turbulent flows when $\alpha(x)=x$, $\beta(y)=y$, and $\gamma(z)=z$.

The Euler-type equations of the complex fluid-flows

From eqs. (60) and (61) we have, by using $\beta=0$, that:

$$\nabla^{(\alpha,\beta,\gamma)}\cdot\boldsymbol{\upsilon}=0 \tag{64}$$

and

$$\eta^{(1)}(t)\partial_{t,\eta}^{(1)}(\rho\boldsymbol{\upsilon})+\boldsymbol{\upsilon}\cdot\nabla^{(\alpha,\beta,\gamma)}(\rho\boldsymbol{\upsilon})=-\nabla^{(\alpha,\beta,\gamma)}p \tag{65}$$

The Euler equations of the complex fluid-flow, proposed by Euler in 1757 [26], are the special cases of the Euler-type equations of the complex fluid-flow when $\alpha(x) = x$, $\beta(y) = y$, and $\gamma(z) = z$.

Conclusion

The present work had addressed the theory of the vector calculus with respect to monotone functions. Moreover, we discussed the material Leibniz-type derivative, transport theorem, Cauchy-type strain tensor, Stokes-type strain tensor, Stokes-type velocity gradient tensor, Navier-Stokes-type equations of the complex fluid-flow, equations of the complex turbulent flows, and Euler-type equations of the complex fluid-flow. The obtained results are important for modeling the complex fluid-flows.

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Nomenclature

t – time, [s]
$\boldsymbol{U}$ – specific body force, [Mm^{-3}]
x, y, z – co-ordinates, [m]

Greek symbol

$\boldsymbol{v}$ – velocity vector, [ms^{-1}]

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