THE VECTOR POWER-LAW CALCULUS WITH APPLICATIONS IN POWER-LAW FLUID FLOW

by

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In this article, based on the Leibniz derivative and Stieltjes-Riemann integral, we suggest the vector power-law calculus to consider the conservations of the mass and angular momentums for the power-law fluid.

Key words: *power-law fluid, Leibniz derivative, Stieltjes-Riemann integral, vector power-law calculus*

Introduction

Fractals are beautiful mathematical constructs, described by the scaling law [1], which is a mathematical relationship to the complex behaviors in the nature phenomena [2], where this description is related to the theory of dynamical systems [3] in condensed matter problems [4], and in the statistical mechanics of disordered systems [5].

In 1967, Mandelbrot structured the Mandelbrot scaling law, given [6]:

$$\phi(t) = \kappa t^{1-D} \tag{1}$$

where $\kappa \in (0,+\infty)$ is the normalization constant, $t \in (0,+\infty)$ is the radius or scale, and $D \in (0,+\infty)$ is the fractal dimension.

The theory of the functions related to fractals [7] has been developed in the different descriptions for the scaling law, which is considered by the measures [8], which is relevant to Hausdorff measure [9], where this quantitation is related to the power-law. The Hausdorff derivative in the Hausdorff space was proposed in [10]. The fractal derivative in the scaling law was considered in [11]. The metric derivative in the metric space was proposed in [11]. The vector calculus based on the Riemann-Liouville fractional derivative was proposed in [12] and further developed in [13, 14].

The scaling-law calculus is one of the hot topics on the theory of the calculus with respect to monotone functions, which includes the Leibniz derivative [15] and Stieltjes-Riemann integral [16] based on the Riemann work [17]. The power-law calculus was proposed in 2019 [18] and further developed in 2020 [19] based on the Leibniz derivative [15] and

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Stieltjes-Riemann integral [16]. Moreover, the power-law derivative is equal to the Hausdorff derivative without the strict proof that was proposed in [20, 21].

The target of the paper is to propose the theory of the vector power-law calculus based on the calculus with respect to monotone functions, to give the generalized integral transforms, and to present the novel application to the power-law fluid-flow.

The results on the calculus with respect to monotone functions

In this section, we introduce the power-law differential calculus and power-law integral calculus.

Let $\Phi_\omega(t)=(\Phi\circ\omega)(t)=\Phi[\omega(t)]$, where $\omega(t)$ is the monotone function, *e. g.*, $\omega^{(1)}(t)=\mathrm{d}\omega(t)/\mathrm{d}t>0$. Let $\mathfrak{A}(\Phi)$ be the set of the continuous functions $\Phi(\omega)$ in the domain $\aleph$. Let $\mathfrak{M}(\omega)$ be the set of the continuous derivatives of the functions $\omega(t)$ in the domain $\mathfrak{I}$. Let $\Phi_\varphi(t)=(\Phi\circ\varphi)(t)=\Phi[\varphi(t)]$.

Let us consider the sets of the composite functions, given as below:

$$\aleph(\Phi_\omega)=\left[\Phi_\omega(t):\Phi_\omega(t)\in\mathfrak{M}(\omega),\omega\in\mathfrak{A}(\varphi)\right] \tag{2}$$

The calculus with respect to monotone functions

Let $\Phi_\omega\in\aleph(\Phi_\omega)$. The Leibniz derivative of the function $\Phi_\omega(t)$ is defined [15, 18, 20]:

$${}^{L}D_t^{(1)}\Phi_\omega(t)=\frac{1}{\omega^{(1)}(t)}\frac{\mathrm{d}\Phi_\omega(t)}{\mathrm{d}t} \tag{3}$$

The geometric interpretations of the Leibniz derivative is the rate of change of the functional $\Phi(\omega)$ with the function $\omega(t)$ in the independent variable t [18, 19]. It is to say, the slope of the functional $\Phi(\omega)$ with the function $\omega(t)$ in the independent variable t [18, 19].

Let $\Phi_\omega\in\aleph(\Phi_\omega)$. The Stieltjes-Riemann integral of the function $\Theta_\omega(t)$ is defined [16, 18, 19]:

$${}_{a}^{L}I_b^{(1)}\Theta_\omega(t)=\int_a^b\Theta_\omega(t)\omega^{(1)}(t)\,\mathrm{d}t \tag{4}$$

Similarly, the geometric interpretations of the Stieltjes-Riemann integral is the area enclosed by the integrand function $\Phi(\omega)$ and the function $\omega(t)$ in the independent variable $t\in[a,b]$[18, 19].

The power-law calculus

Let $\omega(t)=t^D$, where D is the fractal dimension.

Let $\Phi_\omega\in\aleph(\Phi_\omega)$. The power-law derivative of the function $\Phi_\omega(t)$ is defined [18, 20]:

$${}^{PL}D_t^{(1)}\Phi_\omega(t)=\frac{t^{1-D}}{D}\frac{\mathrm{d}\Phi_\omega(t)}{\mathrm{d}t} \tag{5}$$

It is not difficult to show that the geometric interpretations of the power-law derivative is the rate of change of the functional $\Phi(\omega)$ with the function $\omega(t)=t^D$ in the independent variable t [21].

Let $\Phi_\omega\in\aleph(\Phi_\omega)$. The power-law differential of the function $\Phi_\omega(t)$, denoted by $\mathrm{d}\Phi_\omega(t)$, is given:

$$\mathrm{d}\Phi_\omega(t)=Dt^{D-1}\,{}^{PL}D_t^{(1)}\Phi_\omega(t)\,\mathrm{d}t \tag{6}$$

Let $\Phi_\omega \in \aleph(\Phi_\omega)$. The power-law integral of the function $\Theta_\omega(t)$ is defined [18, 19]:

$$ {}^{PL}_{a}I_{b}^{(1)}\Theta_\omega(t) = D\int_a^b \Theta_\omega(t)t^{D-1}\mathrm{d}t \tag{7}$$

Similarly, it is shown that the geometric interpretations of the power-law integral is the area enclosed by the integrand function $\Phi(\omega)$ and the function $\omega(t) = t^D$ in the independent variable $t \in [a,b]$ [21].

Let $\Phi_\omega \in \aleph(\Phi_\omega)$. The indefinite power-law integral of the function $\Theta_\varphi(t)$ is defined:

$$ {}^{PL}I_t^{(1)}\Theta_\omega(t) = D\int \Theta_\omega(t)Dt^{D-1}\mathrm{d}t \tag{8}$$

Let $\Theta_\omega \in \aleph(\Phi_\omega)$ and $\Pi_\omega \in \aleph(\Pi_\omega)$. The properties of the power-law differential calculus used in this paper can be presented as follows [18,19]:

(Y1) The product rule for the power-law derivative [15]:

$$ {}^{PL}D_t^{(1)}[\Theta_\omega(t)\cdot\Pi_\omega(t)] = \Pi_\omega(t)\, {}^{PL}D_t^{(1)}\,\Theta_\omega(t) + \Theta_\omega(t)\, {}^{PL}D_t^{(1)}\,\Pi_\omega(t) \tag{9}$$

(Y2) The chain rule for the power-law derivative:

$$ {}^{PL}D_t^{(1)}\,\{w[\Theta_\omega(t)]\} = w^{(1)}(\Theta_\omega)\cdot {}^{PL}D_t^{(1)}\,\Theta_\omega(t) \tag{10}$$

where $w^{(1)}(\Theta_\omega) = \mathrm{d}w(\Theta_\omega)/\mathrm{d}\Theta_\omega$ exists.

The power-law partial derivatives, power-law gradients and directional power-law derivative

Let us consider the power-law co-ordinate system, given as $ix^{D_1} + jy^{D_2} + kz^{D_3} = [x^{D_1}, y^{D_2}, z^{D_3}]$, where D_1, D_2 and D_3 are the fractal dimensions, and i, j, and k are the unit vector in the Cartesian co-ordinate system.

Let us consider the function, defined by $\psi_\omega = \psi_\omega^{(D_1,D_2,D_3)}(x,y,z) = \psi(x^{D_1}, y^{D_2}, z^{D_3})$.

The power-law partial derivatives of the function $\psi_\omega = \psi_\omega(x,y,z)$ are defined:

$$ {}^{PL}\partial_x^{(1)}\psi_\omega = \left(\frac{x^{1-D_1}}{D_1}\frac{\partial}{\partial x}\right)\psi_\omega,\quad {}^{PL}\partial_y^{(1)}\psi_\omega = \left(\frac{y^{1-D_2}}{D_2}\frac{\partial}{\partial y}\right)\psi_\omega,\quad {}^{PL}\partial_z^{(1)}\psi_\omega = \left(\frac{z^{1-D_3}}{D_2}\frac{\partial}{\partial z}\right)\psi_\omega, \tag{11a,b,c}$$

where $\psi = \psi_\omega^{(D,D,D)}(x,y,z) = \psi(x^D, y^D, z^D)$.

The total power-law differential of the function $\psi_\omega = \psi_\omega(x,y,z)$ is defined:

$$ \mathrm{d}\psi_\omega = [D_1x^{D_1-1}\,{}^{PL}\partial_x^{(1)}\psi_\omega]\mathrm{d}x + [D_2y^{D_2-1}\,{}^{PL}\partial_y^{(1)}\psi_\omega]\mathrm{d}y + [D_3z^{D_3-1}\,{}^{PL}\partial_z^{(1)}\psi_\omega]\mathrm{d}z \tag{12}$$

Thus, the power-law derivative with respect to the time t is given:

$$ \frac{\mathrm{d}\psi_\omega}{\mathrm{d}t} = \left[D_1x^{D_1-1}\,{}^{PL}\partial_x^{(1)}\psi_\omega\right]\frac{\mathrm{d}x}{\mathrm{d}t} + \left[D_2x^{D_2-1}\,{}^{PL}\partial_y^{(1)}\psi_\omega\right]\frac{\mathrm{d}y}{\mathrm{d}t} + \left[D_3x^{D_3-1}\,{}^{PL}\partial_z^{(1)}\psi_\omega\right]\frac{\mathrm{d}z}{\mathrm{d}t} \tag{13}$$

The power-law gradient of first type in the Cartesian co-ordinate system is defined:

$$ \nabla^{(D_1,D_2,D_3)} = i(D_1x^{D_1-1})\,{}^{PL}\partial_x^{(1)} + j(D_2y^{D_2-1})\,{}^{PL}\partial_y^{(1)} + k(D_3z^{D_3-1})\,{}^{PL}\partial_z^{(1)} \tag{14}$$

which deduces that the power-law gradient of second type in the Cartesian co-ordinate system:

$$ \nabla^{(D)} = i(Dx^{D-1})\,{}^{PL}\partial_x^{(1)} + j(Dy^{D-1})\,{}^{PL}\partial_y^{(1)} + k(Dz^{D-1})\,{}^{PL}\partial_z^{(1)} \tag{15}$$

The power-law gradient of first type of the scalar field $\psi_\omega = \psi_\omega(x, y, z)$ reads:

$$\nabla^{(D_1,D_2,D_3)}\psi_\omega = i(D_1 x^{D_1-1})^{PL}\partial_x^{(1)}\psi_\omega + j(D_2 y^{D_2-1})^{PL}\partial_y^{(1)}\psi_\omega + k(D_3 z^{D_3-1})^{PL}\partial_z^{(1)}\psi_\omega \tag{16}$$

Similarly, the power-law gradient of second type of the scalar field ψ can be written:

$$\nabla^{(D)}\psi = i(Dx^{D-1})^{PL}\partial_x^{(1)}\psi + j(Dy^{D-1})^{PL}\partial_y^{(1)}\psi + k(Dz^{D-1})^{PL}\partial_z^{(1)}\psi \tag{17}$$

From eqs. (16) and (17) we have that:

$$\mathrm{d}\psi_\omega = \nabla^{(D_1,D_2,D_3)}\psi_\omega \bullet \mathbf{d}\boldsymbol{r} = \nabla^{(D_1,D_2,D_3)}\psi_\omega \bullet \boldsymbol{n}\mathrm{d}r \text{ and } \mathrm{d}\psi = \nabla^{(D)}\psi \bullet \mathbf{d}\boldsymbol{r} = \nabla^{(D)}\psi \bullet \boldsymbol{n}\mathrm{d}r, \tag{18a,b}$$

with $\mathbf{d}\boldsymbol{r} = \boldsymbol{n}\mathrm{d}r = i\mathrm{d}x + j\mathrm{d}y + k\mathrm{d}z$, where $\boldsymbol{n}$ is the unit normal, and $\mathrm{d}r$ is a distance measured along the normal $\boldsymbol{n}$.

The directional power-law derivative of the function $\psi_\omega = \psi_\omega(x, y, z)$, denoted by $\nabla_n^{(D_1,D_2,D_3)}\psi_\omega$, is defined:

$$\frac{\mathrm{d}\psi_\omega}{\mathrm{d}r} = \nabla^{(D_1,D_2,D_3)}\psi_\omega \bullet \boldsymbol{n} = \partial_n^{(D_1,D_2,D_3)}\psi_\omega \tag{19}$$

which leads to:

$$\frac{\mathrm{d}\psi}{\mathrm{d}r} = \nabla^{(D)}\psi \bullet \boldsymbol{n} = \partial_n^{(D)}\psi \tag{20}$$

where $\mathrm{d}\psi_\omega/\mathrm{d}r$ and $\mathrm{d}\psi/\mathrm{d}r$ are the rates of changes of ψ_ω and ψ along the normal $\boldsymbol{n}$, respectively.

The power-law Laplace-like operator of first type, denoted as $\nabla^{(D_1,D_2,D_3)} \bullet \nabla^{(D_1,D_2,D_3)} = \nabla^{(2D_1,2D_2,2D_3)}$, of the scalar field ψ_ω is defined:

$$\nabla^{(2D_1,2D_2,2D_3)}\psi_\omega = \left[(D_1 x^{D_1-1})^{PL}\partial_x^{(1)}\right]^2\psi_\omega + \left[(D_2 y^{D_2-1})^{PL}\partial_y^{(1)}\right]^2\psi_\omega + \left[(D_3 z^{D_3-1})^{PL}\partial_z^{(1)}\right]^2\psi_\omega \tag{21}$$

In a similar way, the power-law Laplace-like operator of second type, denoted as $\nabla^{(2D)} = \nabla^{(D)} \bullet \nabla^{(D)}$, of the scalar field ψ is defined:

$$\nabla^{(2D)}\psi = \left[(Dx^{D-1})^{PL}\partial_x^{(1)}\right]^2\psi + \left[(Dy^{D-1})^{PL}\partial_y^{(1)}\right]^2\psi + \left[(Dz^{D-1})^{PL}\partial_z^{(1)}\right]^2\psi \tag{22}$$

The properties for the power-law gradient of first type read:

$$\left[\nabla^{(D_1,D_2,D_3)} \bullet \nabla^{(D_1,D_2,D_3)}\right]\psi_\omega = \nabla^{(2D_1,2D_2,2D_3)}\psi_\omega \tag{23}$$

$$\nabla^{(D_1,D_2,D_3)}(\psi_\omega\Theta_\omega) = \psi_\omega\nabla^{(D_1,D_2,D_3)}\Theta_\omega + \Theta_\omega\nabla^{(D_1,D_2,D_3)}\psi_\omega \tag{24}$$

$$\nabla^{(D_1,D_2,D_3)} \bullet \left(\Theta_\omega\nabla^{(D_1,D_2,D_3)}\psi_\omega\right) = \Theta_\omega\nabla^{(2D_1,2D_2,2D_3)}\psi_\omega + \nabla^{(D_1,D_2,D_3)}\psi_\omega \bullet \nabla^{(D_1,D_2,D_3)}\Theta_\omega \tag{25}$$

where ψ_ω and Θ_ω are the scalar fields.

The properties for the power-law gradient of second type can be given:

$$\left[\nabla^{(D)} \bullet \nabla^{(D)}\right]\psi = \nabla^{(2D)}\psi, \quad \nabla^{(D)}(\psi\Theta) = \psi\nabla^{(D)}\Theta + \Theta_\omega\nabla^{(D)}\psi \tag{26a,b}$$

$$\nabla^{(D)} \bullet \left(\Theta\nabla^{(D)}\psi\right) = \Theta\nabla^{(2D)}\psi + \nabla^{(D)}\psi \bullet \nabla^{(D)}\Theta \tag{27}$$

where ψ and Θ are the scalar fields.

Theory of the vector power-law calculus

The element of the vector line $\ell = \ell_{\omega}(x, y, z) = \ell(x^{D_1}, y^{D_2}, z^{D_3})$ is given:

$$\mathbf{d}\boldsymbol{l} = \boldsymbol{m}\mathrm{d}\ell = i(D_1 x^{D_1 - 1})\mathrm{d}x + j(D_2 y^{D_2 - 1})\mathrm{d}y + k(D_3 z^{D_3 - 1})\mathrm{d}z \tag{28}$$

and

$$\mathrm{d}\ell = |\mathbf{d}\boldsymbol{l}| = \sqrt{(D_1 x^{D_1 - 1})^2 (\mathrm{d}x)^2 + (D_2 y^{D_2 - 1})^2 (\mathrm{d}y)^2 + (D_3 z^{D_3 - 1})^2 (\mathrm{d}z)^2} \tag{29}$$

where $\boldsymbol{m}$ is the vector with $|\boldsymbol{m}| = 1$.

The arc length $\ell = \int_0^{\ell} \mathrm{d}\ell$ from $t = a$ to $t = b$ is given:

$$\ell = \int_a^b \sqrt{\left(D_1 x^{D_1 - 1}\right)^2 \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(D_2 y^{D_2 - 1}\right)^2 \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(D_3 z^{D_3 - 1}\right)^2 \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\, \mathrm{d}t \tag{30}$$

The line power-law integral of the vector field

The line power-law integral of the vector field $\boldsymbol{T} = \boldsymbol{T}_{\omega}(x, y, z)$ along the curve $L(x, y, z) = L(x^{D_1}, y^{D_2}, z^{D_3})$, denoted by $\mathfrak{B}$, is defined:

$$\mathfrak{B} = \int_{L(x,y,z)} \boldsymbol{T}_{\omega}(x, y, z) \bullet \mathbf{d}\boldsymbol{l} \tag{31}$$

where $\boldsymbol{T} = \boldsymbol{T}_{\omega}(x, y, z) = \boldsymbol{T}(x^{D_1}, y^{D_2}, z^{D_3}) = T_x i + T_y j + T_z k$, and the element of the vector line is:

$$\mathbf{d}\boldsymbol{l} = i(D_1 x^{D_1 - 1})\mathrm{d}x + j(D_2 y^{D_2 - 1})\mathrm{d}y + k(D_3 z^{D_3 - 1})\mathrm{d}z = i\mathrm{d}\ell(x) + j\mathrm{d}\zeta(y) + k\mathrm{d}\xi(z) \tag{32}$$

With use of eq. (61), we get:

$$\int_{L(x,y,z)} \boldsymbol{T} \bullet \mathbf{d}\boldsymbol{l} = \int_{L(x,y,z)} \boldsymbol{T}_{\omega}(x, y, z) \bullet \mathbf{d}\boldsymbol{l} = \int_{L(t)} \boldsymbol{T} \bullet \frac{\mathbf{d}\boldsymbol{l}}{\mathrm{d}t} \mathrm{d}t \tag{33}$$

where $\mathbf{d}\boldsymbol{l}/\mathrm{d}t = i(D_1 x^{D_1 - 1})\mathrm{d}x/\mathrm{d}t + j(D_2 y^{D_2 - 1})\mathrm{d}y/\mathrm{d}t + k(D_3 z^{D_3 - 1})\mathrm{d}z/\mathrm{d}t$.

Therefore, by using eqs. (32), (31) can be presented as follows:

$$\int_{L(x,y,z)} \boldsymbol{T} \bullet \mathbf{d}\boldsymbol{l} = \int_{L(x,y,z)} T_x (D_1 x^{D_1 - 1})\mathrm{d}x + T_y (D_2 y^{D_2 - 1})\mathrm{d}y + T_z (D_3 z^{D_3 - 1})\mathrm{d}z \tag{34}$$

The vector field $\boldsymbol{T} = \boldsymbol{T}_{\omega}(x, y, z)$ in $L(x, y, z) = L(x^{D_1}, y^{D_2}, z^{D_3})$ is said to be conservative if:

$$\oint_{L(x,y,z)} \boldsymbol{T} \bullet \mathbf{d}\boldsymbol{l} = 0 \tag{35}$$

The double power-law integral of the scalar field

The double power-law integral of the scalar field $\Theta_{\omega}(x, y) = \Theta(x^{D_1}, y^{D_2})$ on the region $S(x, y) = S(x^{D_1}, y^{D_2})$, denoted by $A(\Theta)$, is defined:

$$A(\Theta) = \iint_{S(x,y)} \Theta_{\omega}(x, y)\, \mathrm{d}S \tag{36}$$

where $\mathrm{d}S = (D_1 x^{D_1 - 1})(D_2 y^{D_2 - 1})\mathrm{d}x\,\mathrm{d}y$.

When $\ell(x)=x^{D_1}$ and $\zeta(y)=y^{D_2}$, we have:

$$\mathrm{d}S=(D_1x^{D_1-1})(D_2y^{D_2-1})\,\mathrm{d}x\,\mathrm{d}y=\mathrm{d}\ell(x)\,\mathrm{d}\zeta(y) \tag{37}$$

It is shown from eqs. (36) and (37) that:

$$\iint\limits_{S(x,y)}\Theta_\omega(x,y)\mathrm{d}S=\int_c^d\left[\int_a^b\Theta_\omega(x,y)(D_1x^{D_1-1})\,\mathrm{d}x\right](D_2y^{D_2-1})\,\mathrm{d}y$$
$$=\int_a^b\left[\int_c^d\Theta_\omega(x,y)(D_2y^{D_2-1})\,\mathrm{d}y\right](D_1x^{D_1-1})\,\mathrm{d}x \tag{38}$$

where $x\in[a,b]$ and $y\in[c,d]$.

The volume power-law integral of the scalar field

The volume power-law integral of the scalar field $\Theta_\omega(x,y,z)=\Theta(x^{D_1},y^{D_2},z^{D_3})$ is defined:

$$V(\Theta)=\iiint\limits_{\Omega(x,y,z)}\Theta_\omega(x,y,z)\mathrm{d}V \tag{39}$$

with

$$\mathrm{d}V=(D_1x^{D_1-1})(D_2y^{D_2-1})(D_3z^{D_3-1})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\mathrm{d}\ell(x)\,\mathrm{d}\zeta(y)\,\mathrm{d}\xi(z)$$

where $\ell(x)=x^{D_1}$, $\zeta(y)=y^{D_2}$, and $\xi(z)=z^{D_3}$.

Thus, we have:

$$\iiint\limits_{\Omega(x,y,z)}\Theta_\omega(x,y,z)\,\mathrm{d}V=\int_\alpha^\beta(D_3z^{D_3-1})\,\mathrm{d}z\int_c^d(D_2y^{D_2-1})\,\mathrm{d}y\int_a^b\Theta_\omega(x,y,z)(D_1x^{D_1-1})\,\mathrm{d}x$$
$$=\int_c^d(D_1x^{D_1-1})\,\mathrm{d}x\int_a^b(D_3z^{D_3-1})\,\mathrm{d}z\int_\alpha^\beta\Theta_\omega(x,y,z)(D_2y^{D_2-1})\,\mathrm{d}y$$
$$=\int_c^d(D_2y^{D_2-1})\,\mathrm{d}y\int_a^b(D_1x^{D_1-1})\,\mathrm{d}x\int_\alpha^\beta\Theta_\omega(x,y,z)(D_3z^{D_3-1})\mathrm{d}z \tag{40}$$

where $x\in[a,b]$, $y\in[c,d]$, and $z\in[\alpha,\beta]$.

The surface power-law integral of the vector field

The surface power-law integral of the vector field $\boldsymbol{\psi}_\omega(x,y,z)=\boldsymbol{\psi}(x^{D_1},y^{D_2},z^{D_3})$ is defined:

$$\iint\limits_{\boldsymbol{S}(x,y,z)}\boldsymbol{\psi}_\omega(x,y,z)\bullet\mathbf{d}\boldsymbol{S}=\iint\limits_{\boldsymbol{S}(x,y,z)}\boldsymbol{\psi}_\omega(x,y,z)\bullet\boldsymbol{n}\mathrm{d}S \tag{41}$$

where $\boldsymbol{n}=\mathbf{d}\boldsymbol{S}/\mathrm{d}S$ is the unit normal vector to the surface $\boldsymbol{S}(x,y,z)=\boldsymbol{S}(x^{D_1},y^{D_2},z^{D_3})$.

Let us consider that $\boldsymbol{n}=\mathbf{d}\boldsymbol{S}/|\mathbf{d}\boldsymbol{S}|=\mathbf{d}\boldsymbol{S}/\mathrm{d}S$, $\mathrm{d}S=|\mathbf{d}\boldsymbol{S}|$, and

$$\mathbf{d}\boldsymbol{S}=\mathrm{d}\zeta(y)\,\mathrm{d}\xi(z)i+\mathrm{d}\ell(x)\,\mathrm{d}\xi(z)\,j+\mathrm{d}\ell(x)\,\mathrm{d}\zeta(y)k$$
$$=i(D_2y^{D_2-1})(D_3z^{D_3-1})\,\mathrm{d}y\,\mathrm{d}z+j(D_1x^{D_1-1})(D_3z^{D_3-1})\,\mathrm{d}x\,\mathrm{d}z+k(D_1x^{D_1-1})(D_2y^{D_2-1})\,\mathrm{d}x\,\mathrm{d}y \tag{42}$$

where $\mathrm{d}\zeta(y)\mathrm{d}\xi(z)=(D_2y^{D_2-1})(D_3z^{D_3-1})\mathrm{d}y\mathrm{d}z$, $\mathrm{d}\ell(x)\mathrm{d}\xi(z)=(D_1x^{D_1-1})(D_3z^{D_3-1})\mathrm{d}x\mathrm{d}z$, and $\mathrm{d}\ell(x)\mathrm{d}\zeta(y)=(D_1x^{D_1-1})(D_2y^{D_2-1})\mathrm{d}x\mathrm{d}y$.

Thus, we may have from eqs. (41) and (42) that:

$$\iint\limits_{\boldsymbol{S}(x,y,z)} \boldsymbol{\psi}_\omega(x,y,z)\bullet\mathbf{d}\boldsymbol{S}=\iint\limits_{S(x,y,z)} \psi_x\mathrm{d}\zeta(y)\mathrm{d}\xi(z)+\psi_y\mathrm{d}\ell(x)\mathrm{d}\xi(z)+\psi_z\mathrm{d}\ell(x)\mathrm{d}\zeta(y) \tag{43}$$

where $\boldsymbol{\psi}=\boldsymbol{\psi}_\omega(x,y,z)=\boldsymbol{\psi}(x^{D_1},y^{D_2},z^{D_3})=i\psi_x+j\psi_y+k\psi_z$.

The flux of the vector field $\boldsymbol{\psi}=\boldsymbol{\psi}_\omega(x,y,z)$ across the surface $\mathbf{d}\boldsymbol{S}$, denoted by $\boldsymbol{\Phi}$, is defined:

$$\boldsymbol{\Phi}=\oint\limits_{\boldsymbol{S}(x,y,z)} \boldsymbol{\psi}\bullet\mathbf{d}\boldsymbol{S} \tag{44}$$

The power-law divergence of the vector field

The power-law divergence of the vector field $\boldsymbol{\psi}$ is defined:

$$\nabla^{(D_1,D_2,D_3)}\bullet\boldsymbol{\psi}=\lim_{\Delta V_m\to 0}\frac{1}{\Delta V_m}\oiint\limits_{\Delta\boldsymbol{S}_m(x,y,z)} \boldsymbol{\psi}\bullet\mathbf{d}\boldsymbol{S} \tag{45}$$

where the volume V is divided into a large number of small subvolumes ΔV_m with surfaces $\Delta\boldsymbol{S}_m(x,y,z)$, $\boldsymbol{\psi}$ is a continuously differentiable vector field, and $\mathbf{d}\boldsymbol{S}$ is an element of the surface $\boldsymbol{S}(x,y,z)$ bounding the solid $\Omega(x,y,z)$.

With use of (14), (45) can be written:

$$\nabla^{(D_1,D_2,D_3)}\bullet\boldsymbol{\psi}=i(D_1x^{D_1-1})^{PL}\partial_x^{(1)}\psi_x+j(D_2y^{D_2-1})^{PL}\partial_y^{(1)}\psi_y+k(D_3z^{D_3-1})^{PL}\partial_z^{(1)}\psi_z \tag{46}$$

where $\boldsymbol{\psi}=\boldsymbol{\psi}_\omega(x,y,z)=\boldsymbol{\psi}(x^{D_1},y^{D_2},z^{D_3})=i\psi_x+j\psi_y+k\psi_z$.

The power-law curl of the vector field

The *power-law* curl of the vector field $\boldsymbol{T}$ is defined:

$$\nabla^{(D_1,D_2,D_3)}\times\boldsymbol{T}=\lim_{\Delta S_m(x,y,z)\to 0}\frac{1}{\Delta S_m(x,y,z)}\oint\limits_{\Delta L_m(x,y,z)} \boldsymbol{T}_\omega(x,y,z)\bullet\mathbf{d}\boldsymbol{l} \tag{47}$$

where $\boldsymbol{T}=\boldsymbol{T}_\omega(x,y,z)=\boldsymbol{T}(x^{D_1},y^{D_2},z^{D_3})=T_xi+T_yj+T_zk$ be a continuously differentiable vector field, $\mathbf{d}\boldsymbol{l}$ – the element of the vector line, $\Delta S_m(x,y,z)$ is a small surface element perpendicular to $\boldsymbol{n}$, $\Delta L_m(x,y,z)$ – the closed curve of the boundary of $\Delta S_m(x,y,z)$, and $\boldsymbol{n}$ are oriented in a positive sense.

Similarly, eq. (47) can be represented:

$$\nabla^{(D_1,D_2,D_3)}\times\boldsymbol{T}=\begin{pmatrix} i & j & k \\ (D_1x^{D_1-1})^{PL}\partial_x^{(1)} & (D_2y^{D_2-1})^{PL}\partial_y^{(1)} & (D_3z^{D_3-1})^{PL}\partial_z^{(1)} \\ T_x & T_y & T_z \end{pmatrix} \tag{48}$$

where $\boldsymbol{T}=\boldsymbol{T}_\omega(x,y,z)=\boldsymbol{T}(x^{D_1},y^{D_2},z^{D_3})=T_xi+T_yj+T_zk$.

The Gauss-like theorem

From the definition of eq. (45), we present the Gauss-like theorem as follows.

Let us consider that:

$$\oiint\limits_{\boldsymbol{S}(x,y,z)} \boldsymbol{\psi}\bullet\boldsymbol{n}\mathrm{d}S = \oiint\limits_{\boldsymbol{S}(x,y,z)} \psi_x \mathrm{d}\ell(y)\mathrm{d}\xi(z) + \psi_y \mathrm{d}\ell(x)\mathrm{d}\xi(z) + \psi_z \mathrm{d}\ell(x)\mathrm{d}\zeta(y)$$

The Gauss-like theorem states that:

$$\iiint\limits_{\Omega(x,y,z)} \nabla^{(D_1,D_2,D_3)}\bullet\boldsymbol{\psi}\mathrm{d}V = \oiint\limits_{\boldsymbol{S}(x,y,z)} \boldsymbol{\psi}\bullet\boldsymbol{n}\,\mathrm{d}S \tag{49}$$

where $\boldsymbol{\psi}$ is a continuously differentiable vector field, $\mathrm{d}V$ denotes an element of volume $\Omega(x,y,z)$, $\boldsymbol{n}$ is the unit outward normal to $\boldsymbol{S}(x,y,z)$, and $\mathrm{d}S$ is an element of the surface area of the surface $\boldsymbol{S}(x,y,z)$ bounding the solid $\Omega(x,y,z)$.

Taking $\mathbf{d}\boldsymbol{S} = \boldsymbol{n}\,\mathrm{d}S$, we have from eq. (49) that:

$$\iiint\limits_{\Omega(x,y,z)} \nabla^{(D_1,D_2,D_3)}\bullet\boldsymbol{\psi}\mathrm{d}V = \oiint\limits_{\boldsymbol{S}(x,y,z)} \boldsymbol{\psi}\bullet\mathbf{d}\boldsymbol{S} \quad \text{and} \quad \iiint\limits_{\Omega(x,y,z)} \nabla^{(D)}\bullet\boldsymbol{\psi}\mathrm{d}V = \oiint\limits_{\boldsymbol{S}(x,y,z)} \boldsymbol{\psi}\bullet\mathbf{d}\boldsymbol{S} \tag{50a,b}$$

It is illustrated that eq. (50b) is the case of eq. (50a) when $D_1 = D_2 = D_3 = D$.

From the definition of eq. (48), we present the Stokes-like theorem as follows.

The Stokes-like theorem

Let us consider that:

$$\oint\limits_{L(x,y,z)} \boldsymbol{\psi}\bullet\mathbf{d}\boldsymbol{l} = \oint\limits_{L(x,y,z)} T_x(D_1 x^{D_1-1})\mathrm{d}x + T_y(D_2 y^{D_2-1})\mathrm{d}y + T_z(D_3 z^{D_3-1})\mathrm{d}z$$

The Stokes-like theorem states that:

$$\iint\limits_{S(x,y,z)} \left[\nabla^{(D_1,D_2,D_3)} \times \boldsymbol{\psi}\right]\bullet\boldsymbol{n}\,\mathrm{d}S = \oint\limits_{L(x,y,z)} \boldsymbol{\psi}\bullet\mathbf{d}\boldsymbol{l} \tag{51}$$

where $\boldsymbol{\psi}$ is a constant vector field, $S(x,y,z)$ denotes an open, two sided curve surface, $L(x,y,z)$ represents the closed contour bounding *S*, and $\mathbf{d}\boldsymbol{l}$ denotes the element of the vector line.

Taking $\mathbf{d}\boldsymbol{S} = \boldsymbol{n}\,\mathrm{d}S$, we show from eq. (51) that:

$$\iint\limits_{S(x,y,z)} \left[\nabla^{(D_1,D_2,D_3)} \times \boldsymbol{\psi}\right]\bullet\boldsymbol{n}\,\mathrm{d}S = \oint\limits_{L(x,y,z)} \boldsymbol{\psi}\bullet\mathbf{d}\boldsymbol{l} \text{ and } \iint\limits_{S(x,y,z)} \left[\nabla^{(D)} \times \boldsymbol{\psi}\right]\bullet\boldsymbol{n}\,\mathrm{d}S = \oint\limits_{L(x,y,z)} \boldsymbol{\psi}\bullet\mathbf{d}\boldsymbol{l} \tag{52a,b}$$

It is shown that eq. (52b) is the case of eq. (52a) when $D_1 = D_2 = D_3 = D$.

The Green-like theorem

The Green-like theorem states:

$$\oint\limits_{L(x,y)} T_x(D_1 x^{D_1-1})\mathrm{d}x + T_y(D_2 y^{D_2-1})\mathrm{d}y =$$
$$= \iint\limits_{S(x,y)} \left\{[D_1 x^{D_1-1}]^{PL}\partial_x^{(1)} T_y - [D_2 y^{D_2-1}]^{PL}\partial_y^{(1)} T_x\right\}\mathrm{d}\ell(x)\mathrm{d}\zeta(y) \tag{53}$$

where $S(x,y)$ is the domain bounded by the contour $L(x,y)$, and $\boldsymbol{T} = T_x i + T_y j$.

When $D_1 = D_2 = D_3 = D$, we have that:

$$\oint\limits_{L(x,y,z)} T_x x^{D-1}\mathrm{d}x + T_y y^{D-1}\mathrm{d}y = D \iint\limits_{S(x,y)} \left[x^{D-1\ PL}\partial_x^{(1)} T_y - y^{D-1\ PL}\partial_y^{(1)} T_x\right] x^{D-1} y^{D-1}\,\mathrm{d}x\,\mathrm{d}y \tag{54}$$

The Green-like identities

Taking $\Phi = \Theta\nabla^{(D_1,D_2,D_3)}\Phi$, we have that:

$$\nabla^{(D_1,D_2,D_3)} \bullet \left[\Theta\nabla^{(D_1,D_2,D_3)}\Phi\right] = \Theta\nabla^{(2D_1,2D_2,2D_3)}\Phi + \nabla^{(D_1,D_2,D_3)}\Phi \bullet \nabla^{(D_1,D_2,D_3)}\Theta \tag{55}$$

and

$$\nabla^{(D_1,D_2,D_3)} \bullet \left[\Phi\nabla^{(D_1,D_2,D_3)}\Theta\right] = \Phi\nabla^{(2D_1,2D_2,2D_3)}\Theta + \nabla^{(D_1,D_2,D_3)}\Phi \bullet \nabla^{(D_1,D_2,D_3)}\Theta \tag{56}$$

where $\Phi = \Phi_\omega(x,y,z) = \Phi(x^{D_1}, y^{D_2}, z^{D_3})$ and $\Theta = \Theta_\omega(x,y,z) = \Theta(x^{D_1}, y^{D_2}, z^{D_3})$ are the scalar fields.

With the use of eq. (49), the Green-like identity of first type can be given:

$$\iiint\limits_{\Omega(x,y,z)} \nabla^{(D_1,D_2,D_3)} \bullet \left[\Theta\nabla^{(2D_1,2D_2,2D_3)}\Phi + \nabla^{(D_1,D_2,D_3)}\Phi \bullet \nabla^{(D_1,D_2,D_3)}\Theta\right] \mathrm{d}V = = \oiint\limits_{\boldsymbol{S}(x,y,z)} \Theta\partial_n^{(D_1,D_2,D_3)}\Phi \mathrm{d}S \tag{57}$$

In a similar way, we have that:

$$\iiint\limits_{\Omega(x,y,z)} \nabla^{(D_1,D_2,D_3)} \bullet \left[\Phi\nabla^{(2D_1,2D_2,2D_3)}\Theta + \nabla^{(D_1,D_2,D_3)}\Phi \bullet \nabla^{(D_1,D_2,D_3)}\Theta\right] \mathrm{d}V = = \oiint\limits_{\boldsymbol{S}(x,y,z)} \Phi\partial_n^{(D_1,D_2,D_3)}\Theta \mathrm{d}S \tag{58}$$

which reduces to the Green-like identity of second type, given:

$$\iiint\limits_{\Omega(x,y,z)} \nabla^{(D_1,D_2,D_3)} \bullet \left[\Theta\nabla^{(2D_1,2D_2,2D_3)}\Phi - \Phi\nabla^{(2D_1,2D_2,2D_3)}\Theta\right] \mathrm{d}V = = \oiint\limits_{\boldsymbol{S}(x,y,z)} \left[\Theta\partial_n^{(D_1,D_2,D_3)}\Phi - \Phi\partial_n^{(D_1,D_2,D_3)}\Theta\right] \mathrm{d}S \tag{59}$$

Taking $D_1 = D_2 = D_3 = D$, we have from eqs. (57) and (59) that:

$$\iiint\limits_{\Omega(x,y,z)} \nabla^{(D)} \bullet \left[\Theta\nabla^{(2D)}\Phi + \nabla^{(D)}\Phi \bullet \nabla^{(D)}\Theta\right] \mathrm{d}V = \oiint\limits_{\boldsymbol{S}(x,y,z)} \Theta\partial_n^{(D)}\Phi \,\mathrm{d}S \tag{60}$$

and

$$\iiint\limits_{\Omega(x,y,z)} \nabla^{(D)} \bullet \left[\Theta\nabla^{(2D)}\Phi - \Phi\nabla^{(2D)}\Theta\right] \mathrm{d}V = \oiint\limits_{\boldsymbol{S}(x,y,z)} \left[\Theta\partial_n^{(D)}\Phi - \Phi\partial_n^{(D)}\Theta\right] \mathrm{d}S \tag{61}$$

Taking $D_1 = D_2 = D_3 = D = 1$, the Gauss-like, Stokes-like and Green-like theorems and Green-like identities become the Gauss [22], Stokes [23], Green theorems and Green identities [24], respectively.

Applied to describe the power-law fluid flow

Let us consider the power-law co-ordinate system, given as $(t^{D_0}, x^{D_1}, y^{D_2}, z^{D_3}) = t^D + +ix^{D_1} + jy^{D_2} + kz^{D_3}$, where D_0, D_1, D_2, and D_3 are the fractal dimensions, and i, j, and k are the unit vector in the Cartesian co-ordinate system.

The material power-law derivative of the power-law fluid field.

Let $\Phi = \Phi_\omega(t,x,y,z) = \Phi(t^D, x^{D_1}, y^{D_2}, z^{D_3})$ be the power-law type fluid field.

The total power-law differential of the power-law type scalar field is given:

$$\mathrm{d}\Phi = \left[D_1 x^{D_1 - 1\ PL}\partial_x^{(1)}\Phi\right]\mathrm{d}x + \left[D_2 y^{D_2 - 1\ PL}\partial_y^{(1)}\Phi\right]\mathrm{d}y + \\ + \left[D_3 z^{D_3 - 1\ PL}\partial_z^{(1)}\Phi\right]\mathrm{d}z + \left[D_0 t^{D_0 - 1\ PL}\partial_t^{(1)}\Phi\right]\mathrm{d}t \tag{62}$$

which leads to:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \left[D_1 x^{D_1 - 1\ PL}\partial_x^{(1)}\Phi\right]\frac{\partial x}{\partial t} + \left[D_2 y^{D_2 - 1\ PL}\partial_y^{(1)}\Phi\right]\frac{\partial y}{\partial t} + \\ + \left[D_3 z^{D_3 - 1\ PL}\partial_z^{(1)}\Phi\right]\frac{\partial z}{\partial t} + D_0 t^{D_0 - 1\ PL}\partial_t^{(1)}\Phi \tag{63}$$

The material power-law derivative

The material power-law derivative of the power-law fluid density ϕ is defined:

$$\frac{D\phi}{Dt} = D_0 t^{D_0 - 1\ PL}\partial_t^{(1)}\phi + \boldsymbol{\upsilon}\cdot\nabla^{(D_1, D_2, D_3)}\phi \tag{64}$$

where $\boldsymbol{\upsilon} = (\partial x/\partial t, \partial y/\partial t, \partial z/\partial t) = i\upsilon_x + j\upsilon_y + k\upsilon_z$ is the velocity vector.

For $D_0 = 1$ the material power-law-space derivative of the power-law fluid density ϕ, reads:

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \boldsymbol{\upsilon}\cdot\nabla^{(D_1, D_2, D_3)}\phi \tag{65}$$

which, by using $D_1 = D_2 = D_3 = D$, leads to:

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \boldsymbol{\upsilon}\cdot\nabla^{(D)}\phi \tag{66}$$

For $D_1 = D_2 = D_3 = 1$ the material power-law-time derivative of the power-law fluid density, denoted as, can be given:

$$\frac{D\phi}{Dt} = D_0 t^{D_0 - 1\ PL}\partial_t^{(1)}\phi + \boldsymbol{\upsilon}\cdot\nabla\phi \tag{67}$$

It is not difficult to show that the Stokes material derivative, proposed by Stokes in 1845 to consider the velocity [25] and further developed in 1851 [26], is one of the special cases of eqs. (64)-(67) when $D_1 = D_2 = D_3 = D_0 = 1$, and it illustrates the relationship among the change in Lagrangian co-ordinate $\boldsymbol{X} = (t, X, Y, Z)$, Eulerian co-ordinate $\boldsymbol{x} = (t, x, y, z)$, and Eulerian-like co-ordinate $\tilde{\boldsymbol{x}} = (t^{D_0}, x^{D_1}, y^{D_2}, z^{D_3})$ for any fluid field.

The transport theorem for the power-law fluid

From eq. (64) we have that the transport theorem for the power-law fluid, *e. g.*:

$$\frac{D}{Dt}\iiint\limits_{\Omega(t)}\Xi\,\mathrm{d}V = \iiint\limits_{\Omega(t)}\left[D_0 t^{D_0 - 1\ PL}\partial_t^{(1)}\Xi + \boldsymbol{\upsilon}\cdot\nabla^{(D_1, D_2, D_3)}\Xi\right]\mathrm{d}V \tag{68}$$

which, by using eq. (52a), leads to:

$$\frac{D}{Dt}\iiint\limits_{\Omega(t)}\Xi\,\mathrm{d}V = \iiint\limits_{\Omega(t)} D_0 t^{D_0 - 1\ PL}\partial_t^{(1)}\Xi\,\mathrm{d}V + \oiint\limits_{\boldsymbol{S}(t)}\Xi\boldsymbol{\upsilon}\cdot\mathbf{d}\boldsymbol{S} \tag{69}$$

since

$$\iiint_{\Omega(t)} \boldsymbol{\upsilon}\cdot\nabla^{(D_1,D_2,D_3)}\Xi\,\mathrm{d}V = \oiint_{\boldsymbol{S}(t)} \Xi(\boldsymbol{\upsilon}\bullet\boldsymbol{n})\mathrm{d}S = \oiint_{\boldsymbol{S}(t)} \Xi\boldsymbol{\upsilon}\bullet\mathbf{d}\boldsymbol{S} \tag{70}$$

where $\boldsymbol{S}(t)$ is the surface of $\Omega(t)$, $\boldsymbol{n}$ is the unit normal to the surface, $\boldsymbol{\upsilon}$ is the velocity vector, and $\Xi = \Xi_\omega(t,x,y,z) = \Xi(t^{D_0}, x^{D_1}, y^{D_2}, z^{D_3})$ is the power-law fluid.

Here, the Reynolds transport theorem, proposed by in 1903 Reynolds [27], is the special case of eqs. (68) and (69) when $D_1 = D_2 = D_3 = D_0 = 1$.

Let us define the mass of the power-law fluid is defined:

$$\iiint_{\Omega(t)} \mathfrak{J}\mathrm{d}V = \mathfrak{H} \tag{71}$$

where $\mathfrak{J} = \mathfrak{J}_\omega(t,x,y,z) = \mathfrak{J}(t^D, x^{D_1}, y^{D_2}, z^{D_3})$ and $\mathfrak{H} = \mathfrak{H}_\omega(t,x,y,z) = \mathfrak{H}(t^D, x^{D_1}, y^{D_2}, z^{D_3})$.

The conservation of the mass of the power-law fluid is given:

$$D_0 t^{D_0-1}\,{}^{PL}\partial_t^{(1)}\mathfrak{J} + \boldsymbol{\upsilon}\cdot\nabla^{(D_1,D_2,D_3)}\mathfrak{J} = 0 \quad\text{and}\quad D_0 t^{D_0-1}\,{}^{PL}\partial_t^{(1)}\mathfrak{J} + \nabla^{(D_1,D_2,D_3)}\cdot(\mathfrak{J}\boldsymbol{\upsilon}) = 0 \tag{72a,b}$$

since $\boldsymbol{\upsilon}$ is the velocity vector (a constant vector), and there is from eqs. (68) and (71):

$$\frac{D}{Dt}\iiint_{\Omega(t)} \mathfrak{J}\mathrm{d}V = \iiint_{\Omega(t)}\left[D_0 t^{D_0-1}\,{}^{PL}\partial_t^{(1)}\mathfrak{J} + \boldsymbol{\upsilon}\cdot\nabla^{(D_1,D_2,D_3)}\mathfrak{J}\right]\mathrm{d}V = 0 \tag{73}$$

Here, the conservation of the mass of the fluid without power-law, proposed by Euler in 1757 [28], is the special case of eqs. (72b) and (73), and proposed by Lagrange in 1781 [29], is the special case of eq. (72b), where $D_1 = D_2 = D_3 = D_0 = 1$.

Let us consider the velocity gradient tensor for the power-law fluid, defined:

$$\nabla^{(D_1,D_2,D_3)}\bullet\boldsymbol{\upsilon} = \frac{1}{2}(\varsigma+\tau) + \frac{1}{2}(\varsigma-\tau) = \boldsymbol{h} + \frac{1}{2}(\varsigma-\tau) \quad\text{and}\quad \varsigma = \nabla^{(D_1,D_2,D_3)}\cdot\boldsymbol{\upsilon} = 0 \tag{74a,b}$$

where the strain tensor for the power-law fluid is defined as $\boldsymbol{h} = (\varsigma+\tau)/2$ with velocity gradient $\varsigma = \nabla^{(D_1,D_2,D_3)}\cdot\boldsymbol{\upsilon}$ and $\tau = \boldsymbol{\upsilon}\cdot\nabla^{(D_1,D_2,D_3)}$.

The stress tensor for the power-law fluid is defined:

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\beta\boldsymbol{h} \tag{75}$$

where β are the shear moduli of viscosity, and $\boldsymbol{I}$ is the unit tensor.

It is noted that the strain tensor is the special case, proposed by Cauchy in [30, 31], and the Stokes decompose term [25] is the special case of eq. (74), the stress tensor, proposed by Stokes in 1845 [25], is the special case of eq. (75) where $D_1 = D_2 = D_3 = D_0 = 1$.

The conservation of the momentums for the power-law fluid

The conservation of the linear and angular momentums for the power-law fluid is:

$$\frac{D}{Dt}\iiint_{\Omega(t)} \mathfrak{J}\boldsymbol{\upsilon}\mathrm{d}V = \iiint_{\Omega(t)} \boldsymbol{b}\mathrm{d}V + \oiint_{\boldsymbol{S}(t)} \boldsymbol{T}\bullet\mathbf{d}\boldsymbol{S} \tag{76}$$

where $\boldsymbol{b}$ is the specific body force.

Thus, we have:

$$D_0 t^{D_0-1}\,{}^{PL}\partial_t^{(1)}(\mathfrak{J}\boldsymbol{\upsilon}) + \boldsymbol{\upsilon}\cdot\nabla^{(D_1,D_2,D_3)}(\mathfrak{J}\boldsymbol{\upsilon}) = \nabla^{(D_1,D_2,D_3)}\bullet\boldsymbol{T} + \boldsymbol{b} \tag{77}$$

since

$$\iiint_{\Omega(t)}\left[D_0 t^{D_0-1\ PL}\partial_t^{(1)}(\Im\boldsymbol{\upsilon})+\boldsymbol{\upsilon}\cdot\nabla^{(D_1,D_2,D_3)}(\Im\boldsymbol{\upsilon})-\boldsymbol{b}-\nabla^{(D_1,D_2,D_3)}\bullet\boldsymbol{T}\right]\mathrm{d}V=0 \tag{78}$$

where

$$\frac{D}{Dt}\iiint_{\Omega(t)}\Im\boldsymbol{\upsilon}\mathrm{d}V=\iiint_{\Omega(t)}\left[D_0 t^{D_0-1\ PL}\partial_t^{(1)}\Im(\Im\boldsymbol{\upsilon})+\boldsymbol{\upsilon}\cdot\nabla^{(D_1,D_2,D_3)}(\Im\boldsymbol{\upsilon})\right]\mathrm{d}V \tag{79}$$

and

$$\oiint_{\mathbf{S}(t)}\boldsymbol{T}\bullet\mathbf{d}\boldsymbol{S}=\iiint_{\Omega(t)}\nabla^{(D_1,D_2,D_3)}\bullet\boldsymbol{T}\mathrm{d}V \tag{80}$$

From eqs. (74a), (74b) and (77) we have:

$$\nabla^{(D_1,D_2,D_3)}\bullet\boldsymbol{T}=-\nabla^{(D_1,D_2,D_3)}p+\beta\nabla^{(2D_1,2D_2,2D_3)}\boldsymbol{\upsilon} \tag{81}$$

such that

$$\Im\left[D_0 t^{D_0-1\ PL}\partial_t^{(1)}\boldsymbol{\upsilon}+\boldsymbol{\upsilon}\cdot\nabla^{(D_1,D_2,D_3)}\boldsymbol{\upsilon}\right]=-\nabla^{(D_1,D_2,D_3)}p+\beta\nabla^{(2D_1,2D_2,2D_3)}\boldsymbol{\upsilon}+\boldsymbol{b} \tag{82}$$

From eqs. (74a) and (82) we have for $\beta=0$:

$$\Im\left[D_0 t^{D_0-1\ PL}\partial_t^{(1)}\boldsymbol{\upsilon}+\boldsymbol{\upsilon}\cdot\nabla^{(D_1,D_2,D_3)}\boldsymbol{\upsilon}\right]=-\nabla^{(D_1,D_2,D_3)}p+\boldsymbol{b}\quad\text{and}\quad\nabla^{(D_1,D_2,D_3)}\cdot\boldsymbol{\upsilon}=0 \tag{83a,b}$$

Here, the Navier-Stokes equations for the fluid, proposed by Navier in 1822 [32] and by Stokes in 1845 [25] are the special cases of eqs. (74b) and (77), and the Euler equations for the fluid, proposed by Euler in 1757 [28], are the special cases of eqs. (83a) and (83b), where $D_1=D_2=D_3=D_0=1$.

Similarly, from eq. (67) we have that:

$$\frac{D\boldsymbol{\upsilon}}{Dt}=D_0 t^{D_0-1\ PL}\partial_t^{(1)}\boldsymbol{\upsilon}+\boldsymbol{\upsilon}\cdot\nabla\boldsymbol{\upsilon} \tag{84}$$

Here, the Stokes formula for the fluid, proposed by Stokes in 1845 [25], is the special case of eq. (84) for $D_1=D_2=D_3=D_0=1$.

Conclusion

In the present work, we have proposed the theory of the vector power-law calculus based on the Leibniz, Stieltjes, and Riemann tasks. The Navier-Stokes-like and Euler-like equations for the power-law fluid were presented based on the conservations of the mass and angular momentums for the power-law fluid. The proposed results are proposed as an advanced mathematical tool for decryptions for the power-law physical phenomenon.

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Nomenclature

$\boldsymbol{b}$ – specific body force, [Nm^{-3}]
t – time, [s]
x, y, z – co-ordinates, [m]

Greek symbol

$\boldsymbol{\upsilon}$ – velocity vector, [ms^{-1}]

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