APPROXIMATE ANALYTIC SOLUTION FOR MULTI-DIMENSIONAL FRACTIONAL WAVE-LIKE EQUATION

by

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Original scientific paper
https://doi.org/10.2298/TSCI2004645S

The fractional power series method is used to solve 2- and 3-D fractional wave-like models with variable coefficients. The fractional derivatives are described in the Caputo sense. Two examples are considered to show the effectiveness and convenience of the method.

Key words: *fractional power series, fractional series expansion method, fractal calculus, fractal derivative*

Introduction

Many phenomena in engineering, physics, chemistry, and other sciences can be described extremely successfully by models using fractional calculus [1-6]. In this work, we will consider the fractional wave-like equations which model anomalous diffusive and sub-diffusive systems [7-10]:

$$\begin{gathered}\mathrm{D}_t^{\alpha} u = f(x,y,z)\mathrm{D}_x^{\beta_1} u + g(x,y,z)\mathrm{D}_y^{\beta_2} u + h(x,y,z)\mathrm{D}_z^{\beta_3} u \\ 0<x<a, \quad 0<y<b, \quad 0<z<c, \quad 1<\alpha\le 2, \quad t>0\end{gathered} \tag{1}$$

subject to the Neumann boundary conditions:

$$\begin{aligned} u(0,y,z,t) &= \mu_1(y,z,t), \quad u(a,y,z,t) = \mu_2(y,z,t), \\ u(x,0,z,t) &= \nu_1(y,z,t), \quad u(x,b,z,t) = \nu_2(x,z,t), \\ u(x,y,0,t) &= \gamma_1(x,y,t), \quad u(x,y,c,t) = \gamma_2(x,y,t) \end{aligned} \tag{2}$$

and the initial conditions:

$$u(x,y,z,0) = \phi(x,y,z), \quad u_t(x,y,z,0) = \psi(x,y,z) \tag{3}$$

where α ($1 < \alpha \le 2$) is a parameter describing the Caputo fractional derivative, β_j ($1 < \beta_j \le 2$) (j = 1, 2, 3) are three parameter describing the order of the Caputo fractional space derivative, u_t – the rate of change of temperature at a point over time, $u = u(x, y, z, t)$ – temperature as a function of time and space, $f(x, y, z)$, $g(x, y, z)$, and $h(x, y, z)$ – any functions in x, y, and z.

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In this paper, we will use fractional power series method (FPSM) [11] to solve 2- and 3-D fractional wave-like models with variable coefficients [7-9, 11, 12].

Basic definitions of fractional calculus

In this section, we review some basic definitions and properties of fractional calculus theory which are useful of remainder of this paper. See [12-14] for details.

Definition 1. The fractional derivative of $f(x)$ in Riemann-Liouville sense is defined:

$$\mathrm{D}_*^{\alpha} f(x)=\frac{\mathrm{d}^m}{\mathrm{d}x^m}\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-s)^{m-\alpha-1} f(s)\mathrm{d}s \tag{4}$$

for $m-1<\alpha\leq m$, $m\in N^+$, $x>0$, and $f\in C_{-1}^m$.

The Riemann-Liouville fractional derivative has certain disadvantages when trying to model real world phenomena with fractional differential equations. Therefore, we shall use a modified fractional differential D^{α} proposed by Caputo in his work on the theory of viscoelasticity [1, 4-6, 14].

Definition 2. The fractional derivative of $f(x)$ in Caputo sense is defined:

$$\mathrm{D}^{\alpha} f(x)=\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-s)^{m-\alpha-1} f^{(m)}(\alpha s)\mathrm{d}s \tag{5}$$

for $m-1<\alpha\leq m$, $m\in N^+$, $x>0$, and $f\in C_{-1}^m$.

The following basic properties hold true [11]:

$$\mathrm{D}^{\alpha} J^{\alpha} f(x)=f(x) \tag{6}$$

$$J^{\alpha}\mathrm{D}^{\alpha} f(x)=f(x)-\sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!},\quad x>0 \tag{7}$$

Fractional power series representation

In this section, we will recall important definition and theorem related with the classical power series into the fractional case in the sense of the Caputo definition [1, 4-6,11, 14], the method can be considered as an extension of the Taylor series method [15,16].

Definition 3. A power series representation of the form:

$$\sum_{n=0}^{\infty} c_n (t-t_0)^{n\alpha}=c_0+c_1(t-t_0)^{\alpha}+c_2(t-t_0)^{2\alpha}+\cdots \tag{8}$$

is called a FPS about t_0, where $0\leq m-1<\alpha\leq m$, $m\in N^+$, and $t\geq t_0$ is a variable and c_n – the coefficients of the series.

Theorem 1. Suppose that the FPS $\Sigma_{n=0}^{\infty} c_n t^{n\alpha}$ has radius of convergence $r>0$. If $f(t)$ is a function defined $f(t)=\Sigma_{n=0}^{\infty} c_n t^{na}$ on $0\leq t<r$, then for $m-1<\alpha\leq m$ and $0\leq t<r$, we have:

$$\mathrm{D}^{\alpha} f(t)=\sum_{n=0}^{\infty} c_n \frac{\Gamma(n\alpha+1)}{\Gamma[(n-1)\alpha+1]} t^{(n-1)\alpha} \tag{9}$$

Analysis of the FPSM

In this section, we present the FPSM method for multi-dimensional fractional wave-like models with variable coefficients.

Suppose that the solution of eq. (1) takes the form:

$$u(x,y,z,t)=\sum_{k=0}^{\infty}\omega_k(x,y,z)t^{\alpha k} \tag{10}$$

The components $\omega_k(x, y, z)$ ($k = 1, 2,\dots$) will be determined recursively. Using eq. (3), we can get:

$$\omega_0(x,y,z)=\phi(x,y,z) \tag{11}$$

From *Theorem 1*, one gets:

$$\mathrm{D}_t^{\alpha}\,u(x,y,z,t)=\sum_{k=1}^{\infty}\frac{\omega_k(x,y,z)\Gamma(\alpha k+1)}{\Gamma[\alpha(k-1)+1]}\,t^{\alpha(k-1)} \tag{12}$$

From eq. (10), it is easy to see that:

$$\mathrm{D}_x^{\beta_1}\,u=\mathrm{D}_x^{\beta_1}\,\omega_0(x,y,z)+t^{\alpha}\,\mathrm{D}_x^{\beta_1}\,\omega_1(x,y,z)+t^{2\alpha}\,\mathrm{D}_x^{\beta_1}\,\omega_2(x,y,z)+\cdots \tag{13}$$

$$\mathrm{D}_y^{\beta_2}\,u=\mathrm{D}_y^{\beta_2}\,\omega_0(x,y,z)+t^{\alpha}\,\mathrm{D}_y^{\beta_2}\,\omega_1(x,y,z)+t^{2\alpha}\,\mathrm{D}_y^{\beta_2}\,\omega_2(x,y,z)+\cdots \tag{14}$$

$$\mathrm{D}_z^{\beta_3}\,u=\mathrm{D}_z^{\beta_3}\,\omega_0(x,y,z)+t^{\alpha}\,\mathrm{D}_z^{\beta_3}\,\omega_1(x,y,z)+t^{2\alpha}\,\mathrm{D}_z^{\beta_3}\,\omega_2(x,y,z)+\cdots \tag{15}$$

Substituting eqs. (12)-(15) into eq. (1), we obtain:

$$\sum_{k=1}^{\infty}\omega_k(x,y,z)\frac{\Gamma(k\alpha+1)}{\Gamma[(k-1)\alpha+1]}t^{(k-1)\alpha}=f(x,y,z)\sum_{k=0}^{\infty}t^{k\alpha}\,\mathrm{D}_x^{\beta_1}\,\omega_k(x,y,z)+$$

$$+g(x,y,z)\sum_{k=0}^{\infty}t^{k\alpha}\,\mathrm{D}_y^{\beta_2}\,\omega_k(x,y,z)+h(x,y,z)\sum_{k=0}^{\infty}t^{k\alpha}\,\mathrm{D}_z^{\beta_3}\,\omega_k(x,y,z) \tag{16}$$

Comparing the coefficients of $t^{k\alpha}$ in eq. (16), we get:

$$\omega_k(x,y,z)=\frac{\Gamma[\alpha(k-1)+1]}{\Gamma(\alpha k+1)}(f\,\mathrm{D}_x^{\beta_1}\,\omega_{k-1}+g\,\mathrm{D}_y^{\beta_2}\,\omega_{k-1}+h\,\mathrm{D}_z^{\beta_3}\,\omega_{k-1}),\quad (k=1,2,\cdots) \tag{17}$$

We, therefore, obtain the solution:

$$u(x,y,z,t)=\sum_{k=0}^{\infty}t^{k\alpha}\,\frac{\Gamma[\alpha(k-1)+1]}{\Gamma(\alpha k+1)}(f\,\mathrm{D}_x^{\beta_1}\,\omega_{k-1}+g\,\mathrm{D}_y^{\beta_2}\,\omega_{k-1}+h\,\mathrm{D}_z^{\beta_3}\,\omega_{k-1}) \tag{18}$$

Worked examples

We will use the FPSM method to two different examples about 2- and 3-D wave-like models to illustrate the accuracy and efficiency of the method.

Example 1. Consider the 2-D initial boundary value problems (IBVP):

$$\mathrm{D}_t^\alpha u = \frac{1}{2}\left(y^2 \frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial^2 u}{\partial y^2} \right), \quad 0 < x, \quad y < 1, \quad 1 < \alpha \le 2, \quad t > 0 \tag{19}$$

subject to the Neumann boundary conditions:

$$\begin{aligned} &u_x(0, y, t) = 0, \quad u(1, y, t) = 2\cosh t, \\ &u_y(x, 0, t) = 0, \quad u(x, 1, t) = 2\cosh t \end{aligned} \tag{20}$$

and the initial conditions:

$$u(x, y, 0) = x^2 + y^2, \quad u_t(x, y, 0) = 0 \tag{21}$$

To solve the problem, we write the solution of eq. (19) as:

$$u(x, y, t) = \sum_{k=0}^{\infty} \omega_k(x, y) t^{k\alpha} \tag{22}$$

From eq. (9):

$$\begin{aligned} \mathrm{D}_t^\alpha u &= \sum_{k=1}^{\infty} \frac{\Gamma(k\alpha + 1)}{\Gamma[(k-1)\alpha + 1]} \omega_k(x, y) t^{(k-1)\alpha} = \\ &= \Gamma(\alpha + 1)\omega_1(x, y) + \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)} \omega_2(x, y) t^\alpha + \frac{\Gamma(3\alpha + 1)}{\Gamma(2\alpha + 1)} \omega_3(x, y) t^{2\alpha} + \cdots \end{aligned} \tag{23}$$

From eq. (18), it is easy to see that:

$$\frac{\partial^2 u}{\partial x^2} = \sum_{k=0}^{\infty} \frac{\partial^2 \omega_k}{\partial x^2} t^{k\alpha} = \frac{\partial^2 \omega_0}{\partial x^2} + \frac{\partial^2 \omega_1}{\partial x^2} t^\alpha + \frac{\partial^2 \omega_2}{\partial x^2} t^{2\alpha} + \cdots \tag{24}$$

$$\frac{\partial^2 u}{\partial y^2} = \sum_{k=0}^{\infty} \frac{\partial^2 \omega_k}{\partial y^2} t^{k\alpha} = \frac{\partial^2 \omega_0}{\partial y^2} + \frac{\partial^2 \omega_1}{\partial y^2} t^\alpha + \frac{\partial^2 \omega_2}{\partial y^2} t^{2\alpha} + \cdots \tag{25}$$

Substituting the expansion of eqs. (23)-(25) into eq. (19) yields that:

$$\begin{aligned} &\Gamma(\alpha + 1)\omega_1(x, y) + \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)} \omega_2(x, y) t^\alpha + \frac{\Gamma(3\alpha + 1)}{\Gamma(2\alpha + 1)} \omega_3(x, y) t^{2\alpha} + \cdots = \\ &= \frac{1}{2}\left(y^2 \frac{\partial^2 \omega_0}{\partial x^2} + x^2 \frac{\partial^2 \omega_0}{\partial y^2} \right) + \frac{1}{2}\left(y^2 \frac{\partial^2 \omega_1}{\partial x^2} + x^2 \frac{\partial^2 \omega_1}{\partial y^2} \right) t^\alpha + \frac{1}{2}\left(y^2 \frac{\partial^2 \omega_2}{\partial x^2} + x^2 \frac{\partial^2 \omega_2}{\partial y^2} \right) t^{2\alpha} + \cdots \end{aligned} \tag{26}$$

By comparing the coefficients of $t^{k\alpha}$ in both sides of the equation, one gets:

$$\omega_0(x, y) = x^2 + y^2$$

$$\omega_k(x, y) = \frac{\Gamma[\alpha(k-1) + 1]}{\Gamma(\alpha k + 1)} \left(\frac{1}{2} y^2 \frac{\partial^2 \omega_{k-1}}{\partial x^2} + \frac{1}{2} x^2 \frac{\partial^2 \omega_{k-1}}{\partial y^2} \right), \quad (k = 1, 2, \cdots) \tag{27}$$

Thus

$$\omega_1(x,y)=\frac{x^2+y^2}{\Gamma(\alpha+1)} \tag{28}$$

$$\omega_2(x,y)=\frac{x^2+y^2}{\Gamma(2\alpha+1)} \tag{29}$$

$$\omega_3(x,y)=\frac{x^2+y^2}{\Gamma(3\alpha+1)} \tag{30}$$

$$\omega_4(x,y)=\frac{x^2+y^2}{\Gamma(4\alpha+1)} \tag{31}$$

$$\vdots$$

Therefore, we can obtain the following series expansion solution:

$$u(x,y,t)=(x^2+y^2)\left[1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots\right] \tag{32}$$

So, the solution for the standard wave-like model (α = 2) is given by:

$$u(x,y,t)=(x^2+y^2)\left[1+\frac{t^2}{2!}+\frac{t^4}{4!}+\frac{t^6}{6!}+\frac{t^8}{8!}+\cdots+\frac{t^{2n}}{(2n)!}+\cdots\right] \tag{33}$$

This series has the closed form:

$$u(x,y,t)=(x^2+y^2)\cosh t \tag{34}$$

which is the exact solution of the problem (19)-(21) when α = 2.

Example 2. In this example, we consider the 3-D inhomogeneous IBVP:

$$\mathrm{D}_t^\alpha u=x^2y^2z^2-\frac{1}{3}\left(\frac{u_{xx}}{y^2z^2}+\frac{u_{yy}}{x^2z^2}+\frac{u_{zz}}{x^2y^2}\right),\quad 0<x,y,z<1,\quad 1<\alpha\le 2,\quad t>0 \tag{35}$$

subject to the Neumann boundary conditions

$$\begin{aligned}&u(0,y,z,t)=\cosh t,\quad u(1,y,z,t)=\cosh t\cos yz,\\&u(x,0,z,t)=\cosh t,\quad u(x,1,z,t)=\cosh t\cos xz,\\&u(x,y,0,t)=\cosh t,\quad u(x,y,1,t)=\cosh t\cos xy\end{aligned} \tag{36}$$

and the initial conditions

$$u(x,y,z,0)=\cos xyz,\quad u_t(x,y,z,0)=0 \tag{37}$$

To solve the problem, proceeding as in the previous example, the solution of eq. (35) can be written:

$$u(x,y,z,t)=\sum_{k=0}^{\infty}\omega_k(x,y,z)t^{k\alpha} \tag{38}$$

Obviously, $\omega_0(x, y, z) = \cos xyz$. In order to complete the formulation of the FPS method, we may compute the functions $\mathrm{D}_t^\alpha u, u_{xx}, u_{yy}$, and u_{zz}.

From eq. (9), we can get:

$$\mathrm{D}_t^\alpha u=\sum_{k=1}^{\infty}\frac{\Gamma(k\alpha+1)}{\Gamma[(k-1)\alpha+1]}\omega_k(x,y,z)t^{(k-1)\alpha} \tag{39}$$

From eq. (38), it is easy to see that:

$$\frac{\partial^2 u}{\partial x^2}=\sum_{k=0}^{\infty}\frac{\partial^2\omega_k}{\partial x^2}t^{k\alpha}=\frac{\partial^2\omega_0}{\partial x^2}+\frac{\partial^2\omega_1}{\partial x^2}t^{\alpha}+\frac{\partial^2\omega_2}{\partial x^2}t^{2\alpha}+\cdots \tag{40}$$

$$\frac{\partial^2 u}{\partial y^2}=\sum_{k=0}^{\infty}\frac{\partial^2\omega_k}{\partial y^2}t^{k\alpha}=\frac{\partial^2\omega_0}{\partial y^2}+\frac{\partial^2\omega_1}{\partial y^2}t^{\alpha}+\frac{\partial^2\omega_2}{\partial y^2}t^{2\alpha}+\cdots \tag{41}$$

$$\frac{\partial^2 u}{\partial z^2}=\sum_{k=0}^{\infty}\frac{\partial^2\omega_k}{\partial z^2}t^{k\alpha}=\frac{\partial^2\omega_0}{\partial z^2}+\frac{\partial^2\omega_1}{\partial z^2}t^{\alpha}+\frac{\partial^2\omega_2}{\partial z^2}t^{2\alpha}+\cdots \tag{42}$$

Now, substituting eqs. (39)-(42) back into eq. (35), equating the coefficients of $t^{k\alpha}$ to zero in the resulting equation, we get:

$$\omega_k(x,y,z)=\frac{\Gamma[(k-1)\alpha+1]}{\Gamma(k\alpha+1)}\left(-\frac{1}{3y^2z^2}\frac{\partial^2\omega_{k-1}}{\partial x^2}-\frac{1}{3x^2z^2}\frac{\partial^2\omega_{k-1}}{\partial y^2}-\frac{1}{3x^2y^2}\frac{\partial^2\omega_{k-1}}{\partial z^2}\right),(k=1,2,\cdots) \tag{43}$$

Thus

$$\omega_1(x,y,z)=\frac{\cos xyz}{\Gamma(\alpha+1)} \tag{44}$$

$$\omega_2(x,y,z)=\frac{\cos xyz}{\Gamma(2\alpha+1)} \tag{45}$$

$$\omega_3(x,y,z)=\frac{\cos xyz}{\Gamma(3\alpha+1)} \tag{46}$$

$$\vdots$$

Therefore, we can obtain the following series expansion solution:

$$u(x,y,z,t)=\left[1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots\right]\cos xyz \tag{47}$$

So, the solution for the standard wave-like model ($\alpha = 2$) is given by:

$$u(x,y,t)=\left[1+\frac{t^2}{2!}+\frac{t^4}{4!}+\frac{t^6}{6!}+\frac{t^8}{8!}+\cdots+\frac{t^{2n}}{(2n)!}+\cdots\right]\cos xyz \tag{48}$$

This series has the closed form:

$$u(x,y,z,t) = \cosh t \cos xyz \tag{49}$$

which is the exact solution of the problem (35)-(37).

Discussion and conclusion

This paper applies the FPSM to solve 2- and 3-D fractional wave-like models with variable coefficients with great success. This method is a natural extension of the Taylor series method [15, 16] to fractional calculus.

Yang *et al.* [17] suggested the fractional series expansion method for local fractional calculus, and it was extended to general fractional calculus in 2015 by Li and Zhu [18]. The both methods can be extended to fractional differential equations and fractal differential equations [19-32].

To be concluded, this paper shows that fractional power series method (FPSM) is a simple but effective method for fractional calculus, the results given in Examples 1 and 2 are in excellent agreement with the exact ones, and this paper concludes that FPSM method is a straightforward and convenient method for fractional calculus.

Acknowledgment

This work is supported by the key scientific research projects of Henan higher education institutions (Grant No. 20B110009 and Grant No. 17A880017)

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Paper submitted: April 4, 2019
Paper revised: October 20, 2019
Paper accepted: October 20, 2019