THE LOCAL FRACTIONAL VARIATIONAL ITERATION METHOD A Promising Technology for Fractional Calculus

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In order to make the local variational iteration algorithm converge faster and more effective, the Sumudu transform is adopted and a proper initial solution is chosen. Some examples are given to show that the presented method is reliable, efficient and easy to implement from a computational viewpoint.

Key words: *local fractional derivative, local fractional integral, variational iteration method, Sumudu transform*

Introduction

Fractional calculus is a generalization of the ordinary differentiation to a non-integer order. Many problems in thermal science can be effectively modeled by fractional differential equations (see [1-13] and references therein), and many analytical methods have been developed to solve these fractional differential equations. Among these methods, the variational iteration method and the homotopy perturbation method are widely used [14-19].

Recently, the local fractional derivative [20, 21] was developed to process the non-differential problems defined on Cantor sets, which can also be used to deal with practical problems. In this paper, by coupling the local fractional variational iteration method with local fractional Sumudu transform method, we reduce the volume of the computational work of the classical variational iteration method and still maintain its high accuracy. This technique enriches the local fractional variational iteration algorithm for solving the exact or approximate solutions of differential equations or systems.

Mathematical fundamentals

In this section, we introduce the preliminaries of local fractional continuity, local fractional derivative and local fractional Sumudu transform in a fractal space.

Definition 1 [20, 21]. In a fractal space, let $f(x) \in C_\alpha(a, b)$ the local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by:

$$\mathrm{D}_x^{(\alpha)} f(x_0) = f^{(\alpha)}(x_0) = \frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha}\Big|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha [f(x) - f(x_0)]}{(x - x_0)^\alpha} \tag{1}$$

where $\Delta^\alpha [f(x) - f(x_0)] \cong \Gamma(1+\alpha)\Delta[f(x) - f(x_0)]$.

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Definition 2 [20, 21]. Let a function $f(x)$ satisfy the condition (1), the local fractional integral of $f(x)$ of order α in the interval $|a, b|$ is defined by:

$$ {}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(\mathrm{d}t)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^\alpha \tag{2} $$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, \ldots\}$ and $[t_j, t_{j+1}]$ ($j = 0, \ldots, N-1$, $t_0 = a$, $t_N = b$) is a partition of the interval $[a, b]$.

Definition 3 [22]. The local fractional Sumudu transform of $f(x)$ with order $\alpha(0 < \alpha \le 1)$ is defined:

$$ D_\alpha\{f(x)\} = F_\alpha(z) =: \frac{1}{\Gamma(1+\alpha)} \int_0^{+\infty} E_\alpha(-z^{-\alpha} x^\alpha) \frac{f(x)}{z^\alpha} (\mathrm{d}x)^\alpha \tag{3} $$

Theorem 1 [22]. Assuming $D_\alpha\{f(x)\} = F_\alpha(z)$ and assuming:

$$ f(x) = \sum_{n=0}^{\infty} a_n \frac{x^{n\alpha}}{\Gamma(1+\alpha n)} $$

is bounded and continuous, then we have:

$$ F_\alpha(z) = \sum_{n=0}^{\infty} a_n z^{n\alpha} \tag{4} $$

Theorem 2 [22]. Assuming $D_\alpha\{f(x)\} = F_\alpha(z)$, the local fractional Sumudu transform of $[d^\alpha f(x)]/(dx^\alpha)$ is given:

$$ D_\alpha\left\{\frac{d^{n\alpha} f(x)}{dx^{n\alpha}}\right\} = \frac{1}{z^{n\alpha}}\left[F_\alpha(z) - \sum_{k=0}^{n-1} z^{k\alpha} f^{(k\alpha)}(0)\right] \tag{5} $$

Local fractional variational iteration method

The variational iteration method [23] was first proposed in 1998 to solve fractional differential equations, and has becomes a famous mathematical tool for solving various non-linear problem. The fractional variational iteration method [24] is especially suitable for fractional differential equations. In order to illustrate the solution procedure of this method, we consider the following local fractional differential system on a fractal set:

$$ \frac{\partial^{k_0\alpha_i} u_i(x,y)}{\partial x^{k_0\alpha_i}} = L_i[u_1(x,y),\ldots,u_n(x,y)] + R_i[u_1(x,y),\ldots,u_{n_0}(x,y)] + g_i(x,y), \quad i = 1,2,\ldots,n_0 \tag{6} $$

with boundary and initial conditions:

$$ B_i[u_{i,0}(x,y)] = 0, \quad i = 1,2,\ldots,n_0 \tag{7} $$

where k_0 is the term of the highest order derivative with respect to variable x, L_i – the linear differential operator, R_i – the non-linear differential operator, and $g_i(x, y)$ – the source term.

We can construct $u_{i,0}(x, y)$ by the following equations:

$$C_{i1}\frac{\partial^{k_0\alpha_i}u_i(x,y)}{\partial x^{k_0\alpha_i}}=C_{i2}L_i[u_1(x,y),...,u_n(x,y)]+$$

$$+C_{i3}R_i[u_1(x,y),...u_{n_0}(x,y)]C_{i4}g_i(x,y),\quad i=1,2,...,n_0 \tag{8}$$

and $B_i[u_{i,0}(x, y)]$, where C_{i1}, C_{i2}, C_{i3}, $C_{i4}\in$ {0, 1} are constants to be determined according with the specific conditions.

By means of the initial value $u_{i,0}(x, y)$, i = 1, 2,…n_0 and according to the rule of local fractional variational iteration method [25], we can construct correction functional for eq. (6):

$$u_{i,n+1}(x,\mathrm{y})=u_{i,n}(x,y)+{}_0I_x^{\alpha_i}\left\{\lambda_i(x-t)\cdot\right.$$

$$\left.\cdot\left[\frac{\partial^{k_0\alpha_i}u_i(x,y)}{\partial x^{k_0\alpha_i}}-L_i(u_{1,k},...,u_{n_0,k})-R_i(u_{1,k},...,u_{n_0,k})-g_i(x,\mathrm{y})\right]\right\},\quad i=1,2,...,n_0 \tag{9}$$

where $\lambda_i(x - t)$, i = (1, 2,… n) are fractal Lagrange multipliers and where the terms $L_i(u_{1,k},\ldots, u_{n0,k})$ and $R_i(u_{1,k},\ldots, u_{n0,k})$ are considered as restricted local fractional variation, *i. e.*, $\delta^\alpha L_i(u_{1,k},\ldots, u_{n0,k}) = 0$, $\delta^\alpha R_i(u_{1,k},\ldots, u_{n0,k}) = 0$. Anjum and He [26] found that the Laplace transform makes the identification of the Lagrange multiplier extremely easy, in this paper, we use the Sumudu transform [22] instead of Laplace transform.

Applying the local fractional Sumudu transform [22] on both sides of eq. (9) gives:

$$D_\alpha\{u_{i,n+1}\}=D_\alpha\{u_{i,n}\}+$$

$$D_\alpha\left({}_0I_x^{\alpha_i}\left\{\lambda_i(x-t)\left[\frac{\partial^{k_0\alpha_i}u_i}{\partial x^{k_0\alpha_i}}-L_i(u_{1,k},...,u_{n_0,k})-R_i(u_{1,k},...,u_{n_0,k})-g_i(x,y)\right]\right\}\right),i=1,2,...,n_0 \tag{10}$$

According to [22], we can get:

$$D_\alpha\{u_{i,n+1}\}=D_\alpha\{u_{i,n}\}+z^\alpha D_\alpha\{\lambda_i(x)\}D_\alpha\cdot$$

$$\cdot\left\{\left[\frac{\partial^{k_0\alpha_i}u_i}{\partial x^{k_0\alpha_i}}-L_i-R_i(u_{1,n},...,u_{n_0,n})-g_i(x,\mathrm{y})\right]\right\},\quad i=1,2,...,n_0 \tag{11}$$

For the determination of the fractal Lagrange multiplier, the extremum condition of $u_{i,n+1}$ leads us to $\delta^\alpha u_{i,n+1} = 0$.

Using $\delta^\alpha L_i(u_{1,k},\ldots, u_{n0,k}) = 0$, $\delta^\alpha R_i(u_{1,k},\ldots, u_{n0,k}) = 0$ and eq. (11), we get:

$$\delta^\alpha\{D_\alpha\{u_{i,n+1}\}\}=\delta^\alpha\{D_\alpha\{u_{i,n}\}\}+z^\alpha\delta^\alpha\{D_\alpha\{\lambda_i(x)\}\}\delta^\alpha\left\{D_\alpha\left[\frac{\partial^{k_0\alpha_i}u_i}{\partial x^{k_0\alpha_i}}\right]\right\},\quad i=1,2,...,n_0 \tag{12}$$

From the *Theorem 2* and eq. (12), it is easy to verify that:

$$1+D_\alpha\{\lambda_i(x)\}\frac{1}{z^{(k-1)\alpha}}=0,\quad i=1,2,...,n_0 \tag{13}$$

According to the *Theorem 1*, we obtain the following fractal Lagrange multipliers:

$$\lambda_i(x-t) = -\frac{(x-t)^{(k-1)\alpha_i}}{\Gamma[1+(k-1)\alpha_i]}, \quad i=1,2,...,n_0 \tag{14}$$

Substituting eq. (14) into eq. (9) gives the following relationship:

$$u_{i,n+1}(x,y) = u_{i,n}(x,y) -$$
$$-{}_0I_x^{\alpha_i}\left\{\frac{(x-t)^{(k-1)\alpha_i}}{\Gamma[1+(k-1)\alpha_i]}\left[\frac{\partial^{k_0\alpha_i}u_i}{\partial x^{k_0\alpha_i}} - L_i(u_1,...,u_n) - R_i(u_1,...,u_n) - g_i(x,y)\right]\right\}, i=1,2,...,n_0 \tag{15}$$

Taking the local fractional Sumudu transform on both sides of eq. (15), we get:

$$D_\alpha\{u_{i,n+1}(x,y)\} = D_\alpha\{u_{i,n}(x,y)\} -$$
$$-D_\alpha\left({}_0I_x^{\alpha_i}\left\{\frac{(x-t)^{(k-1)\alpha_i}}{\Gamma[1+(k-1)\alpha_i]}\left[\frac{\partial^{k_0\alpha_i}u_i}{\partial x^{k_0\alpha_i}} - L_i(u_{1,n},...,u_{n_0,n}) - R_i(u_{1,n},...,u_{n_0,n}) - g_i(x,\mathrm{y})\right]\right\}\right),$$
$$i=1,2,...,n_0 \tag{16}$$

This is:

$$D_\alpha\{u_{i,n+1}\} = D_\alpha\{u_{i,n}\} - z^{\alpha_i}D_\alpha\left[\frac{x^{(k-1)\alpha_i}}{\Gamma[1+(k-1)\alpha_i]}\right]\cdot$$
$$\cdot D_\alpha\left[\frac{\partial^{k_0\alpha_i}u_i}{\partial x^{k_0\alpha_i}} - L_i(u_{1,n},...,u_{n_0,n}) - R_i(u_{1,n},...,u_{n_0,n}) - g_i(x,y)\right], \quad i=1,2,...,n_0 \tag{17}$$

Then, by the *Theorem 2*, we can arrive:

$$D_\alpha\{u_{i,n+1}\} = D_\alpha\{u_{i,n}\} - z^{k_i\alpha_i}D_\alpha\,\cdot$$
$$\cdot\left[\frac{\partial^{k_0\alpha_i}u_i}{\partial x^{k_0\alpha_i}} - L_i(u_{1,n},...,u_{n_0,n}) - R_i(u_{1,n},...,u_{n_0,n}) - g_i(x,y)\right] =$$
$$= D_\alpha\{u_{i,n}\} - z^{k_i\alpha_i}\left\{\frac{1}{z^{k_i\alpha}}\left[D_\alpha\{u_{i,n}\} - \sum_{k'=0}^{k_i-1}z^{k'\alpha}u_{i,n}{}^{(k'\alpha)}(0)\right]\right\} +$$
$$+z^{k\alpha_i}D_\alpha\{L_i(u_{1,n},...,u_{n_0,n}) + R_i(u_{1,n},...,u_{n_0,n}) + g_i(x,y)\} =$$
$$= \sum_{k'=0}^{k_i-1}z^{k'\alpha}u_{i,n}{}^{(k'\alpha)}(0) + z^{k\alpha_i}D_\alpha\{L_i(u_{1,n},...,u_{n_0,n}) + R_i(u_{1,n},...,u_{n_0,n}) + g_i(x,y)\} \tag{18}$$

Finally, the exact solution of eq. (6) is:

$$u_i(x,y) = \lim_{n\to\infty}D_\alpha^{-1}\{D_\alpha\{u_{i,n}(x,y)\}\}, \quad i=1,2,...,n_0 \tag{19}$$

Illustrative examples

In this section, we solve some non-linear fractional differential equations or systems in order to exhibit the accuracy and efficiency of the proposed method.

Example 1. Consider the fourth-order singular parabolic fractional partial differential equation:

$$\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} - \left[\frac{y^\alpha}{E_\alpha(y^\alpha)} - 1\right]\frac{\partial^{4\alpha} u}{\partial y^{4\alpha}} = 1 \tag{20}$$

with the initial conditions:

$$u(0,y) = y^\alpha - E_\alpha(y^\alpha), \quad \frac{\partial^\alpha u}{\partial x^\alpha}(0,y) = -[y^\alpha - E_\alpha(y^\alpha)] \tag{21}$$

Letting:

$$\frac{\partial^{2\alpha} u_0}{\partial x^{2\alpha}} - 1 = 0 \tag{22}$$

satisfying the following conditions:

$$u_0(0, y = y^\alpha - E_\alpha(y^\alpha), \quad \frac{\partial^\alpha u_0}{\partial x^\alpha}(0,y) = -[y^\alpha - E_\alpha(y^\alpha)] \tag{23}$$

From eqs. (22) and (23), we obtain:

$$u_0(x,y) = [y^\alpha - E_\alpha(y^\alpha)]\left[1 - \frac{x^\alpha}{\Gamma(1+\alpha)}\right] + \frac{x^{2\alpha}}{\Gamma(1+\alpha)} \tag{24}$$

In light of the local fractional variational iteration method algorithm given in eq. (18), the first few terms of $u_i(x, y)$ are given by the following expressions:

$$u_1(x,y) = D_\alpha^{-1}\left\{z^{2\alpha_i}\sum_{k'=0}^{1} z^{k'\alpha} u_0^{(k'\alpha)}(0,y) + z^{2\alpha_i} D_\alpha\left[\left(\frac{y^\alpha}{E_\alpha(y^\alpha)} - 1\right)\frac{\partial^{4\alpha} u_0}{\partial y^{4\alpha}} + 1\right]\right\} =$$

$$= [y^\alpha - E_\alpha(y^\alpha)]\left[1 - \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)}\right] + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \tag{25}$$

$$u_2(x,y) = D_\alpha^{-1}\left\{z^{2\alpha_i}\sum_{k'=0}^{1} z^{k'\alpha} u_1^{(k'\alpha)}(0,y) + z^{2\alpha_i} D_\alpha\left[\left(\frac{y^\alpha}{E_\alpha(y^\alpha)} - 1\right)\frac{\partial^{4\alpha} u_1}{\partial y^{4\alpha}} + 1\right]\right\} =$$

$$= u_1(x,y) + [y^\alpha - E_\alpha(y^\alpha)]\left[\frac{x^{4\alpha}}{\Gamma(1+4\alpha)} - \frac{x^{5\alpha}}{\Gamma(1+5\alpha)}\right] \tag{26}$$

$$u_3(x,y) = u_2(x,y) + [y^\alpha - E_\alpha(y^\alpha)]\left[\frac{x^{6\alpha}}{\Gamma(1+6\alpha)} - \frac{x^{7\alpha}}{\Gamma(1+7\alpha)}\right] \tag{27}$$

$$u_4(x,y)=u_3(x,y)+[y^\alpha-E_\alpha(y^\alpha)]\left[\frac{x^{8\alpha}}{\Gamma(1+8\alpha)}-\frac{x^{9\alpha}}{\Gamma(1+9\alpha)}\right] \tag{28}$$

Proceeding in the same manner, the rest of the components can be also obtained:

$$u_{n+1}(x,y)=u_n(x,y)+[y^\alpha-E_\alpha(y^\alpha)]\left\{\frac{x^{2(n+1)\alpha}}{\Gamma[1+2(n+1)\alpha]}-\frac{x^{(2n+3)\alpha}}{\Gamma[1+(2n+3)\alpha]}\right\} \tag{29}$$

Then, the exact solution of eq. (20) is thus entirely determined:

$$u(x,y)=[y^\alpha-E_\alpha(y^\alpha)]\sum_{k=0}^{\infty}(-1)^{n-1}\frac{x^{n\alpha}}{\Gamma(1+\text{n}\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}=$$

$$=[y^\alpha-E_\alpha(y^\alpha)]E_\alpha(-x^\alpha)+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \tag{30}$$

Example 2. Consider the following fractional non-homogeneous non-linear gas dynamic equation:

$$\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}+u\frac{\partial^\alpha u}{\partial y^\alpha}-u(1-u)=-E_\alpha(x^\alpha-y^\alpha) \tag{31}$$

with the following initial condition:

$$u(0,0)=0,\quad \frac{\partial^\alpha u}{\partial x^\alpha}(0,0)=-1 \tag{32}$$

Letting:

$$u_0\frac{\partial^\alpha u_0}{\partial y^\alpha}-u_0^2=0 \tag{33}$$

satisfying the condition:

$$u_0(0,0)=0,\quad \frac{\partial^\alpha u_0}{\partial x^\alpha}(0,0)=-1 \tag{34}$$

from eqs. (33) and (34), we obtain:

$$u_0(x,y)=1+E_\alpha(-y^\alpha)\left[-1-\frac{x^\alpha}{\Gamma(1+\alpha)}\right] \tag{35}$$

In light of the algorithm given in eq. (18), the first few terms of $u_i(x, y)$ are given by the following expressions:

$$u_1(x,y)=$$

$$=D_\alpha^{-1}\left\{\sum_{k'=0}^{1}z^{k'\alpha}u_0^{(k'\alpha)}(0,y)+z^{2\alpha_i}D_\alpha\left[-u_0\frac{\partial^\alpha u_0}{\partial y^\alpha}+u_0(1-u_0)-E_\alpha(x^\alpha-y^\alpha)\right]\right\}=$$

$$=1-E_\alpha(x^\alpha-y^\alpha) \tag{36}$$

$$u_2(x,y)=$$
$$=D_\alpha^{-1}\left\{\sum_{k'=0}^{1} z^{k'\alpha}u_1^{(k'\alpha)}(0,y)+z^{2\alpha_i}D_\alpha\left[-u_1\frac{\partial^\alpha u_1}{\partial y^\alpha}+u_1(1-u_1)-E_\alpha(x^\alpha-y^\alpha)\right]\right\}=$$
$$=1-E_\alpha(x^\alpha-y^\alpha) \tag{37}$$

$$u_3(x,y)=$$
$$=D_\alpha^{-1}\left\{\sum_{k'=0}^{1} z^{k'\alpha}u_2^{(k'\alpha)}(0,y)+z^{2\alpha_i}D_\alpha\left[-u_2\frac{\partial^\alpha u_2}{\partial y^\alpha}+u_2(1-u_2)-E_\alpha(x^\alpha-y^\alpha)\right]\right\}=$$
$$=1-E_\alpha(x^\alpha-y^\alpha) \tag{38}$$

Proceeding in the same manner, the rest of the components can be completely obtained:

$$u_{n+1}(x,y)=$$
$$=D_\alpha^{-1}\left\{\sum_{k'=0}^{1} z^{k'\alpha}u_n^{(k'\alpha)}(0,y)+z^{2\alpha_i}D_\alpha\left[-u_n\frac{\partial^\alpha u_n}{\partial y^\alpha}+u_n(1-u_n)-E_\alpha(x^\alpha-y^\alpha)\right]\right\}=$$
$$=1-E_\alpha(x^\alpha-y^\alpha) \tag{39}$$

Therefore, the series solution of eq. (31) is determined:

$$u(x,t)=\lim_{n\to\infty}u_n(x,y)=1-E_\alpha(x^\alpha-y^\alpha) \tag{40}$$

Example 3. Consider the following local fractional partial differential system:

$$\begin{cases}\dfrac{\partial^{2\alpha}u(x,y)}{\partial x^{2\alpha}}+\dfrac{\partial^{2\alpha}v(x,y)}{\partial y^{2\alpha}}+u(x,y)+v(x,y)=0\\[2ex]\dfrac{\partial^{2\alpha}v(x,y)}{\partial x^{2\alpha}}+\dfrac{\partial^{2\alpha}u(x,y)}{\partial y^{2\alpha}}+u(x,y)+v(x,y)=0\end{cases} \tag{41}$$

with the initial conditions:

$$u(0,0)=0,\quad \frac{\partial^\alpha u}{\partial x^\alpha}(0,0)=-1,\quad \frac{\partial^\alpha u}{\partial \mathrm{y}^\alpha}(0,0)=1,$$
$$v(0,0)=1,\quad \frac{\partial^\alpha v}{\partial x^\alpha}(0,0)=0,\quad \frac{\partial^\alpha u}{\partial x^\alpha}(0,0)=0 \tag{42}$$

Letting:

$$\begin{cases}\dfrac{\partial^{2\alpha}v_0(x,y)}{\partial y^{2\alpha}}+v_0(x,y)=0\\[2ex]\dfrac{\partial^{2\alpha}u_0(x,y)}{\partial y^{2\alpha}}+u_0(x,y)=0\end{cases} \tag{43}$$

satisfying the condition eqs. (42), we can get:

$$u_0(x,y)=\sin_\alpha y^\alpha-\frac{x^\alpha}{\Gamma(1+\alpha)}\cos_\alpha y^\alpha,\quad v_0(x,y)=\cos_\alpha y^\alpha+\frac{x^\alpha}{\Gamma(1+\alpha)}\sin_\alpha y^\alpha \tag{44}$$

In light of the algorithm given in eq. (18), the first few terms of $u_i(x, y)$ and $v_i(x, y)$ are given by the following expressions, respectively:

$$u_1(x,y)=D_\alpha^{-1}\left\{z^{2\alpha_i}\sum_{k'=0}^{1}z^{k'\alpha}u_0^{(k'\alpha)}(0,y)-z^{2\alpha_i}D_\alpha\left[\frac{\partial^{2\alpha}v_0(x,y)}{\partial y^{2\alpha}}+u_0(x,y)+v_0(x,y)\right]\right\}=$$
$$=\sin_\alpha y^\alpha\left[1-\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\right]+\left[-\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}\right]\cos_\alpha y^\alpha \tag{45}$$

$$v_1(x,y)=D_\alpha^{-1}\left\{z^{2\alpha_i}\sum_{k'=0}^{1}z^{k'\alpha}v_0^{(k'\alpha)}(0,y)-z^{2\alpha_i}D_\alpha\left[\frac{\partial^{2\alpha}u_0(x,y)}{\partial y^{2\alpha}}+v_0(x,y)+u_0(x,y)\right]\right\}=$$
$$=\cos_\alpha y^\alpha\left[1-\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\right]+\left[\frac{x^\alpha}{\Gamma(1+\alpha)}-\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}\right]\sin_\alpha y^\alpha \tag{46}$$

$$u_2(x,y)=D_\alpha^{-1}\left\{z^{2\alpha_i}\sum_{k'=0}^{1}z^{k'\alpha}u_1^{(k'\alpha)}(0,y)-z^{2\alpha_i}D_\alpha\left[\frac{\partial^{2\alpha}v_1(x,y)}{\partial x^{2\alpha}}+u_1(x,y)+v_1(x,y)\right]\right\}=$$
$$=\sin_\alpha y^\alpha\left[1-\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}\right]+\left[-\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}-\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}\right]\cos_\alpha y^\alpha \tag{47}$$

$$v_2(x,y)=D_\alpha^{-1}\left\{z^{2\alpha_i}\sum_{k'=0}^{1}z^{k'\alpha}v_1^{(k'\alpha)}(0,y)-z^{2\alpha_i}D_\alpha\left[\frac{\partial^{2\alpha}u_1(x,y)}{\partial y^{2\alpha}}+v_1(x,y)+u_1(x,y)\right]\right\}=$$
$$=\cos_\alpha y^\alpha\left[1-\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}\right]+\left[\frac{x^\alpha}{\Gamma(1+\alpha)}-\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}\right]\sin_\alpha y^\alpha \tag{48}$$

Proceeding in this same manner, the rest of the components can be completely obtained. Then, the series solution of eq. (41) is given:

$$\begin{gathered}u(x,y)=\sin_\alpha y^\alpha\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n\alpha}}{\Gamma(1+2n\alpha)}+\\+\cos_\alpha y^\alpha\sum_{n=1}^{\infty}(-1)^n\frac{x^{(2n-1)\alpha}}{\Gamma[1+(2n-1)\alpha]}=\sin_\alpha(x-y)^\alpha,\\v(x,y)=\cos_\alpha y^\alpha\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n\alpha}}{\Gamma(1+2n\alpha)}+\\+\sin_\alpha y^\alpha\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{(2n-1)\alpha}}{\Gamma[1+(2n-1)\alpha]}=\cos_\alpha(x-y)^\alpha\end{gathered} \tag{49}$$

Conclusions

Fractal calculus has been becoming the hottest topic in both mathematics and engineering, the fractal order is generally related to the value of fractional dimensions [27-30], and new analytical methods soar, the Taylor method, the oldest one, might be also a powerful tool to fractional calculus, the simpler, the better [31, 32].

In this paper, we proposed the local fractional variational iteration method in details. Some examples are used to illustrate that the method is a straightforward and concise mathematical method for solving a wide variety of fractional differential equation or systems.

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