FRACTIONAL FOKKER-PLANCK EQUATION IN A FRACTAL MEDIUM

by

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This paper studies a fractal modification of Fokker-Planck equation for a heat conduction in a fractal medium. Fourier transform and Darboux transformation are used to solve the equation, some new results are obtained.

Key words: *fractional Fokker-Planck equation, fractal Fourier transform, fractal media, Darboux transformation*

Introduction

The following classical Fokker-Planck equation was first proposed by Fokker and Planck to model the Brownian motion of particles:

$$\frac{\partial W}{\partial t} = u\frac{\partial W}{\partial x} + \gamma \frac{kT}{m}\frac{\partial^2 W}{\partial x^2} \tag{1}$$

where u is the velocity for the Brownian motion of a small particle, t – the time, γ – the fraction constant, k – the Boltzmann's constant, and T – the temperature of fluid [1, 2].

It has been found that the Fokker-Planck equation can be applied to numerous scientific fields, including the thermal physics, chemical physics, quantum optics, circuit theory and theoretical biology [1-12]. Recently, some authors argued that chaotic Hamiltonian dynamics of particles can be described by fractional differential equation involving Riemann-Liouville derivative or Caputo derivative [3-12]. However, these fractional derivatives are non-local and thus such equations are not suitable to study the heat conduction in fractal media. In order to overcome this shortcoming, many new modifications have been developed, for examples, the fractal derivative and local fractional derivative [4, 13-25]. Tarasov [4] introduced the following Fokker-Planck equation in a fractal medium:

$$\frac{\partial \rho(x,t)}{\partial t} + k\frac{\partial \rho(x,t)}{\partial x^\alpha} + \frac{D}{2}\frac{\partial^2 \rho(x,t)}{(\partial x^\alpha)^2} = 0 \tag{2}$$

where $\dfrac{\partial}{\partial x^\alpha} = \dfrac{|x|^{1-\alpha}}{\alpha}\dfrac{\partial}{\partial x}$, and obtained the following stationary solution:

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$$\rho(x,t) = N\exp\left(-\frac{2k|x|^{\alpha}}{D}\right)$$

In the present work, we consider the following fractional Fokker-Planck equation in a fractal medium:

$$\frac{\partial W(x,t)}{\partial t^{\alpha}} = \frac{\partial[a(x)W(x,t)]}{\partial x^{\beta}} + \mu^2\frac{\partial}{\partial x^{\beta}}\left[\frac{\partial W(x,t)}{\partial x^{\beta}}\right] \tag{3}$$

with the conditions $W(x, 0) = \delta(x)$, where $0 < \alpha$, $\beta \le 1$ are constants, the symbol $\partial/[\partial(\cdot)^{\lambda}]$ denotes the fractal derivative operator of order λ, $a(x)$ are given functions, μ is a constant, and $\delta(x)$ represents the delta function.

Some basic results

In this section, let us recall some basic results of fractal derivative and the theory of fractal Fourier transform [24].

The fractal derivative is a natural extension of classical derivative for discontinuous fractal media.

Definition 1. Chen's definition is [16]:

$$\frac{\mathrm{d}u}{\mathrm{d}x^{\alpha}} = \lim_{s\to x}\frac{u(x)-u(s)}{x^{\alpha}-s^{\alpha}} \tag{4}$$

where α is the order of the fractal derivative.

Definition 2. Consider a fractal media and assume the smallest measure is L_0 (any discontinuity less than L_0 is ignored). He's fractal derivative has the form [18]:

$$\frac{\partial u}{\partial x^{\alpha}}(x_0) = \Gamma(1+\alpha)\lim_{x-x_0\to L_0}\frac{u(x)-u(x_0)}{(x-x_0)^{\alpha}} \tag{5}$$

When $\alpha \to 1$ and $L_0 \to 0$, eq. (5) turns out to be the ordinary differentiation.

Definition 3 [24]. A partition of the interval $[a, b]$ is denoted as (t_j, t_{i+1}), $j = 0, 1,\ldots, N-1$, $t_0 = a$ and $t_N = b$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_0, \Delta t_1, \ldots, \Delta t_N\}$. The fractal integral of $f(x)$ in the interval $[a, b]$ is defined by:

$${}_aI_b^{(\alpha)}f(x) = \frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)(\mathrm{d}x)^{\alpha} = \lim_{\Delta t\to 0}\sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^{\alpha} \tag{6}$$

The following two formulas of fractal derivatives and integral hold true:

$$\frac{\partial(x^{n\alpha})}{\partial x^{\alpha}} = \frac{\Gamma(1+n\alpha)x^{(n-1)\alpha}}{\Gamma[1+(n-1)\alpha]} \tag{7}$$

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b x^{n\alpha}(\mathrm{d}x)^{\alpha} = \frac{\Gamma(1+n\alpha)[b^{(n+1)\alpha}-a^{(n+1)\alpha}]}{\Gamma[1+(n+1)\alpha]} \tag{8}$$

Definition 4 [24]. In the fractal space, the Mittag-Leffler function is defined by:

$$E_\lambda(x^\alpha)=\sum_{n=0}^{\infty}\frac{x^{(n\alpha)}}{\Gamma(1+n\lambda)},\quad 0<\alpha\le 1 \tag{9}$$

Definition 5 [24]. Let $f(x)\in C_\alpha(a, b)$. On the fractal media, the fractal Fourier transform is defined:

$$F_\alpha[f(x);\omega]=\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{+\infty}E_\alpha(-i^\alpha\omega^\alpha x^\alpha)f(x)(\mathrm{d}x)^\alpha \tag{10}$$

and its inverse Fourier transform:

$$f(x)=F_\alpha^{-1}[f_\alpha^{F,\alpha}(\omega)]=\frac{1}{(2\pi)^\alpha}\int_{-\infty}^{+\infty}E_\alpha(i^\alpha\omega^\alpha x^\alpha)f_\alpha^{F,\alpha}(\omega)(\mathrm{d}\omega)^\alpha \tag{11}$$

where $f_\alpha^{F,\alpha}(\omega)=F_\alpha[f(x);\omega]$.

The following properties hold true:

$$F_\alpha[af(x)+bg(x);\omega]=aF_\alpha[f(x);\omega]+bF_\alpha[g(x);\omega] \tag{12}$$

$$F_\alpha\left[\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}f(x);\omega\right]=i^\alpha\omega^\alpha F[f(x);\omega] \tag{13}$$

where $\lim_{x\to\pm\infty}f(x)=0$.

$$F_\alpha^{-1}[E_\alpha(-a^2\omega^2 t)]=\frac{1}{2a\sqrt{\pi t}}E_\alpha\left(-\frac{x^2}{4a^2t}\right) \tag{14}$$

Fractional Fokker-Planck equation

We first consider a simple case when $a(x) = k$, $\mu = 1$:

$$\frac{\partial W(x,t)}{\partial t^\alpha}=k\frac{\partial W(x,t)}{\partial x^\beta}+\frac{\partial}{\partial x^\beta}\frac{\partial W(x,t)}{\partial x^\beta} \tag{15}$$

To solve this equation, we use the following transformation:

$$u=x^\beta,\quad v=t^\alpha \tag{16}$$

which is the fractional complex transform [25], and it is also called as the two-scale transform [26, 27]. On a large scale (u, v) the problem can be considered as a continuous one, while on a smaller scale (x, t), it can be considered a fractal one. By the two-scale transform, we have:

$$\frac{\partial W(x,t)}{\partial v}=k\frac{\partial W(x,t)}{\partial u}+\frac{\partial}{\partial u}\frac{\partial W(x,t)}{\partial u} \tag{17}$$

Taking the Fourier transform with respect to u on the both sides of eq. (17), we have:

$$\frac{\mathrm{d}\bar{W}_\beta(\omega,v)}{\mathrm{d}v}=k(i\omega)\bar{W}_\beta(\omega,v)+(i\omega)^2\bar{W}_\beta(\omega,v)=(ik\omega-\omega^2)\bar{W}_\beta(\omega,v) \tag{18}$$

where

$$F_\beta[W(u,v);\omega]=\bar{W}_\beta(\omega,v) \tag{19}$$

Solving eq. (18) results in:

$$\bar{W}_\beta(\omega,v)=E_\beta(\omega^2 v-ik\omega v) \tag{20}$$

By taking the inverse Fourier transform on the both sides of eq. (20) yields:

$$W(u,v)=\delta(u)*F_\beta^{-1}[E_\beta(\omega^2 v-ik\omega v)]$$

From the relations (14), we can get:

$$F_\beta^{-1}\{E_\beta[(\omega^2-ik\omega)v]\}=\frac{1}{\sqrt{4\pi v}}E_\beta\left(\frac{k^2v-2kv}{4}-\frac{u^2}{4v}\right) \tag{21}$$

Thus, we have:

$$W(u,v)=\frac{1}{\sqrt{4\pi v}}E_\beta\left(\frac{k^2v-2kv}{4}\right)E_\beta\left(\frac{-u^2}{4v}\right) \tag{22}$$

Finally, we get fundamental solution of (15):

$$W(x,t)=\frac{1}{\sqrt{4\pi t^\alpha}}E_\beta\left(\frac{k^2-2k}{4}t^\alpha\right)E_\beta\left(\frac{x^{2\beta}}{4t^\alpha}\right) \tag{23}$$

Note that if $k=0$, $\alpha=\beta=1$, then from previous formula we get:

$$W(x,t)=\frac{1}{\sqrt{4\pi t}}\exp\left(\frac{-x^2}{4t}\right) \tag{24}$$

which is a classical result on the heat conduction equation.

New we consider the following fractional Fokker-Planck equation with the variable coefficient:

$$\frac{\partial W(x,t)}{\partial t^\alpha}=\frac{\partial[a(x)W(x,t)]}{\partial x^\beta}+\mu^2\frac{\partial}{\partial x^\beta}\frac{\partial[W(x,t)]}{\partial x^\beta} \tag{25}$$

By the two-scale transform [26, 27], eq. (25) can be written as:

$$\frac{\partial W(u,v)}{\partial v}=\frac{\partial[a(u)W(u,v)]}{\partial u}+\mu^2\frac{\partial^2[W(u,v)]}{\partial u^2} \tag{26}$$

We first let $p(u)=\dfrac{1}{4\mu^2}a^2(u)-\dfrac{1}{2}a'(u)$, and $y(u)$ is a positive solution of:

$$-\mu^2 y'' + p(u)y = \lambda y \tag{27}$$

Then we can construct a Darboux transformation [28]:

$$T = \mu\left[-\frac{\partial}{\partial u} + \frac{y'(u)}{y(u)}\right] \tag{28}$$

which has the following properties: if $\omega(x, t)$ satisfies:

$$\frac{\partial \omega}{\partial v} = \mu^2 \frac{\partial^2 \omega}{\partial u^2} - p(u)\omega \tag{29}$$

then $T\omega$ satisfies:

$$\frac{\partial T\omega}{\partial v} = \mu^2 \frac{\partial^2 T\omega}{\partial u^2} - Tp(u)T\omega \tag{30}$$

where

$$Tp = p - 2\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{y'}{y}\right) \tag{31}$$

When $p(u) = 0$, by (24), we have:

$$\omega(u,v) = \frac{1}{2\mu\sqrt{\pi v}}\exp\left(-\frac{u^2}{4\mu^2 v}\right) \tag{32}$$

Based on eq. (32) and the previous Darboux transformation, we can obtain some new results, for example:

When $a(x) = -1/(2\mu^2 x)$, by eq. (32), we get the solution of eq. (25):

$$W(x,t) = \frac{1}{2\mu\sqrt{\pi t^\alpha}}\exp\left(-\frac{x^{2\beta}}{4\mu^2 t^\alpha} - \frac{\ln x^\beta}{2\mu^2}\right) \tag{33}$$

When $a^2(x) - 2\mu^2 a'(x) = 2\csc^2 x,$ by eqs. (28) and (33), we have:

$$W(x,t) = \left(\frac{x^\beta}{4\mu^2 t^\alpha} + \frac{\cot x^\beta}{2\sqrt{\pi t^\alpha}}\right)\exp\left(-\frac{x^{2\beta}}{4\mu^2 t^\alpha} - \frac{\ln x^\beta}{2\mu^2}\right) \tag{34}$$

When $a(x) = -2\tanh x$, by eqs. (28) and (32), we obtain:

$$W(x,t) = \frac{\cosh x^\beta}{\sqrt{1+\mu^2 t^\alpha}}\exp\left[-\frac{x^{2\beta}}{2(1+\mu^2 t^\alpha)} - \mu^2 t^\alpha\right] \tag{35}$$

Discussion and conclusion

Fokker-Planck equation admits a variational formulation [2], and the variational principle for the present fractional Fokker-Planck equation is still unknown. Wang *et al.* [29] and

Wang and He [30] applied the semi-inverse method and a obtained a fractal variational principle in a fractal medium by the fractal derivative [31], which might be extended to the present study.

The Fourier transform and the Darboux transformation are the powerful tool to the fractional Fokker-Planck equation. By Fourier transform method, we have solved the linear fractional Fokker-Planck equation. For the fractional Fokker-Planck equation with the variable coefficient, we obtain some new results by using the Darboux transformation.

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