THE REPRODUCING KERNEL FOR THE REACTION-DIFFUSION MODEL WITH A TIME VARIABLE FRACTIONAL ORDER

by

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The variable-order fractional calculus has become a useful mathematical framework to describe a complex reaction-diffusion process. It is very hard to solve the problem, and there is almost no analytical method available in open literature. In this article, the reproducing kernel method is proposed for this purpose, and some examples show that the method is of high precision.

Key words: *reaction-diffusion model, variable fractional derivative, reproducing kernel space*

Introduction

Recently one of the most important and interesting problems is to search for an approximate solution of a reaction-diffusion model with a time variable fractional order [1-3]:

$$\begin{cases} D_t^{\alpha(x,t)}u(x,t)+\beta_1(x,t)\dfrac{\partial^2 u(x,t)}{\partial x^2}+\beta_2(x,t)\dfrac{\partial u(x,t)}{\partial x}=F(x,t) \\ u(x,0)=u_0(x), \quad u(0,t)=\phi_1(t), \quad u(1,t)=\phi_2(t) \end{cases} \qquad 0\le x, \quad t\le 1 \tag{1}$$

where $\alpha(x,t), \beta_1(x,t), \beta_2(x,t), f(x,t)$ are known functions, $0<\alpha(x,t)\le 1$, $D_t^{\alpha(x,t)}u(x,t)$ is the Caputo fractional derivative.

Definition 1. The Caputo fractional derivative operator of order $0<\alpha(x,t)\le 1$ is defined:

$$D_t^{\alpha(x,t)}u(x,t)=\begin{cases} \dfrac{1}{\Gamma[1-\alpha(x,t)]}\displaystyle\int_0^t (t-\tau)^{-\alpha(x,t)}\dfrac{\partial u(x,\tau)}{\partial \tau}\mathrm{d}\tau, & 0<\alpha(x,t)<1 \\ \dfrac{\partial u(x,t)}{\partial t}, & \alpha(x,t)=1 \end{cases} \tag{2}$$

In recent years, there has been a growing interest in using the reproducing kernel Hilbert space method to solve operator equations [4-12]. In this paper, we use the method to solve eq. (1). Its analytical solution is represented in the form of series, which uniformly converges to the exact one. The reproducing kernel Hilbert space is defined as follows.

Definition 2. Let H be a real Hilbert space of function $f:\Omega\to R.$ A function $K:\Omega\times\Omega\to R$ is called a reproducing kernel for H if:

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– $K(X, \cdot) \in H$ for all $x \in \Omega$,
– $f(x) = \langle f, K(\cdot, x)\rangle_H$ for all $f \in H$ and all $x \in \Omega$.

It is known that a real Hilbert space H on a set Ω is called a reproducing kernel Hilbert space if there exists a reproducing kernel K of H. The existence of a reproducing kernel is due to the Riesz representation theorem, and the reproducing kernel of a Hilbert space is unique, the reproducing kernel K of a Hilbert space H completely determines the space H.

For solving eq. (1), let:

$$v(x,t) = u(x,t) - \omega(x,t)$$

where

$$v(x,t) = u_0(x) + (1-x)\varphi_1(t) + x\varphi_2(t) - (1-x)\varphi_1(0) - x\varphi_2(0)$$

So, eq. (1) can be turn into eq. (3):

$$\begin{cases} \mathrm{D}_t^{\alpha(x,t)} v(x,t) + \beta_1(x,t)\dfrac{\partial^2 v(x,t)}{\partial x^2} + \beta_2(x,t)\dfrac{\partial v(x,t)}{\partial x} = f(x,t) \\ u(x,0) = u(0,t) = u(1,t) = 0 \end{cases} \quad 0 \le x, \quad t \le 1 \tag{3}$$

where

$$f(x,t) = -\mathrm{D}_t^{\alpha(x,t)}\omega(x,t) - \beta_1(x,t)\frac{\partial^2 \omega(x,t)}{\partial x^2} - \beta_2(x,t)\frac{\partial \omega(x,t)}{\partial x} + F(x,t)$$

Using the method of [13-18], we can get a reproducing kernel of eq. (3), which is:

$$K_{(x,t)}(y,s) = R_1(t,s) \times R_2(x,y)$$

where

$$R_1(t,s) = \begin{cases} -\dfrac{s^3}{6} + \dfrac{1}{2}ts(2+s), & s \le t \\ -\dfrac{1}{6}t(s^2 - 6s - 3ts), & t < s \end{cases} \tag{4}$$

$$R_2(x,y) = \begin{cases} \dfrac{\begin{aligned} &y\{-x(120+30x+10x^2-5x^3+x^4)(120+30y+10y^2-5y^3+y^4)+ \\ &+156[y^4+10x^2y(3+y)-5x(-24+y^3)]\} \end{aligned}}{18720}, & y \le x \\ \dfrac{\begin{aligned} &x[156(x^4+120y-5x^3y+30xy^2+10x^2y^2)-(120+30x+10x^2-5x^3+x^4) \\ &y(120+30y+10y^2-5y^3+y^4)] \end{aligned}}{18720}, & x < y \end{cases} \tag{5}$$

Let

$$Lv(x,t) = \mathrm{D}_t^{\alpha(x,t)} v(x,t) + \beta_1(x,t)\frac{\partial^2 v(x,t)}{\partial x^2} + \beta_2(x,t)\frac{\partial v(x,t)}{\partial x}$$

Representation of analytical solution and approximate solution

Theorem 1. If $\{x_i, t_i\}_{i=1}^{\infty}$ is countable dense points in [0, 1]×[0, 1]. Let $\bar{\varphi}_i(x,t) = \Sigma_{k=1}^{i}\beta_{ik}\psi_k(x,t)$, where β_{ik} are the coefficients resulting from Gram-Schmid orthogonalization, $\psi_i(x,t)=L_{(y,s)}K_{(x,t)}(y,s)|_{(y,s)=(x_i,t_i)}$, then $v(x,t)=\Sigma_{i=1}^{\infty}\Sigma_{k=1}^{i}\beta_{ik}f(x_k,t_k)\bar{\psi}_i(x,t)$ is an analytical solution of eq. (3).

Proof. The function $v(x, t)$ can be expanded to Fourier series in terms of normal orthogonal basis $\{\bar{\psi}_i(x,t)\}_{i=1}^{\infty}$:

$$\begin{aligned}
v(x,t) &= \sum_{i=1}^{\infty}\langle v(x,t),\bar{\psi}_i(x,t)\rangle\bar{\psi}_i(x,t)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle v(x,t),\psi_k(x,t)\rangle\bar{\psi}_i(x,t)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle v(x,t),(L_{(y,s)}K_{(x,t)}(y,s)(x_k,t_k)\right\rangle\bar{\psi}_i(x,t)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(L_{(y,s)}\left\langle v(x,t),K_{(x,t)}(y,s)\right\rangle)(x_k,t_k)\bar{\psi}_i(x,t)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(L_{(y,s)}\left\langle v(x,t),K_{(x,t)}(y,s)\right\rangle)(x_k,t_k)\bar{\psi}_i(x,t)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(L_{(y,s)}\left\langle v(x,t),K_{(x,t)}(y,s)\right\rangle)(x_k,t_k)\bar{\psi}_i(x,t)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(L_{(y,s)}v(y,s))(x_k,t_k)\bar{\psi}_i(x,t)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k,t_k)\bar{\psi}_i(x,t)
\end{aligned}$$

we define an approximate solution $v_n(x,t)$ by:

$$v_n(x,t)=\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}f(x_k,t_k)\bar{\psi}_i(x,t) \tag{6}$$

In calculation, let $\varphi_i(x,t)=K_{(x_i,t_i)}(x,t), v_n(x,t)=\Sigma_{n=1}^{n}c_i\varphi_i(x,t)$ The coefficient c_i, $i = 1, \ldots n$ are determined by eqs. (16)-(18).

$$\sum_{i=1}^{m}c_iL\varphi_i(x,t)|_{(x,t)=(x_j,t_j)}=f(x_j,t_j), \quad j=1,2,\ldots n \tag{7}$$

Numerical experiments

*Example 1***.** Consider the following reaction-diffusion model [1]:

$$\begin{cases}\mathrm{D}_t^{\alpha(x,t)}u(x,t)+\dfrac{x^2}{200}\dfrac{\partial^2u(x,t)}{\partial x^2}+\dfrac{3x}{50}\dfrac{\partial u(x,t)}{\partial x}-\dfrac{3}{50}u(x,t)=0 & 0\le x, \quad t\le 1\\ u(x,0)=e^x+x+1, \quad u(0,t)=t^{\alpha}+2, \quad u(1,t)=t^{\alpha}+e+2\end{cases} \tag{8}$$

The exact solution is $u(x,t)=t^{\alpha(x,t)}+e^x+x+1.$. Numerical results are given in tab. 1 and figs. 1 and 2.

Table 1. Comparison of absolute errors for *Example 1*

$\alpha(x, t)$	[17]	Present method
$\alpha(x, t) = 0.1$	$1.5110\cdot10^{-5}$	$2.43467\cdot10^{-15}$
$\alpha(x, t) = 0.3$	$2.9460\cdot10^{-5}$	$2.42733\cdot10^{-15}$
$\alpha(x, t) = 0.5$	$3.2753\cdot10^{-5}$	$2.40098\cdot10^{-15}$
$\alpha(x, t) = 0.7$	$3.2559\cdot10^{-5}$	$2.37873\cdot10^{-15}$
$\alpha(x, t) = 0.9$	$3.0953\cdot10^{-5}$	$2.32409\cdot10^{-15}$

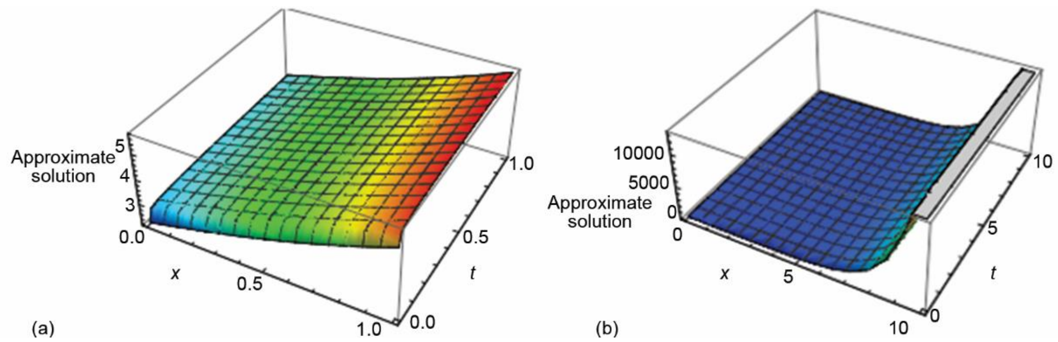

Figure 1. The approximate solution $u_{25}(x, t)$ obtained by present method for *Example 1* with $\alpha(x, t) = 0.1$

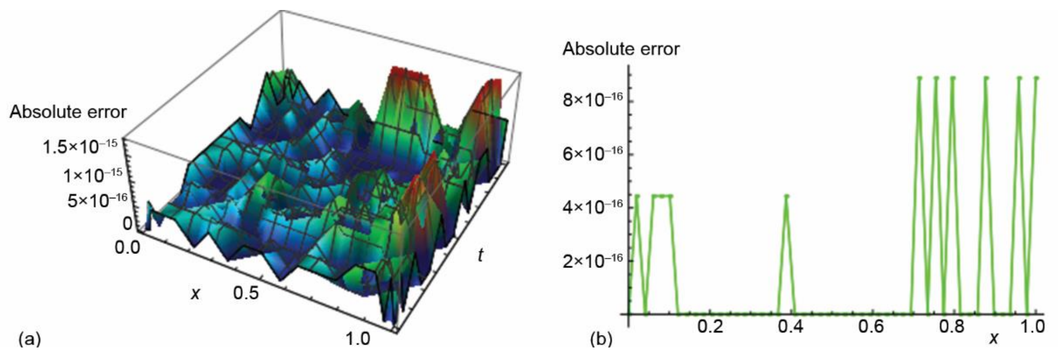

Figure 2. Absolute errors obtained by present method for *Example 1* with $\alpha(x, t)= 0.1$; (a) $|u(x, t) - u_{25}(x, t)|$, (b) $|u(x, 1) - u_{25}(x, 1)|$

Example 2. Consider the following time-fractional reaction-diffusion model [2]

$$\begin{cases} D_t^{\alpha(x,t)}u(x,t) + \dfrac{x^2}{50}\dfrac{\partial^2 u(x,t)}{\partial x^2} + \dfrac{x}{20}\dfrac{\partial u(x,t)}{\partial x} - \dfrac{1}{20}u(x,t) = 0 \\ u(x,0) = \sin(x^\alpha) + 1, \quad u(0,t) = \cos(t^3), \quad u(1,t) = \sin(1) + \cos(t^3) \end{cases} \quad 0 \le x, \quad t \le 1 \qquad (9)$$

The exact solution is $u(x, t) = \sin(x^\alpha) + \cos(t^3)$. Numerical results are given in fig. 3.

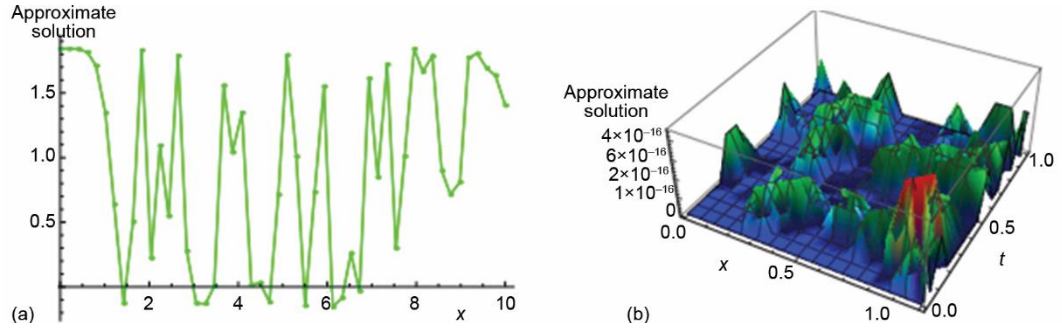

Figure 3. The results obtained by present method for *Example 2* with *α*(*x, t*)= 0.5; (a) the approximate solution $u_{100}(1, t)$, (b) the absolute error $|u(x, t) - u_{100}(x, t)|$

Example 3. Consider the following time variable fractional order reaction-diffusion model:

$$\begin{cases} D_t^{\alpha(x,t)}u(x,t) - \dfrac{\partial^2 u(x,t)}{\partial x^2} + \dfrac{\partial u(x,t)}{\partial x} = f(x,t) & 0 \le x, \quad t \le 1 \\ u(x,0) = 5x(1-x), \quad u(0,t) = 0, \quad u(1,t) = 0 \end{cases} \tag{10}$$

where

$$\alpha(x,t) = 0.8 + 0.005\cos(xt)\sin(x).$$

$$f(x,t) = 5x(1-x) + \frac{5x(1-x)t^{1-\alpha(x,t)}}{\alpha[2-\alpha(x,t)]} + 5(t+1)(1-2x) + 10(t+1)$$

The exact solution is $u(x,t) = 5(t+1)x(1-x)$. Numerical results of *Example 3* are given in fig. 4.

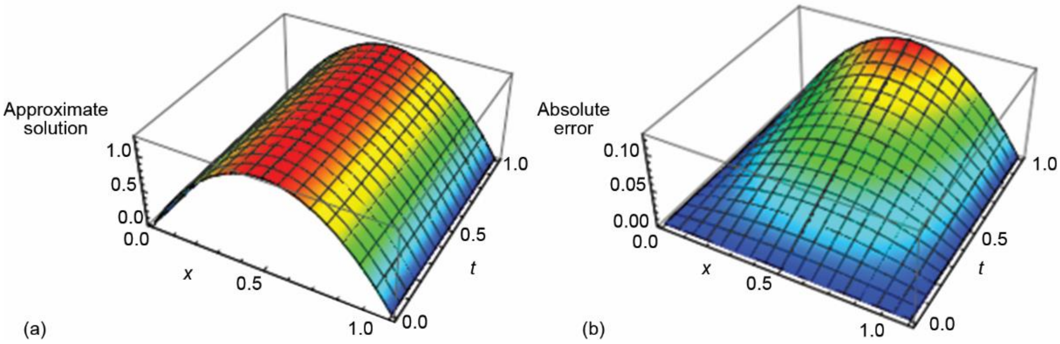

Figure 4. The result obtained by present method for *Example 3*; (a) The approximate solution $u_{25}(x, t)$, (b) The absolute error $|u(x, t) - u_{25}(x, t)|$

Example 4. Consider the following time-fractional reaction-diffusion model [3]:

$$\begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial x^\alpha} + \dfrac{\partial u(x,t)}{\partial x} - xu(x,t) = f(x,t), & 0 \le x, t \le 1 \\ u(x,0) = 0, u(0,t) = 0, \end{cases} \tag{11}$$

when

$$f(x,t)=t^{\frac{1}{10}}(2-3x-x^2+x^3)+\frac{t^2x\left[231(1+\sqrt{5})\pi(-1+x)x\right]}{200\,\Gamma[0.1]},$$

the exact solution is $u(x,t)=x^2(1-x)t^{(2+\alpha)}$. Numerical results are given in figs. 5 and 6.

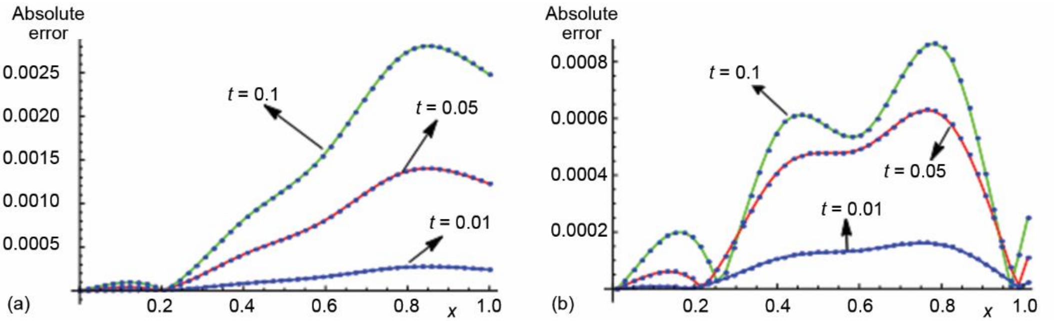

Figure 5. Absolute errors of *u* obtained by present method for *Example 4*, (a) α = 0.8, *m* = 5, (b) α = 0.01, *m* = 5

Conclusion

A new method is provided to solve a class of reaction-diffusion models with time variable fractional orders in a very favorable reproducing kernel space. Numerical experiments show that the algorithm is accurate and efficient. The present method can be extended to fractal calculus with variable fractal orders [19-28].

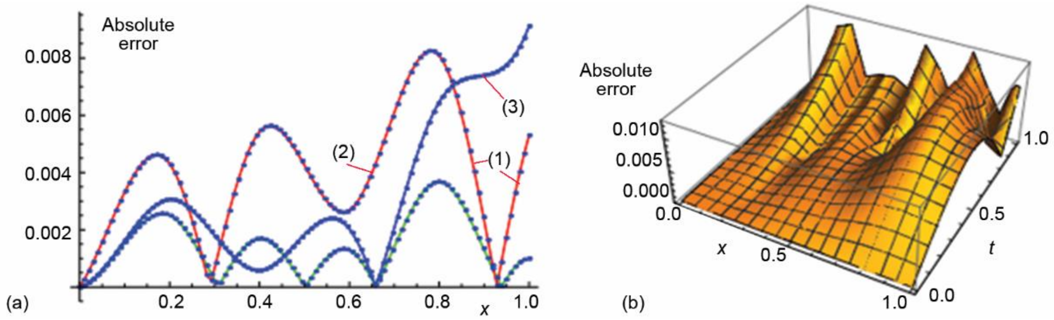

Figure 6. Absolute errors of *u* obtained by present method for *Example 4*; (a) α = 0.8 (1), α = 0.5 (2), α = 0.01 (3), *t* = 0.5, *m* = 5, (b) α = 0.8, *m* = 5

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