THE FRACTIONAL RESIDUAL METHOD FOR SOLVING THE LOCAL FRACTIONAL DIFFERENTIAL EQUATIONS

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This paper proposes a new method to solve local fractional differential equation. The method divides the studied equation into a system, where the initial solution is obtained from a residual equation. The new method is therefore named as the fractional residual method. Examples are given to elucidate its efficiency and reliability.

Key words: *local fractional derivative, residual method*

Introduction

Fractal calculus and fractional calculus have seen remarkable development due to their exact description of non-differentiable problems. There are many definitions of the fractional differential derivative in open literature, for example, the Caputo derivative, the Riemann-Liouville derivative, the Grunwald-Letnikov derivative, He's fractional derivative, and fractal derivative [1-10]. This paper will adopt the local fractional derivative [11, 12], which could describe the non-differential functions defined on Cantor sets.

Preliminaries of local fractional calculus

In this section, we introduce some mathematical preliminaries of the local fractional calculus in a fractal space for our subsequent development [11].

Definition 1. Suppose that there is [11]:

$$|u(t)-u(t_0)|<\varepsilon^{\alpha} \tag{1}$$

with $|t-t_0|<\delta,$ for $\varepsilon,\delta>0$ and $\varepsilon,\delta\in R,$ then $u(t)$ is called local fractional continuous at $t = t_0$ and it is denoted by $\lim_{t\to t_0} u(t)=u(t_0).$

Definition 2. Suppose that the function $u(t)$ is satisfied the condition (1) for $t\in(a,b),$ it is called local fractional continuous on the interval (a, b), denoted by:

$$u(t)\in C_{\alpha}(a,b) \tag{2}$$

Definition 3. In fractal space, let $u(t)\in C_{\alpha}(a,b),$ the local fractional derivative of $u(t)$ of order α at $t = t_0$ is given by [11]:

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$$D_t^{(\alpha)}u(t_0) = u^{(\alpha)}(t_0) = \frac{d^\alpha u(t)}{dt^\alpha}\Big|_{t=t_0} = \lim_{t\to t_0}\frac{\Delta^\alpha[u(t)-u(t_0)]}{(t-t_0)^\alpha} \tag{3}$$

where $\Delta^\alpha[u(t)-u(t_0)] \cong \Gamma(1+\alpha)\Delta[u(t)-u(t_0)]$.

Definition 4. [11] Let the function $u(t)$ satisfied the condition (2), the local fractional integral of $u(t)$ of order α in the interval $[a, b]$ is defined by:

$${}_aI_b^{(\alpha)}u(t) = \frac{1}{\Gamma(1+\alpha)}\int_a^b u(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1} u(t_j)(\Delta t_j)^\alpha \tag{4}$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, ...\}$, and $[t_j, t_{j+1}]$, $j = 0,..., N-1$, $t_0 = a$, $t_N = b$, is a partition of the interval $[a,b]$.

The fractional residual method

In order to elucidate the basic solution process of the new method, we consider the following local fractional differential equations on a fractal set:

$$Lu(x,t) + R(u,x,t) = g(x,t) \tag{5}$$

where L is a linear differential operator, R – a linear or non-linear operator, and $g(x, t)$ – an inhomogeneous term.

In order to solve eq. (5), we apply the local fractional reverse operator $L^{-1}(\cdot)$ on both sides of eq. (5), then we obtain:

$$u(x,t) = \tilde{u}(x,t) - L^{-1}[R(u,x,t) - g(x,t)] \tag{6}$$

where $\tilde{u}(x,t)$ is derived from the initial conditions.

Supposing $u_0(x, t)$ is a function to be determined, we can rewrite eq. (6):

$$u(x,t) = \tilde{u}(x,t) + L^{-1}u_0 - L^{-1}[u_0 + R(u,x,t) - g(x,t)] \tag{7}$$

If we let:

$$L^{-1}[u_0 + R(u,x,t) - g(x,t)] = 0 \tag{8}$$

and solve eq. (8), we can determine $u_0(x,t)$. Then substituting $u_0(x,t)$ into eq. (7), we can get the exact solution of eq. (5):

$$u(x,t) = \tilde{u}(x,t) + L^{-1}u_0 \tag{9}$$

In our new method, it is important to choose $u_0(x,t) = 0$ in the solution process. Because the main step of this method is to cut eq. (5) into one equation system, so we call this method as the fractional residual method, it can be easily proved that eq. (9) can be obtained as the first order approximate solution by the homotopy perturbation method [13-17].

Illustrative examples

In this section, to demonstrate the effectiveness of the method, several non-linear partial differential equations are presented.

Example 1. Consider the following local fractional differential equations

$$\frac{\partial^{3\alpha}u}{\partial x^{3\alpha}}(x,t)-\frac{\Gamma(1+\alpha)}{2t^{\alpha}}\frac{\partial^{3\alpha+\alpha}u}{\partial x^{3\alpha}\partial t^{\alpha}}(x,t)-\frac{1}{2^{\alpha}}=0 \tag{10}$$

subject to the initial conditions:

$$u(0,t)=u_x^{(\alpha)}(0,t)=u_{xx}^{(2\alpha)}(0,t)=0 \tag{11}$$

Obviously:

$$R(u,x,t)=-\frac{\Gamma(1+\alpha)}{2t^{\alpha}}\frac{\partial^{3\alpha+\alpha}u}{\partial x^{3\alpha}\partial t^{\alpha}}(x,t)$$

is a non-linear operator and $g(x, t) = -1/2^{\alpha}$ is an in-homogeneous term.

Applying the inverse operator $L^{-1}(\cdot) = {}_0I_x^{(3\alpha)}(\cdot)$ on both sides of eq. (10) and making use of eq. (11), we obtain:

$$u(x,t)={}_0I_x^{(3\alpha)}(u_0)-{}_0I_x^{(3\alpha)}\left[u_0-\frac{\Gamma(1+\alpha)}{2t^{\alpha}}\frac{\partial^{3\alpha+\alpha}u}{\partial x^{3\alpha}\partial t^{\alpha}}(x,t)-\frac{1}{2^{\alpha}}\right] \tag{13}$$

According to eqs. (8) and (9), we let:

$$u(x,t)={}_0I_x^{(3\alpha)}(u_0) \tag{14}$$

and

$${}_0I_x^{(3\alpha)}\left[u_0-\frac{\Gamma(1+\alpha)}{2t^{\alpha}}\frac{\partial^{3\alpha+\alpha}u}{\partial x^{3\alpha}\partial t^{\alpha}}(x,t)-\frac{1}{2^{\alpha}}\right]=0 \tag{15}$$

By virtue of eq. (15), we let:

$$u_0-\frac{\Gamma(1+\alpha)}{2t^{\alpha}}\frac{\partial^{3\alpha+\alpha}u}{\partial x^{3\alpha}\partial t^{\alpha}}(x,t)-\frac{1}{2^{\alpha}}=0 \tag{16}$$

Substituting eq. (14) into eq. (16), we can derive:

$$u_{0t}^{(\alpha)}-2\frac{t^{\alpha}}{\Gamma(1+\alpha)}u_0-t^{\alpha}=0 \tag{17}$$

Solving eq. (17), we can get:

$$u_0=-\frac{1}{2^{\alpha}}+CE_{\alpha}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \tag{18}$$

Substituting eq. (18) into eq. (14), we get the following exact solution of eq. (10):

$$u(x,t)=\frac{x^{3\alpha}}{2\Gamma(1+3\alpha)}-C\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}E_{\alpha}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \tag{19}$$

Example 2. Consider the following Korteweg-de Vries -like equation:

$$u_t^{(\alpha)}+uu_x^{(\alpha)}+u_x^{(3\alpha)}=E_{\alpha}[2(x-t)^{\alpha}] \tag{20}$$

subject to the initial conditions:

$$u(0,t)=u_x^{(\alpha)}(0,t)=u_{xx}^{(2\alpha)}(0,t)=E_\alpha(-t^\alpha) \tag{21}$$

Obviously, $L(u)=u_{xxx}^{(3\alpha)}$ is a linear operator, $N(u,x,t)=u_t^{(\alpha)}+uu_x^{(\alpha)}$ is a non-linear operator and $f(x, t) = E_\alpha[2(x-t)]^\alpha$ is an in-homogeneous term.

Applying the inverse operator $L^{-1}(\cdot) = {}_0I_x^{(3\alpha)}(\cdot)$ on both sides of eq. (20) and making use of eq. (21), we obtain:

$$u(x,t)=E_\alpha(-t^\alpha)\left[1+\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\right]+{}_0I_x^{(3\alpha)}(u_0)-$$

$$-{}_0I_x^{(3\alpha)}\{u_0+u_t^{(\alpha)}+uu_x^{(\alpha)}-E_\alpha[2(x-t)^\alpha]\} \tag{22}$$

According to eqs. (8) and (9), we let:

$$u(x,t)=E_\alpha(-t^\alpha)\left[1+\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\right]+{}_0I_x^{(3\alpha)}(u_0) \tag{23}$$

and

$${}_0I_x^{(3\alpha)}\{u_0+u_t^{(\alpha)}+uu_x^{(\alpha)}-E_\alpha[2(x-t)^\alpha]\}=0 \tag{24}$$

As we know:

$$E_\alpha[2(x-t)^\alpha]=E_\alpha(-2t^\alpha)\sum_{n=0}^{\infty}\frac{2^{n\alpha}}{\Gamma(1+n\alpha)}x^{n\alpha} \tag{25}$$

If we suppose:

$$u_0(x,t)=\sum_{n=1}^{\infty}a_n(t)\frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \tag{26}$$

where $a_n(t)$ are all functions to be determine.

Substituting eqs. (23), (25), and (26) into eq. (24), we can deduce:

$$0=\frac{1}{\Gamma(1+3\alpha)}\{a_0(t)+a_0'(t)+a_0(t)a_1(t)-E_\alpha[2(x-t)^\alpha]\}x^{3\alpha}+$$

$$+\frac{1}{\Gamma(1+4\alpha)}\{a_1(t)+a_1'(t)+a_0(t)a_2(t)+a_1^2(t)-2E_\alpha[2(x-t)^\alpha]\}x^{4\alpha}+$$

$$+\frac{1}{\Gamma(1+5\alpha)}\{a_2(t)+a_2'(t)+a_0(t)a_3(t)+3a_1(t)a_2(t)-4E_\alpha[2(x-t)^\alpha]\}x^{5\alpha}+$$

$$+\frac{1}{\Gamma(1+6\alpha)}\{a_3(t)+a_3'(t)+a_0(t)a_4(t)+4a_1(t)a_3(t)+3a_2^2(t)-8E_\alpha[2(x-t)^\alpha]\}x^{6\alpha}+$$

$$+\ldots \tag{27}$$

According to eq. (27), we can obtain:

$$a_0(t)=a_1(t)=a_2(t)=,\ldots,a_n(t)=E_\alpha(-t^\alpha) \tag{28}$$

Substituting eq. (28) into eq. (26) and then substituting the result into eq. (23), we can obtain the following exact solution of eq. (20):

$$u(x,t) = E_\alpha(x^\alpha)E_\alpha(-t^\alpha) \tag{29}$$

Example 3. Consider the following fractional differential equation:

$$u_{xxt}^{(3\alpha)} - \frac{\Gamma(1+\alpha)}{t^\alpha}u + 2u_t^{(\alpha)} + \frac{\Gamma(1+\alpha)}{t^\alpha}u_{xx}^{(2\alpha)} = 0 \tag{30}$$

subject to the initial conditions:

$$u(0,t) = 0, u_x^{(\alpha)}(0,t) = \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \quad u_{xx}^{(2\alpha)}(0,t) = 0 \tag{31}$$

Obviously, $L(u) = u_{xxt}^{(3\alpha)}$ is a linear operator,

$$R(u,x,t) = -\frac{\Gamma(1+\alpha)}{t^\alpha}u + 2u_t^{(\alpha)} + \frac{\Gamma(1+\alpha)}{t^\alpha}u_{xx}^{(2\alpha)}$$

is a non-linear operator and $f(x, t) = 0$ is an inhomogeneous term.

Applying the inverse operator $L^{-1}(\cdot) = {}_0I_t^{(\alpha)}{}_0I_x^{(2\alpha)}(\cdot)$ on both sides of eq. (30), and making use of eq. (31), we obtain:

$$u(x,t) = \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\frac{x^\alpha}{\Gamma(1+\alpha)} + {}_0I_t^{(\alpha)}\,{}_0I_x^{(2\alpha)}u_0 -$$

$$-{}_0I_t^{(\alpha)}\,{}_0I_x^{(2\alpha)}\left[u_0 - \frac{\Gamma(1+\alpha)}{t^\alpha}u + 2u_t^{(\alpha)} + \frac{\Gamma(1+\alpha)}{t^\alpha}u_{xx}^{(2\alpha)}\right] \tag{32}$$

According to eqs. (8) and (9), we let:

$$u(x,t) = \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\frac{x^\alpha}{\Gamma(1+\alpha)} + {}_0I_t^{(\alpha)}\,{}_0I_x^{(2\alpha)}u_0 \tag{33}$$

and

$${}_0I_t^{(\alpha)}\,{}_0I_x^{(2\alpha)}\left[u_0 - \frac{\Gamma(1+\alpha)}{t^\alpha}u + 2u_t^{(\alpha)} + \frac{\Gamma(1+\alpha)}{t^\alpha}u_{xx}^{(2\alpha)}\right] = 0 \tag{34}$$

We let :

$$u_0(x,t) = \sum_{m.n=0}^{\infty} a_{m,n}\frac{x^{m\alpha}t^{n\alpha}}{\Gamma(1+m\alpha)\Gamma(1+n\alpha)} \tag{35}$$

where $a_{m,n}$ are all constants to be determine.

Substituting eqs. (33) and (35) into eq. (34), we can derive:

$$0 = \sum_{m.n=0}^{\infty} a_{m,n}\frac{x^{(m+2)\alpha}}{\Gamma[1+(m+2)\alpha]}\frac{t^{(n+1)\alpha}}{\Gamma[1+(n+1)\alpha]} - \sum_{m.n=0}^{\infty} a_{m,n}\frac{x^{(m+4)\alpha}}{\Gamma[1+(m+4)\alpha]}\frac{t^{(n+1)\alpha}}{[(n+1)\alpha]\Gamma[1+(n+1)\alpha]} +$$

$$+2\sum_{m.n=0}^{\infty} a_{m,n}\frac{x^{(m+4)\alpha}}{\Gamma[1+(m+4)\alpha]}\frac{t^{(n+1)\alpha}}{\Gamma[1+(n+1)\alpha]} + \sum_{m.n=0}^{\infty} a_{m,n}\frac{x^{(m+2)\alpha}}{\Gamma[1+(m+2)\alpha]}\frac{t^{(n+1)\alpha}}{[(n+1)\alpha]\Gamma[1+(n+1)\alpha]} +$$

$$+\frac{3x^{3\alpha}t^{2\alpha}}{2\Gamma(1+3\alpha)\Gamma(1+2\alpha)} \tag{36}$$

Comparing with the same coefficient of $x^{m\alpha}t^{n\alpha}$ in eq. (36), we can obtain:

$$a_{m,n}=0, \quad (n\neq 1),$$
$$a_{m,1}=\begin{cases}(-1)^k, & m=2k-1\\ 0, & m=2k\end{cases} \tag{37}$$

Substituting eq. (37) into eq. (35), and then, substituting the result into eq. (33), we can get the following exact solution of eq. (30):

$$u(x,t)=\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\frac{x^{\alpha}}{\Gamma(1+\alpha)}+{}_0I_t^{(\alpha)}\,{}_0I_x^{(2\alpha)}u_0=$$
$$=\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\sum_{m.=0}^{\infty}a_{m,1}\frac{x^{(m+2)\alpha}}{\Gamma[1+(m+2)\alpha]}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}=$$
$$=\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\sum_{k=1}^{\infty}(-1)^k\frac{x^{(2k+1)\alpha}}{\Gamma[1+(2k+1)\alpha]}=$$
$$=\frac{t^{2\alpha}}{\Gamma\left(1+2\alpha\right)}\sin_{\alpha}x^{\alpha} \tag{38}$$

Example 4. Consider the following non-linear gas dynamic like equation:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}+u(x,t)\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}-u(x,t)[1-u(x,t)]=0 \tag{39}$$

subject to the boundary and initial conditions:

$$\begin{cases}u(t,0)=u(t,\pi)=0\\ u(x,0)=\dfrac{\partial^{\alpha}u(x,0)}{\partial t^{\alpha}}=\sin_{\alpha}x^{\alpha}\end{cases} \tag{40}$$

Applying the inverse operator $L^{-1}(\cdot)={}_0I_t^{(3\alpha)}(\cdot)$ on both sides of eq. (39), and making use of eq. (40), we obtain:

$$u(x,t)=\sin_{\alpha}x^{\alpha}\left[1+\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right]+{}_0I_t^{(2\alpha)}u_0-$$
$${}_0I_t^{(2\alpha)}\left[u_0+u(x,t)\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}-u(x,t)[1-u(x,t)]\right] \tag{41}$$

According to eqs. (8) and (9), we let:

$$u(x,t)=\sin_{\alpha}x^{\alpha}\left[1+\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right]+{}_0I_t^{(2\alpha)}u_0 \tag{42}$$

and

$$ {}_0I_t^{(2\alpha)}\left[u_0+u(x,t)\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}-u(x,t)[1-u(x,t)]\right]=0 \tag{43}$$

We suppose:

$$u_0(x,t)=\sum_{n=1}^{\infty}u_n(t)\sin_\alpha(nx)^\alpha \tag{44}$$

where $u_n(t)$ are all constants to be determine.

Substituting eqs. (42) and (44) into eq. (43), we can obtain:

$$\sum_{n=1}^{\infty}u_n(t)\sin_\alpha(nx)^\alpha-\sin_\alpha x^\alpha\left[1+\frac{t^\alpha}{\Gamma(1+\alpha)}\right]-\sum_{n=1}^{\infty}\sin_\alpha(nx)^\alpha\,{}_0I_t^{(2\alpha)}u_n(t)+$$

$$+u(x,t)\left[-\sum_{n=1}^{\infty}(n)^{2\alpha}\sin_\alpha(nx)^\alpha\,{}_0I_t^{(2\alpha)}u_n(t)+\sum_{n=1}^{\infty}\sin_\alpha(nx)^\alpha\,{}_0I_t^{(2\alpha)}u_n(t)\right]=0 \tag{45}$$

Comparing the coefficient of like powers of $\sin_\alpha(nx)^\alpha$ of eq. (45), the following equations are obtained, respectively:

$$u_n(t)=0, n=2,3... \tag{46}$$

$$u_1(t)-1-\frac{t^\alpha}{\Gamma(1+\alpha)}-{}_0I_t^{(2\alpha)}u_1(t)=0 \tag{47}$$

Solving eq. (47), we get:

$$u_1(t)=E_\alpha(t^\alpha) \tag{48}$$

Substituting eqs. (46) and (48) into eq. (43), and then, substituting the result into eq. (42), we can get the following exact solution of eq. (39):

$$u(x,t)=E_\alpha(t^\alpha)\sin_\alpha(x^\alpha) \tag{49}$$

Conclusions

Fractal calculus has been attracting much attention from various communities, the fractional order can be determined experimentally by calculating the fractional dimensions of the studied fractal medium [18, 19].

In this article, we have suggested a new method called the fractional residual method for solving the local fractional equation. Our method is very simply and straight-forward as compared with other methods, *e. g.*, the variational iteration method [20, 21]. The test examples are showed that the suggested method can be regarded as a simple and efficient tool for computing local fractional differential equations.

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