DARBOUX TRANSFORM AND CONSERVATION LAWS OF NEW DIFFERENTIAL-DIFFERENCE EQUATIONS

by

Sheng ZHANG * ***and Dongdong LIU***

School of Mathematics and Physics, Bohai University, Jinzhou, China

Original scientific paper
https://doi.org/10.2298/TSCI2004519Z

Darboux transforms, exact solutions and conservation laws are important topics in thermal science and other fields as well. In this paper, the new non-linear differential-difference equations associated a discrete linear spectral problem are studied analytically. Firstly, the Darboux transform of the studied equations is constructed, and exact solutions are then obtained. Finally, infinite many conservation laws are derived.

Key words: *Darboux transform, conservation law, exact solution, discrete linear spectral problem, differential-difference equations*

Introduction

When continuum hypothesis is no longer valid, fractal calculus [1-5], two-scale thermodynamics [6, 7] and differential-difference equations (DDE) [8, 9] often are considered as useful candidates to describe some phenomena like those arising in heat/electron conduction, flow in carbon nanotubes and nanoporous materials. Discussing the conservation laws [10] and variational formulations [11, 12] plays an important role in the study of integrability of soliton equations. For a given (1+1)-D DDE, the conservation law of which can be written as $\partial T/\partial t + (E-1)X = 0$, here T – the conservation density and X – the conservation flow, which are related to the potential function $u(n, t)$. In the field of non-linear mathematical physics, there exist many effective methods, for examples, the exp-function method [13, 14], the variational iteration method [15], the homotopy perturbation method [16, 17], the variational method [11, 12, 18, 19], the Taylor series method [20], He's frequency formulation [21], for solving non-linear PDE. Among them, the Darboux transform (DT) [22] is a gauge transform which transforms the spectral problem into another spectral problem of the same form. Based on a suitable seed solution, the DT can be utilized to construct exact solutions of non-linear PDE.

In this paper, we would like to construct conservation laws [23-25], DT, and exact solutions of the following new non-linear DDE [26]:

$$r_{n,t} = r_n\left(\frac{s_{n+1}}{r_{n+1}} - \frac{s_n}{r_n}\right), \quad s_{n,t} = s_n\left(\frac{s_{n+1}}{r_{n+1}} - \frac{s_n}{r_n}\right) + r_n\left(\frac{1}{r_n} - \frac{1}{r_{n-1}}\right) \tag{1}$$

which are associated with the discrete linear spectral problem:

$$E\varphi_n = \varphi_{n+1} = U_n\varphi_n \tag{2}$$

* Corresponding author, e-mail: szhangchina@126.com

$$\varphi_{n,t} = V_n \varphi_n \tag{3}$$

where

$$U_n = \begin{pmatrix} \lambda r_n + s_n & r_n \\ 1 & 0 \end{pmatrix} \tag{4}$$

$$V_n = \begin{pmatrix} \dfrac{\lambda}{2} + \dfrac{s_n}{r_n} & 1 \\ \dfrac{1}{r_{n-1}} & -\dfrac{\lambda}{2} \end{pmatrix} \tag{5}$$

where $\lambda_t = 0$ is the spectral parameter, $\varphi_n = (\varphi_{1,n}, \varphi_{2,n})^T$ – the eigenfunction vector, T – the transpose of the matrix, E – the shift operator defined by $Ef(n,t) = f(n+1,t) \equiv f_{n+1}$, $E^{-1} f(n,t) = f(n-1,t) \equiv f_{n-1}$, $r_n = r_n(t)$ and $s_n = s_n(t)$ – the potential functions.

Darboux transform of eqs. (1)

Constructing the DT of eq. (1) is to find such a gauge transform:

$$\bar{\varphi}_n = T_n \varphi_n \tag{6}$$

of eqs. (2) and (3) so that $\bar{\varphi}_n$ satisfies another discrete linear spectral problem with the same formal Lax pair:

$$\bar{\varphi}_{n+1} = \bar{U}_n \bar{\varphi}_n \tag{7}$$

$$\bar{\varphi}_{n,t} = \bar{V}_n \bar{\varphi}_n \tag{8}$$

where T_n is a two-order undetermined matrix, and:

$$\bar{U}_n = \begin{pmatrix} \lambda \bar{r}_n + \bar{s}_n & \bar{r}_n \\ 1 & 0 \end{pmatrix} \tag{9}$$

$$\bar{V}_n = \begin{pmatrix} \dfrac{\lambda}{2} + \dfrac{\bar{s}_n}{\bar{r}_n} & 1 \\ \dfrac{1}{\bar{r}_{n-1}} & -\dfrac{\lambda}{2} \end{pmatrix} \tag{10}$$

Namely:

$$T_{n+1} U_n = \bar{U}_n T_n, \quad T_{n,t} + T_n V_n = \bar{V}_n T_n \tag{11}$$

For this purpose, we suppose that:

$$T_n = \begin{bmatrix} (1+b_n)\lambda + a_n & b_n \\ c_n & \lambda + d_n \end{bmatrix} \tag{12}$$

where a_n, b_n, c_n, and d_n are all functions of n and t to be determined later. Take two basic solutions of eqs. (2) and (3) when $\lambda = \lambda_i$:

$$\varphi_n(\lambda_j) = [\varphi_{1,n}(\lambda_j), \varphi_{2,n}(\lambda_j)]^T, \quad \psi_n(\lambda_j) = [\psi_{1,n}(\lambda_j), \psi_{2,n}(\lambda_j)]^T \tag{13}$$

If there exist constants γ_j (j = 1, 2) satisfying:

$$[(1+b_n)\lambda_j+a_n]\varphi_{1,n}(\lambda_j)+b_n\varphi_{2,n}(\lambda_j)-\gamma_j\{[(1+b_n)\lambda_j+a_n]\psi_{1,n}(\lambda_j)+b_n\psi_{2,n}(\lambda_j)\}=0 \tag{14}$$

$$c_n\varphi_{1,n}(\lambda_j)+(\lambda_j+d_n)\varphi_{2,n}(\lambda_j)-\gamma_j[c_n\psi_{1,n}(\lambda_j)+(\lambda_j+d_n)\psi_{2,n}(\lambda_j)]=0 \tag{15}$$

which can be equivalently rewritten as a linear system:

$$[(1+b_n)\lambda_j+a_n]+\sigma_j b_n=0,\quad c_n+\sigma_j(\lambda_j+d_n)=0 \tag{16}$$

where

$$\sigma_j=\sigma_j(n)=\frac{\varphi_{2,n}(\lambda_j)-\gamma_j\psi_{2,n}(\lambda_j)}{\varphi_{1,n}(\lambda_j)-\gamma_j\psi_{1,n}(\lambda_j)},\quad j=1,2 \tag{17}$$

then we solve eq. (16) and obtain:

$$a_n=\frac{\lambda_2\sigma_1(n)-\lambda_1\sigma_2(n)}{\lambda_2-\lambda_1+\sigma_2(n)-\sigma_1(n)},\quad b_n=\frac{\lambda_1-\lambda_2}{\lambda_2-\lambda_1+\sigma_2(n)-\sigma_1(n)} \tag{18}$$

$$c_n=\frac{\sigma_1(n)\sigma_2(n)(\lambda_1-\lambda_2)}{\sigma_1(n)-\sigma_2(n)},\quad d_n=\frac{\sigma_1(n)\lambda_1-\sigma_2(n)\lambda_2}{\sigma_2(n)-\sigma_1(n)} \tag{19}$$

$$\sigma_j(n+1)=\frac{1}{\lambda_j r_n+s_n+r_n\sigma_j(n)},\quad \sigma_j(n-1)=\frac{1-\sigma_j(n)(\lambda_j r_{n-1}+s_{n-1})}{r_{n-1}\sigma_j(n)} \tag{20}$$

From eqs. (12), (16), and (17), we can see that $\det T(\lambda)$ is a quadratic polynomial in λ. When $\det T(\lambda)=0$, eqs. (18)-(20) show a_{n+1}, b_{n+1}, c_{n+1}, and d_{n+1} satisfy the following equations:

$$r_n c_{n+1}=b_n,\quad r_n d_{n+1}=a_n r_n-s_n b_n$$

$$r_n a_{n+1}(1+b_n)=(1+b_{n+1})[(1+b_n)(s_n b_n+d_n r_n)-a_n b_n r_n] \tag{21}$$

$$[(a_{n+1}s_n+b_{n+1})(1+b_n)-r_n a_n a_{n+1}](1+b_n)=(1+b_{n+1})[(a_n s_n+r_n c_n)(1+b_n)-a_n^2 r_n] \tag{22}$$

Theorem 1. When the matrix $\overline{U}_n$ determined by eq. (7) has the same form as U_n in eq. (4), the transforms:

$$\overline{r}_n=\frac{1+b_{n+1}}{1+b_n}r_n,\quad \overline{s}_n=\frac{s_n(1+b_{n+1})+r_n a_{n+1}-a_n\overline{r}_n}{1+b_n} \tag{23}$$

map the old potential functions r_n and s_n into new potential functions $\overline{r}_n$ and $\overline{s}_n$.

Proof. Let $T_n^{-1}=T_n^*/\det T_n$ and:

$$T_{n+1}U_n T_n^*=\begin{bmatrix} f_{11}(\lambda,n) & f_{12}(\lambda,n)\\ f_{21}(\lambda,n) & f_{22}(\lambda,n)\end{bmatrix} \tag{24}$$

then we have:

$$f_{11}(\lambda,n)=\lambda^3 r_n(1+b_{n+1})+\lambda^2[a_{n+1}r_n+(s_n+d_nr_n)(1+b_{n+1})]+$$
$$+\lambda[b_{n+1}+a_{n+1}(s_n+d_nr_n)+(1+b_{n+1})(d_ns_n-c_nr_n)]+(a_{n+1}s_n+b_{n+1})d_n-a_{n+1}c_nr_n \tag{25}$$

$$f_{12}(\lambda,n)=\lambda^2 r_n(b_{n+1}+1)+\lambda[a_{n+1}r_n+(a_nr_n-b_ns_n)(1+b_{n+1})]+(a_na_{n+1}r_n-a_{n+1}b_ns_n-b_nb_{n+1})+$$
$$+\lambda[b_{n+1}+a_{n+1}(s_n+d_nr_n)+(1+b_{n+1})(d_ns_n-c_nr_n)]+(a_{n+1}s_n+b_{n+1})d_n-a_{n+1}c_nr_n \tag{26}$$

$$f_{12}(\lambda,n)=\lambda^2 r_n(b_{n+1}+1)+\lambda[a_{n+1}r_n+(a_nr_n-b_ns_n)(1+b_{n+1})]+$$
$$+(a_na_{n+1}r_n-a_{n+1}b_ns_n-b_nb_{n+1}) \tag{27}$$

$$f_{21}(\lambda,n)=\lambda^2(c_{n+1}r_n+1)+\lambda(c_{n+1}s_n+d_{n+1}+c_{n+1}d_nr_n+d_n)+$$
$$+(c_{n+1}d_ns_n+d_nd_{n+1}-c_nc_{n+1}r_n) \tag{28}$$

$$f_{22}(\lambda,n)=\lambda(c_{n+1}r_n-b_n)+(a_nc_{n+1}r_n-b_nc_{n+1}s_n-b_nd_{n+1}) \tag{29}$$

which hint $f_{11}(\lambda, n)$ is a cubic polynomial in λ, $f_{12}(\lambda, n)$ and $f_{21}(\lambda, n)$ are all quadratic polynomials in λ, and $f_{22}(\lambda, n) = 0$. Thus, eq. (24) can be rewritten:

$$T_{n+1}U_nT_n^*=(\det T_n)P_n \tag{30}$$

where

$$P_n=\begin{bmatrix} P_{11}^{(1)}(n)\lambda+P_{11}^{(0)}(n) & P_{12}^{(0)}(n) \\ P_{21}^{(0)}(n) & 0 \end{bmatrix} \tag{31}$$

From eqs. (30) and (31), we have:

$$T_{n+1}U_n=P_nT_n \tag{32}$$

and determine that:

$$P_{11}^{(1)}(n)=\frac{r_n(1+b_{n+1})}{1+b_n},\quad P_{11}^{(0)}(n)=\frac{s_n(1+b_{n+1})+r_na_{n+1}-\dfrac{a_nr_n(1+b_{n+1})}{1+b_n}}{1+b_n} \tag{33}$$

$$P_{12}^{(0)}(n)=\frac{r_n(1+b_{n+1})}{1+b_n},\quad P_{21}^{(0)}(n)=\frac{r_nc_{n+1}}{b_n} \tag{34}$$

by comparing the coefficients of λ^2, λ, and λ^0 in eq. (32).

In view of eqs. (21)-(23), we obtain:

$$P_{11}^{(1)}(n)=\bar r_n,\quad P_{11}^{(0)}(n)=\bar s_n,\quad P_{12}^{(0)}(n)=\bar r_n,\quad P_{21}^{(0)}(n)=1 \tag{35}$$

That is to say $P_n=\bar U_n$. The proof of *Theorem 1* is end.

Theorem 2. Under the transforms (6) and (23), the matrix $\bar{V}_n$ in eq. (8) can be determined through eq. (10) which has the same form as V_n in eq. (5).

Proof. Supposing $T_n^{-1} = T_n^*/\det T_n$ and letting:

$$(T_{n,t} + T_n V_n)T_n^* = \begin{bmatrix} g_{11}(\lambda,n) & g_{12}(\lambda,n) \\ g_{21}(\lambda,n) & g_{22}(\lambda,n) \end{bmatrix} \tag{36}$$

similarly we have:

$$g_{11}(\lambda,n) = \frac{1+b_n}{2}\lambda^3 + \lambda^2\left[\frac{a_n}{2} + \frac{s_n(1+b_n)}{r_n} + b_{n,t} + \frac{d_n(1+b_n)}{2}\right] + \\ + \lambda\left\{a_{n,t} + \frac{a_n s_n}{r_n} + \frac{b_n}{r_{n-1}} + d_n\left[\frac{a_n}{2} + \frac{s_n(1+b_n)}{r_n} + b_{n,t}\right] - c_n\left(1+\frac{b_n}{2}\right)\right\} + \\ + d_n\left(a_{n,t} + \frac{a_n s_n}{r_n} + \frac{b_n}{r_{n-1}}\right) - c_n(a_n + b_{n,t}) \tag{37}$$

$$g_{12}(\lambda,n) = (1+b_n)\lambda^2 + \lambda\left\{(a_n + b_{n,t})(1+b_n) + a_n\left(1+\frac{b_n}{2}\right) - b_n\left[\frac{a_n}{2} + \frac{s_n(1+b_n)}{r_n} + b_{n,t}\right]\right\} \\ + a_n(a_n + b_{n,t}) - b_n\left(a_{n,t} + \frac{a_n s_n}{r_n} + \frac{b_n}{r_{n-1}}\right) \tag{38}$$

$$g_{21}(\lambda,n) = \left(\frac{1}{r_{n-1}} + c_n\right)\lambda^2 + \lambda\left(c_{n,t} + \frac{c_n s_n}{r_n} + \frac{2d_n}{r_{n-1}} + c_n d_n\right) + \\ + d_n\left(c_{n,t} + \frac{c_n s_n}{r_n} + \frac{d_n}{r_{n-1}}\right) - (d_{n,t} + c_n)c_n \tag{39}$$

$$g_{22}(\lambda,n) = -\frac{1+b_n}{2}\lambda^3 - \left[\frac{d_n(1+b_n)}{2} + \frac{a_n}{2}\right]\lambda^2 + \\ + \left[(d_{n,t} + c_n)(1+b_n) - b_n\left(\frac{c_n}{2} + \frac{1}{r_{n-1}}\right) - \frac{a_n d_n}{2}\right]\lambda \\ - b_n\left(c_{n,t} + \frac{c_n s_n}{r_n} + \frac{d_n}{r_{n-1}}\right) + (d_{n,t} + c_n)a_n \tag{40}$$

which show $g_{11}(\lambda, n)$ and $g_{22}(\lambda, n)$ are all cubic polynomials in λ, $g_{12}(\lambda, n)$ and $g_{21}(\lambda, n)$ are all quadratic polynomials in λ. It is easy to see from eqs. (2) and (3) that:

$$\sigma_{j,t} = \frac{1}{r_{n-1}} - \left(\lambda_j + \frac{s_n}{r_n}\right)\sigma_j(n) - \sigma_j^2(n), \quad j = 1, 2 \tag{41}$$

From eqs. (18), (19) and (41), we have:

$$a_{n,t}=\frac{(\lambda_2\sigma_{1,t}-\lambda_1\sigma_{2,t})(\lambda_2-\lambda_1+\sigma_2-\sigma_1)-(\lambda_2\sigma_1-\lambda_1\sigma_2)(\sigma_{2,t}-\sigma_{1,t})}{(\lambda_2-\lambda_1+\sigma_2-\sigma_1)^2} \tag{42}$$

$$b_{n,t}=\frac{(\lambda_2-\lambda_1)(\sigma_{2,t}-\sigma_{1,t})}{(\lambda_2-\lambda_1+\sigma_2-\sigma_1)^2},\quad c_{n,t}=\frac{(\lambda_1-\lambda_2)(\sigma_1^2\sigma_{2,t}-\sigma_2^2\sigma_{1,t})}{(\sigma_1-\sigma_2)^2}$$

$$d_{n,t}=\frac{(\lambda_1-\lambda_2)(\sigma_2\sigma_{1,t}-\sigma_1\sigma_{2,t})}{(\sigma_2-\sigma_1)^2} \tag{43}$$

On the other hand, we can easy to see that $(T_{n,t}+T_nV_n)T_n^*=(\det T_n)R_n$, here:

$$R_n=\begin{bmatrix} R_{11}^{(1)}(n)\lambda+R_{11}^{(0)}(n) & R_{12}^{(0)}(n) \\ R_{21}^{(0)}(n) & R_{22}^{(1)}(n)\lambda+R_{22}^{(0)}(n)\end{bmatrix} \tag{44}$$

and hence have:

$$T_{n,t}+T_nV_n=R_nT_n \tag{45}$$

Comparing the coefficients of λ^2, λ, and λ^0 in eq. (45) yields:

$$R_{11}^{(0)}(n)=\frac{s_n}{r_n}+\frac{b_{n,t}}{1+b_n},\quad R_{21}^{(0)}(n)=\frac{c_n}{b_n}+\frac{d_{n,t}}{b_n} \tag{46}$$

$$R_{11}^{(1)}(n)=\frac{1}{2},\quad R_{12}^{(0)}(n)=1,\quad R_{22}^{(1)}(n)=-\frac{1}{2},\quad R_{22}^{(0)}(n)=0 \tag{47}$$

and hence we have:

$$a_{n,t}+\frac{a_ns_n}{r_n}+\frac{b_n}{r_{n-1}}=\left(\frac{s_n}{r_n}+\frac{b_{n,t}}{1+b_n}\right)a_n+c_n,\quad \frac{s_n}{r_n}+\frac{b_{n,t}}{1+b_n}=\frac{a_n-d_n}{b_n}+\frac{b_{n,t}}{b_n} \tag{48}$$

$$\frac{c_n}{b_n}+\frac{d_{n,t}}{b_n}=\frac{c_n}{1+b_n}+\frac{1}{(1+b_n)r_{n-1}},\quad \frac{c_n}{b_n}+\frac{d_{n,t}}{b_n}=\frac{c_{n,t}}{a_n}+\frac{c_ns_n}{a_nr_n}+\frac{d_n}{a_nr_{n-1}} \tag{49}$$

Substituting eqs. (42) and (43) into eqs. (48) and (49), we can verify eqs. (48) and (49) all hold. At the same time, by a direct computation we obtain:

$$R_{11}^{(0)}(n)=\frac{\bar{s}_n}{\bar{r}_n},\quad R_{21}^{(0)}(n)=\frac{1}{\bar{r}_{n-1}} \tag{50}$$

Equations (44), (47) and (50) clearly tell that $R_n=\bar{V}_n$. Thus, we finish the proof of *Theorem 2*.

Exact solutions

In this section, we employ the DT (6) and (23) to construct exact solutions of eq. (1). Firstly, we select a pair of seed solutions $r_n = s_n = 1$. Secondly, we obtain two basic solutions of the discrete linear spectral problem (2) and (3):

$$\varphi_n(\lambda_j)=\begin{bmatrix}\varphi_{1,n}(\lambda_j)\\ \varphi_{2,n}(\lambda_j)\end{bmatrix}=\begin{pmatrix}\tau_1^n e^{\rho_1 t}\\ \tau_1^{n-1}e^{\rho_1 t}\end{pmatrix},\quad \psi_n(\lambda_j)=\begin{bmatrix}\psi_{1,n}(\lambda_j)\\ \psi_{2,n}(\lambda_j)\end{bmatrix}=\begin{pmatrix}\tau_2^n e^{\rho_2 t}\\ \tau_2^{n-1}e^{\rho_2 t}\end{pmatrix} \tag{51}$$

where

$$\tau_1=\frac{(\lambda_j+1)+\sqrt{(\lambda_j+1)^2+4}}{2},\quad \tau_2=\frac{(\lambda_j+1)-\sqrt{(\lambda_j+1)^2+4}}{2} \tag{52}$$

$$\rho_1=\frac{1+\sqrt{(\lambda_j+1)^2+4}}{2},\quad \rho_2=\frac{1-\sqrt{(\lambda_j+1)^2+4}}{2} \tag{53}$$

From eqs. (21) and (22), we have:

$$a_{n+1}=\frac{(1+b_{n+1})[(1+b_n)(b_n+d_n)-a_nb_n]}{1+b_n},$$

$$b_{n+1}=\frac{(b_n+d_n-a_n)(1+b_n-a_n)-c_n}{(a_n-b_n-d_n)(1+b_n-a_n)+c_n-(1+b_n)} \tag{54}$$

Substituting the seed solutions $r_n = s_n = 1$ into eq. (23), we have:

$$(\bar{r}_n,\bar{s}_n)=\left(\frac{1+b_{n+1}}{1+b_n},\frac{1+b_{n+1}+a_{n+1}-a_n\bar{r}_n}{1+b_n}\right) \tag{55}$$

where a_n, b_n, c_n, and d_n are determined by eqs. (16), a_{n+1} and b_{n+1} are determined by eq. (55), and σ_j (j = 1, 2) is determined by eq. (17). We, therefore, determine the new solutions $(\bar{r}_n,\bar{s}_n)$ of eqs. (1) by one-fold DT.

Conservation laws

From eqs. (2) and (4), we have $\varphi_{1,n+1}=(\lambda r_n+s_n)\varphi_{1,n}+r_n\varphi_{1,n-1}$, which can be written:

$$1=(\lambda r_n+s_n)\theta_n+r_n\theta_{n-1}\theta_n \tag{56}$$

by introducing $\theta_n=\varphi_{1,n}/\varphi_{1,n+1}$.

On the other hand, from eqs. (3) and (5) we have:

$$\varphi_{1,n,t}=\left(\frac{\lambda}{2}+\frac{s_n}{r_n}\right)\varphi_{1,n}+\varphi_{1,n-1},\quad \varphi_{2,n,t}=\frac{1}{r_{n-1}}\varphi_{1,n}-\frac{\lambda}{2}\varphi_{2,n} \tag{57}$$

which lead to:

$$-(\ln\theta_n)_t=\frac{\varphi_{1,n+1,t}}{\varphi_{1,n+1}}-\frac{\varphi_{1,n,t}}{\varphi_{1,n}} \tag{58}$$

namely

$$-(\ln\theta_n)_t=(E-1)\left(\frac{\lambda}{2}+\frac{s_n}{r_n}+\theta_{n-1}\right) \tag{59}$$

If we set:

$$\theta_n = \sum_{j=1}^{\infty} \theta_n^{(j)} \lambda^{-j} \tag{60}$$

then eq. (56) becomes:

$$1 = (\lambda r_n + s_n) \sum_{j=1}^{\infty} \theta_n^{(j)} \lambda^{-j} + r_n \left[\sum_{j=1}^{\infty} \theta_{n-1}^{(j)} \lambda^{-j} \right] \sum_{j=1}^{\infty} \theta_n^{(j)} \lambda^{-j} \tag{61}$$

Comparing the coefficients with same order of λ in eq. (61), we have:

$$\theta_n^{(1)} = \frac{1}{r_n}, \quad \theta_n^{(2)} = -\frac{s_n}{r_n^2}, \quad \theta_n^{(3)} = \frac{s_n^2}{r_n^3} - \frac{1}{r_n r_{n-1}} \tag{62}$$

and so forth. At the same time, a recursion is derived:

$$\theta_n^{(m+1)} = -\frac{1}{r_n} \left[s_n \theta_n^{(m)} + r_n \sum_{k=1}^{m-1} \theta_{n-1}^{(k)} \theta_n^{(m-k)} \right], \quad m \geq 2 \tag{63}$$

Substituting eq. (60) into eq. (59), we have:

$$-\left[\ln \sum_{j=1}^{\infty} \theta_n^{(j)} \lambda^{-j} \right]_t = (E-1) \left[\frac{\lambda}{2} + \frac{s_n}{r_n} + \sum_{j=1}^{\infty} \theta_{n-1}^{(j)} \lambda^{-j} \right] \tag{64}$$

which can be written as:

$$(\ln r_n)_t + \left\{ \sum_{k=1}^{\infty} (-1)^k \frac{1}{k} \left[r_n \sum_{j=1}^{\infty} \theta_n^{(j+1)} \lambda^{-j} \right]^k \right\}_t = (E-1) \left[\frac{\lambda}{2} + \frac{s_n}{r_n} + \sum_{j=1}^{\infty} \theta_{n-1}^{(j)} \lambda^{-j} \right] \tag{65}$$

Comparing each of the coefficients with same order of λ in eq. (65), we obtain the following infinite many conservation laws of eqs. (1):

$$(\ln r_n)_t = (E-1)\frac{s_n}{r_n}, \quad \left(\frac{s_n}{r_n} \right)_t = (E-1)\frac{1}{r_{n-1}}, \quad \left(\frac{1}{r_{n-1}} - \frac{1}{2}\frac{s_n^2}{r_n^2} \right)_t = -(E-1)\frac{s_{n-1}}{r_{n-1}^2} \tag{66}$$

Acknowledgment

This work was supported by the Natural Science Foundation of China (11547005), the Natural Science Foundation of Liaoning Province of China (20170540007), the Natural Science Foundation of Education Department of Liaoning Province of China (LZ2017002) and Innovative Talents Support Program in Colleges and Universities of Liaoning Province of China (LR2016021).

References

[1] He, J. H., A Tutorial Review on Fractal Spacetime and Fractional Calculus, *International Journal of Theoretical Physics, 53* (2014), 11, pp. 3698-3718
[2] He, J. H., Fractal Calculus and its Geometrical Explanation, *Results in Physics, 10* (2018), Sept., pp. 272-276
[3] Li, X. X., *et al.*, A Fractal Modification of the Surface Coverage Model for an Electrochemical Arsenic Sensor, *Electrochimica Acta, 296* (2019), Feb., pp. 491-493

[4] Wang, Q. L., *et al.*, Fractal Calculus and its Application to Explanation of Biomechanism of Polar Bear Hairs, *Fractals*, *26* (2018), 6, 1850086
[5] Wang, Y., Deng, Q., Fractal Derivative Model for Tsunami Travelling, *Fractals, 27* (2019), 1, 1950017
[6] He, J. H., Ji, F. Y., Two-Scale Mathematics and Fractional Calculus for Thermodynamics, *Thermal Science*, *23* (2019), 4, pp. 2131-2133
[7] Ain, Q. T., He, J. H., On Two-Scale Dimension and its Applications, *Thermal Science*, *23* (2019), 3B, pp. 1707-1712
[8] He, J. H., Zhu, S. D., Differential-Difference Model for Nanotechnology, *Journal of Physics: Conference Series*, *96* (2008), 1, ID 012189
[9] Zhang, S., *et al*., Differential-Difference Equation Arising in Nanotechnology and its Exact Solutions, *Journal of Nano Research*, *23* (2013), 1, pp. 113-116
[10] Zhang, S., Liu, D. D., Infinite Many Conservation Laws of Discrete System Associated with a 3×3 Matrix Spectral Problem, *Thermal Science*, *21* (2017), 4, pp. 1613-1619
[11] He, J. H., A Modified Li-He's Variational Principle for Plasma, *International Journal of Numerical Methods for Heat and Fluid Flow*, On-line first, https://doi.org/10.1108/HFF-06-2019-0523, 2019
[12] He, J. H., Lagrange Crisis and Generalized Variational Principle for 3D unsteady flow, *International Journal of Numerical Methods for Heat and Fluid Flow*, On-line first, https://doi.org/10.1108/HFF-07-2019-0577, 2019
[13] He, J. H., Wu, X. H., Exp-Function Method for Non-Linear Wave Equations, *Chaos, Solitons and Fractals*, *30* (2006), 3, pp. 700-708
[14] Zhang, S., *et al.*, A Direct Algorithm of Exp-Function Method for Non-Linear Evolution Equations in Fluids, *Thermal Science*, *20* (2016), 3, pp. 881-884
[15] Anjum, N., He, J. H., Laplace Transform: Making the Variational Iteration Method Easier, *Applied Mathematics Letters*, *92* **(**2019), June, pp. 134-138
[16] He, J. H., Some Asymptotic Methods for Strongly Nonlinear Equations, *International Journal of Modern Physics B*, *20* (2006), 10, pp. 1141-1199
[17] Wu, Y., He, J. H., Homotopy Perturbation Method for Nonlinear Oscillators with Coordinate Dependent Mass., *Results in Physics*, *10* (2018), Sept., pp. 270-271
[18] Wu, Y., He, J. H., A Remark on Samuelson's Variational Principle in Economics, *Applied Mathematics Letters*, *84* (2018), Oct., pp. 143-147
[19] He, J. H., Hamilton's Principle for Dynamical Elasticity, *Applied Mathematics Letters*, *72* (2017), Oct., pp. 65-69
[20] He, J. H., Ji, F. Y., Taylor Series Solution for Lane-Emden Equation, *Journal of Mathematical Chemistry, 57* (2019), 8, pp. 1932-1934
[21] He, J. H., The Simplest Approach to Nonlinear Oscillators, *Results in Physics, 15* (2019), Dec., ID 102546
[22] Matveev, V. B., Salle, M. A., *Darboux Transformation and Solitons*, Springer-Verlag: Berlin, Germany, 1991
[23] Tian, Y., Symmetry Reduction a Promising Method for Heat Conduction Equations, *Thermal Science*, *23* (2019), 4, pp. 2219-2227
[24] Tian, Y., Diffusion-Convection Equations and Classical Symmetry Classification, *Thermal Science*, *23* (2019), 4, pp. 2151-2156
[25] Bai, Y. S., Zhang, Q., Perturbed Korteweg-de Vries Equations Symmetry Analysis and Conservation Laws, *Thermal Science*, *23* (2019), 4, pp. 2281-2289
[26] Xu, X. X., A Family of Integrable Differential-Difference Equations, its Bi-Hamiltonian Structure and Binary Nonlinearization of the Lax Pairs and Adjoint Lax Pairs, *Chaos, Solitons & Fractals*, *45* (2012), 4, pp. 444-453

Paper submitted: April 28, 2019
Paper revised: August 10, 2019
Paper accepted: September 8, 2019