AFFINE ROTATION SURFACES OF ELLIPTIC TYPE IN AFFINE 3-SPACE

by

Hasan ES *

Department of Mathematics, Faculty of Education, Gazi University, Ankara, Turkey

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In this paper, we classify affine rotation surfaces of elliptic type in affine 3-space satisfying some algebraic equations regarding the co-ordinate functions and the Laplacian operators in relation the first and the second affine fundamental forms of affine rotation surfaces of elliptic type. We also give explicit forms of these surfaces.

Key words: *affine surfaces of revolution, laplacian operator, affine 3-space, affine rotational surfaces*

Introduction

Affine rotation surfaces are a generalization of the well-known surfaces of revolution, affine rotation surfaces arise naturally within the framework of affine differential geometry, a field started by Blaschke in the first decades of the past century. Affine rotations are the affine equivalents of Euclidean rotations. Affine rotation surfaces are surfaces invariant under affine rotations [1].

Affine surfaces of revolution have been studied by many geometricians. Lee [2] and Suss [3] studied non-degenerated surfaces in 3-D affine space with affine rotational symmetry. Manhart gave classified of affine rotational surfaces in affine 3-space, A^3, with vanishing affine Gauss-Kronecker curvature [4]. Yang and Nie [5] showed that if a rotation surface is minimal, then it must be a plane or catenoid and if a catenoid is minimal, then it must be a right helicoid. Krauter [6] gave a complete list of affine minimal surfaces in A^3 with Euclidean rotational symmetry and proved that these surfaces have maximal affine surface area within the class of all affine surfaces of rotation satisfying suitable boundary conditions. Yang *et al*. [7] studied the invariant properties for affine rotation surfaces in A^3 under the centro affine transformation group. Furthermore, they also gave some classification results for centro affine minimal rotation surfaces with the constant tensor norm of the Tchebychev vector field induced by the centro affinemetric. Lehebel investigated affine surfaces (and hypersurfaces) which are affine rotation surfaces. These surfaces can be characterized by the fact that all affine normals (in the Blaschke sense) intersect a fixed straight line (the axis) and the section with planes containing the axis are shadow boundries with respect to parallel light. In case the axis is a proper line (not at infinity) there are three types of surfaces: elliptic, hyperbolic and parabolic. Faghfouri, *et al*. [8] determined a Blaschke structure for affine immersion of Euclidian and hyperbolic type for plane equi-affine curve. Alcazar and Goldman [1] analyzied several properties of algebraic affine rotation surfaces and by using some notions and results from affine differential geometry and also they showed how to find the axis of an affine rotation surface. Karacan *et al*. [9] gave

* Author's e-mail: hasanes@gazi.edu.tr

classification of LCN-translation surfaces in affine 3-space. Lone *et al*. [10] characterized finite type non-degenerate translation Surfaces in affine 3-Space.

In this study, we aim to complete the categorize of affine surfaces of revolution of elliptic type in A^3 with respect to the position vector field and the Laplacian operators in relation the first and second affine fundamental forms of affine surfaces of revolution of elliptic type. We also give explicit forms of affine surfaces of revolution of elliptic type in A^3.

Preliminaries

In this section some definitions of the main affine structures will be given. The conormal and normal vectors and the Gaussian and the mean curvatures. The Berwald-Blaschke metric is invariant for affine tranformations and also independent of system of co-ordinates. This metric is a quadric form. This quadratic form might not be positive definite (non-convex) case. Let $\Psi : \Omega \rightarrow \mathbb{R}^3$ be a regular surface. The Berwald-Blaschke metric is given:

$$h = \frac{L}{\left|LN - M^2\right|^{\frac{1}{4}}} du^2 + \frac{M}{\left|LN - M^2\right|^{\frac{1}{4}}} dudv + \frac{N}{\left|LN - M^2\right|^{\frac{1}{4}}} dv^2 = Edu^2 + \ \ Fdudv + Gdv^2 \tag{1}$$

where

$$L = \left[\Psi_u, \Psi_v, \Psi_{uu}\right],\ N = \left[\Psi_u, \Psi_v, \Psi_{vv}\right],\ M = \left[\Psi_u, \Psi_v, \Psi_{uv}\right]$$

[11-15]. From now on, we shall assume that the surface is non-degenerate, that is, $LN - M^2 \neq 0$. The points $LN - M^2$ can be negative, zero or positive. If $LN - M^2$ is negative we call these points as hyperbolic, if $LN - M^2$ is zero we call these points as parabolic and if $LN - M^2$ is positive we call these points as elliptic.

Definition 1. Let S be a regular surface with non-degenerate points and $\Psi : \Omega \subset \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^3$ be a parametrrization. Then, we define the affine conormal field given by the expression:

$$\nu = \frac{\Psi_u \wedge \Psi_v}{\left|LN - M^2\right|^{1/4}} = \frac{\Psi_u \wedge \Psi_v}{\left[\nu, \nu_u, \nu_v\right]} \tag{2}$$

where L, N, and M are the coefficents of the first affine fundamental form [11-15].

By definition, it can be seen that $\nu d\Psi = 0$:

$$d = \pm\left[\nu, \nu_u, \nu_v\right] = \pm\left(LN - M^2\right)^{1/4}$$

where the signal ± is the point elliptical or hyperbolic. Using this notation, we have:

$$\nu = \frac{\Psi_u \wedge \Psi_v}{d} \tag{3}$$

Definition 2. We define the affine normal vector:

$$\xi = \frac{\nu_u \wedge \nu_v}{\left[\nu, \nu_u, \nu_v\right]} = \frac{\nu_u \wedge \nu_v}{d} \tag{4}$$

or

$$\xi = \frac{1}{2} \frac{\left|LN - M^2\right|^{1/4}}{\sqrt{LN - M^2}} \left[\frac{\partial}{\partial u}\left(\frac{N\Psi_u - M\Psi_v}{\sqrt{LN - M^2}}\right) + \frac{\partial}{\partial v}\left(\frac{L\Psi_v - M\Psi_u}{\sqrt{LN - M^2}}\right)\right] \tag{5}$$

or

$$\xi = \frac{1}{2}\frac{1}{\sqrt{EG-F^2}}\left[\frac{\partial}{\partial u}\left(\frac{G\Psi_u - F\Psi_v}{\sqrt{EG-F^2}}\right)+\frac{\partial}{\partial v}\left(\frac{E\Psi_v - F\Psi_u}{\sqrt{EG-F^2}}\right)\right] \tag{6}$$

where $\nu\xi = 1$, $\nu_u\xi = 0$, and $\nu_v\xi = 0$.

Observe that, the affine normal vector does not belong to the tangent plane to the surfaces, S. The curvatures describe the variation of the normal vector. We see that $\nu\xi_u = 0$, $\nu\xi_v = 0$. That is, the derivatives ξ_u and ξ_v are orthogonal to ν. In particular $\xi_u, \xi_v \in T_pS$. Therefore, we can define the shape operator S:

$$S : T_pS \to T_pS$$

given by $S_p(v) = -D_v\xi$. Since ξ_u and ξ_v are tangents to the surface, we have that there are functions:

$$b_{ij} : \Omega \to \mathbb{R},\ i, j = 1, 2$$

such that

$$\xi_u = b_{11}\Psi_u + b_{12}\Psi_v,\ \ \xi_v = b_{21}\Psi_u + b_{22}\Psi_v \tag{7}$$

where

$$b_{11} = \frac{[\xi_u, \Psi_v, \xi]}{d},\ \ b_{12} = \frac{[\xi_v, \Psi_v, \xi]}{d},\ \ b_{21} = \frac{[\Psi_u, \xi_u, \xi]}{d},\ \ b_{22} = \frac{[\Psi_u, \xi_v, \xi]}{d} \tag{8}$$

This shows that in the basis $\{\Psi_u, \Psi_v\}$, the Shape operator $S_p(v) = -D_v\xi$ is given by the matrix $\mathrm{B} = (b_{ij})$, $i, j = 1, 2$. Notice that this matrix is not necessarily symmetric [11-15].

Definition 3. The coefficients b_{ij} form a matrix $\mathrm{B} = (b_{ij})$, whose determinant and the half of the trace are the Gaussian and the mean curvatures, respectively. Hence, we have [11-15]:

$$\mathrm{K} = \det B = b_{11}b_{22} - b_{12}b_{21},\ \ \mathrm{H} = \frac{1}{2}trB = \frac{b_{11}+b_{22}}{2} \tag{9}$$

The Laplacian operators Δ^I and Δ^h of the first and second affine fundamental forms on S with regard to local co-ordinates $\{u, v\}$ of S are defined:

$$\Delta^I\Psi = -\frac{1}{\sqrt{LN-M^2}}\left[\frac{\partial}{\partial u}\left(\frac{N\Psi_u - M\Psi_v}{\sqrt{LN-M^2}}\right)-\frac{\partial}{\partial v}\left(\frac{M\Psi_u - L\Psi_v}{\sqrt{LN-M^2}}\right)\right] \tag{10}$$

and

$$\Delta^h\Psi = -\frac{1}{\sqrt{EG-F^2}}\left[\frac{\partial}{\partial u}\left(\frac{G\Psi_u - F\Psi_v}{\sqrt{EG-F^2}}\right)-\frac{\partial}{\partial v}\left(\frac{F\Psi_u - E\Psi_v}{\sqrt{EG-F^2}}\right)\right] \tag{11}$$

respectively [9, 10, 16].

Affine rotation surfaces of elliptic type in affine 3-space

In Euclidean geometry, rotation surfaces can be qualified by the property where every Euclidean normal crosses a (proper) straight line g. The affine analogues refer to (non-degenerated) surfaces in A^3 with the property which every affine normal crosses a fixed straight line g, which is named the axis of the surface. The sections with planes that contain g (meridians) refer to shadow lines with regard to parallel light and those surfaces are named affine rotation surfaces. They comprise a one parameter family of conics in parallel planes (parallel curves). Meridians and parallel curves constitute a conjugate net of curves on the surface [4].

Denoting by $(u, v) \in \mathbb{R}^2$ local co-ordinates and by $\Psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ a non-degenerate Blaschke immersion, we can give the representations, in a convenient co-ordinate system (x_1, x_2, x_3) of affine rotation surfaces:

Proper affine surfaces of rotation: The axis g is a proper line (x_3-axis of the co-ordinate system):

– *Elliptic type*: If Ψ is of elliptic type, we have the subsequent representations in local co-ordinates:

$$\Psi(u,v) = \left[f(u)\cos v, f(u)\sin v, g(u)\right] \tag{12}$$

$$\Psi(u,v) = \left[f(u)\cos v, f(u)\sin v,\ u\right] \tag{13}$$

$$\Psi(u,v) = \left[u\cos v, u\sin v, g(u)\right] \tag{14}$$

– *Hyperbolic type*: If Ψ is of hyperbolic type, we get the representations:

$$\Psi(u,v) = \left[f(u)\cosh v, f(u)\ \sinh v, g(u)\right] \tag{15}$$

$$\Psi(u,v) = \left[f(u) cosh v, f(u)\sinh v,\ u\right] \tag{16}$$

$$\Psi(u,v) = \left[u\ \cosh v,\ u\sinh v,\ g(u)\right] \tag{17}$$

In Cases 1 and 2, the parallel curves refer to ellipses and hyperbolas which are centered on g, respectively [4].

– *Parabolic type:* If Ψ is of parabolic type, we have the representation:

$$\Psi(u,v) = \left[u,\ uv,\ \frac{1}{2}uv^2 + g(u)\right] \tag{18}$$

At this point parallel curves (u = constant) refer to parabolas in planes that are parallel to g [4].

Improper affine rotation surfaces: The axis g is at infinity (in the representation given below affine normals are parallel to the constant planes x_2):

$$\Psi(u,v) = \left[u,\ v,\ \frac{1}{2}v^2 + g(u)\right] \tag{19}$$

These are translation surfaces that have plane generating curves and one of those are parabolas [4].

So, we consider a affine surface of revolution of elliptic type defined by a patch:

$$\Psi(u,v) = \left[u\cos v,\ u\sin v,\ g(u)\right] \tag{20}$$

The cofficients of the first affine fundamental form of affine rotation surface of elliptic type in affine 3-space are given:

$$L = u\,g''(u),\ \ N = u^2 g'(u),\ \ M = 0,\ d = \left|LN - M^2\right|^{1/4} = \left(u^3 g' g''\right)^{1/4} \tag{21}$$

Hence the coefficients of the Berwald-Blaschke metric of affine rotation surface of elliptic type or the coefficients of the second affine fundamental form are given:

$$E=\frac{u\,g''}{\left(u^3g'g''\right)^{1/4}},\ \ G=\frac{u^2g'}{\left(u^3g'g''\right)^{1/4}},\ F=0 \tag{22}$$

We suppose that the Berwald-Blaschke metric is non-degenerate, *i. e.*, $d\neq 0$. Geometrically $d>0$ means that the Euclidean Gaussian curvature does not vanish, *i. e*. affine rotation surface of elliptic type is strongly convex. The affine co-normal field of affine rotation surface of elliptic type is given:

$$\nu=\left(-\frac{ug'\cos v}{\left(u^3g'g''\right)^{1/4}},-\frac{ug'\sin v}{\left(u^3g'g''\right)^{1/4}},\frac{u}{\left(u^3g'g''\right)^{1/4}}\right) \tag{23}$$

Thus, we have the affine normal vector given:

$$\xi=\left(-\frac{u^4g'\left[ug''^2+g'\left(-g''+u\,g'''\right)\right]\cos v}{4\left(u^3g'g''\right)^{7/4}}\right.$$
$$\left.-\frac{u^4g'\left[ug''^2+g'\left(-g''+u\,g'''\right)\right]\sin v}{4\left(u^3g'g''\right)^{7/4}},\ \ \frac{\left[3ug''^2+g'\left(g''-u\,g'''\right)\right]}{\left(u^3g'g''\right)^{-1/4}\left(4u^2g''^2\right)}\right) \tag{24}$$

where $\nu.\xi=1$, $\nu_u.\xi=0$, $\nu_v.\xi=0$. Consequently, the coefficients b_{ij} form a matrix B $=[b_{ij}]$:

$$b_{11}=\frac{-5g'^2g''^2+2ug'g''^3+3u^2g''^4-2ug'^2g''g'''}{16g'g''^2\left(u^3g'g''\right)^{3/4}}+\frac{-2u^2g'g''^2g'''+7u^2g'^2g'''^2-4u^2g'^2g''g^{(4)}}{16g'g''^2\,64u^6g'^3g''^5}$$
$$b_{12}=0,\ b_{21}=0,\ b_{22}=-\frac{ug''^2+g'\left(-g''+ug'''\right)}{4g''\left(u^3g'g''\right)^{3/4}} \tag{25}$$

Proposition 1. Let S be a affine rotation surface of elliptic type with non-degenerate in affine 3-space. Then the Gaussian and the mean curvatures of S are given:

$$\mathbf{K}=A(B+C) \tag{26}$$

where

$$A=\frac{u^3\left[ug''^2+g'\left(-g''+ug'''\right)\right]}{64\,g''^2\left(u^3g'g''\right)^{5/2}},\ \ B=-3u^2g''^4+2ug'g''^2\left(-g''+ug'''\right)$$
$$C=g'^2\left\{5\,g''^2-7u^2g'''^2+2u\,g''\left[g'''+2ug^{(4)}\right]\right\}$$

and

$$\mathbf{H}=\frac{1}{32\left(u^3g'g''\right)^{11/4}}\cdot$$
$$\cdot\left[-g'^2g''^2-2ug'g''^3+3u^2g''^4-6ug'^2g''g'''-2u^2g'g''^2g'''+7u^2g'^2g'''^2-4u^2g'^2g''g^{(4)}\right] \tag{27}$$

respectively.

We suppose that the affine rotation surface of elliptic type with non-degenerate given by eq. (14) has zero Gaussian curvature. Then, we obtain:

$$A(B+C)=0 \tag{28}$$

The differential eq. (28) cannot be solved analytically. If the eq. (28) is identically zero, either $A = 0$ or $(B + C) = 0$. If $A = 0$:

$$g(u)=c_1+c_2\left(\frac{1}{2}u\sqrt{u^2+c_3}+\frac{1}{2}\ln\left|u+\sqrt{u^2+c_3}\right|\right) \tag{29}$$

where $c_i \in \mathbb{R}$. If $(B + C) = 0$, then we have

$$g(u)=c_1,\ g(u)=c_1u+c_2,\ g(u)=c_1u^2+c_2,\ g(u)=c_1\ln|u|+c_2 \tag{30}$$

where $c_i \in \mathbb{R}$. In the first and second solutions of $g(u)$ in eq. (30), we have $L = 0$ and $N = 0$, therefore, they give rise a contradiction with assumption that the first affine fundamental form must be non-degenerate. Then S is parametrized:

$$\Psi(u,v)=\left(u\cos v,\ u\sin v, c_1u^2+c_2\right) \tag{31}$$

$$\Psi(u,v)=\left(u\cos v,\ u\sin v, c_1\ln|u|+c_2\right) \tag{32}$$

and

$$\Psi(u,v)=\left[u\cos v,\ u\sin v, c_1+c_2\left(\frac{1}{2}u\sqrt{u^2+c_3}+\frac{1}{2}\ln\left|u+\sqrt{u^2+c_3}\right|\right)\right] \tag{33}$$

The surfaces (31)-(33) can be drawn as in the figs. 1(a)-1(c), respectively.

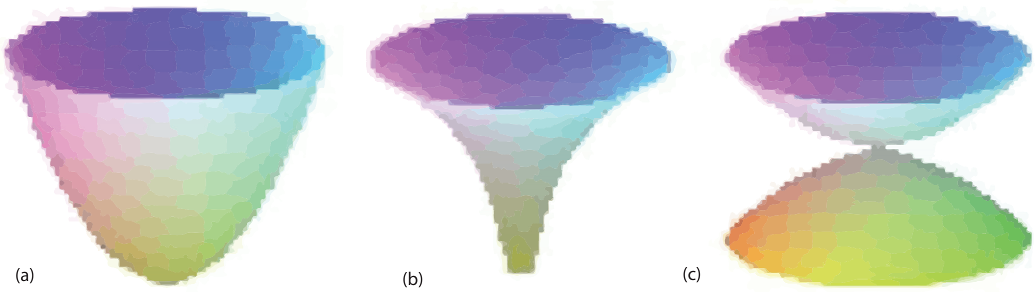

Figure 1. Affine rotation surfaces of elliptic type

Theorem 1. Let S be an affine rotation surface of elliptic type with non-degenerate in affne 3-space. If S has zero Gaussian curvature or affine flat then S is parametrized as eqs. (31)-(33).

We assume that S is affine minimal. Hence, the mean curvature is zero if and only if:

$$\left[-g'^2g''^2-2ug'g''^3+3u^2g''^4-6ug'^2g''g'''-2u^2g'g''^2g'''+7u^2g'^2g'''^2-4u^2g'^2g''g^{(4)}\right]=0 \tag{34}$$

The differential eq. (34) cannot be solved analytically, so the particular solutions of the eq. (34) are given:

$$\begin{gathered}g(u)=c_1\\ g(u)=c_1u+c_2\\ g(u)=c_1u^2+c_2\\ g(u)=c_1+c_2\left(\frac{1}{2}u\sqrt{u^2+c_3}+\frac{1}{2}\ln\left|u+\sqrt{u^2+c_3}\right|\right)\end{gathered} \tag{35}$$

where $c_i \in \mathbb{R}$. In the first and second solutions of $g(u)$ in eq. (34), we have $L = 0$ and $N = 0$, therefore, they give rise a contradiction with assumption that the first affine fundamental form must be non-degenerate. Then S is parametrized by eq. (31) which implies that the surface S and the surface given by eq. (33) are elliptic paraboloids.

Theorem 2. Let S be an affine rotation surface of elliptic type with non-degenerate in affine 3-space. If S is affine minimal, then it is a part of the surfaces given by eqs. (31) and (34).

Corollary 1. Let S be an elliptic paraboloid, then the surface S is affine flat and minimal.

Affine rotation surfaces of elliptic type satisfying $\Delta^I\Psi = \mathbf{A}\Psi$

In this section, we classify affine rotation surface of elliptic type given by eq. (14):

$$\Delta^I\Psi = \mathbf{A}\Psi \tag{36}$$

where $\mathbf{A} = (a_{ij})$, $i, j = 1, 2, 3$, and

$$\Delta^I\Psi = \left(\Delta^I\Psi_1,\ \Delta^I\Psi_2,\ \Delta^I\Psi_3\right) \tag{37}$$

Here:

$$\Psi_1 = u\cos v,\ \Psi_2 = u\sin v,\ \Psi_3 = g(u) \tag{38}$$

where $u \neq 0$.

Suppose that the affine rotation surface of elliptic type is with non-degenerate. Laplacian operator with regard to the first affine fundamental form on S with the help of eqs. (36)-(38), and (11) turns out:

$$\Delta^I\Psi = \left(\frac{\cos v\left[ug''^2+g'\left(-g''+u\,g'''\right)\right]}{2u^2g'g''^2},\ \frac{\sin v\left[ug''^2+g'\left(-g''+u\,g'''\right)\right]}{2u^2g'g''^2},\ \frac{-3ug''^2+g'\left(-g''+u\,g'''\right)}{2u^2g''^2}\right) \tag{39}$$

Suppose that S satisfies eq. (36). Then from eqs. (37) and (38):

$$\begin{aligned}a_{11}u\cos v+a_{12}u\sin v+a_{13}g(u)&=\frac{\cos v\left[ug''^2+g'\left(-g''+u\,g'''\right)\right]}{2u^2g'g''^2}\\ a_{21}u\cos v+a_{22}u\sin v+a_{23}g(u)&=\frac{\sin v\left[ug''^2+g'\left(-g''+u\,g'''\right)\right]}{2u^2g'g''^2}\\ a_{31}u\cos v+a_{32}u\sin v+a_{33}g(u)&=\frac{-3ug''^2+g'\left(-g''+u\,g'''\right)}{2u^2g''^2}\end{aligned} \tag{40}$$

Since the functions cos$v$ and sin$v$ are linearly independent, by eq. (40) we get $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$, $a_{11} = a_{22} = \lambda$, $a_{33} = \mu$. Consequently the matrix **A** can be given:

$$\mathbf{A} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \tag{41}$$

and eq. (40) is re-written

$$\lambda u = \frac{\left[u g''^2 + g'\left(-g'' + u g'''\right)\right]}{2u^2 g' g''^2} \tag{42}$$

$$\mu g(u) = \frac{-3u g''^2 + g'\left(-g'' + u g'''\right)}{2u^2 g''^2} \tag{43}$$

From eqs. (42) and (43) we obtain:

$$\frac{-g'' + u g'''}{2u^2 g''^2} = \lambda u - \frac{1}{2ug'}, \quad \frac{-g'' + u g'''}{2u^2 g''^2} = \frac{3 + 2u\mu g}{2ug'} \tag{44}$$

Combining the first and second equations of eq. (44):

$$-2 + \mu u g + \lambda u^2 g' = 0 \tag{45}$$

Table 1. The solutions of ordinary differential eq. (45)

λ, μ	$g(u)$
$\lambda \neq 0$, $\mu \neq 0$	$u\left[\frac{c_1}{\frac{\mu}{\lambda}} - \frac{2}{u(\lambda-\mu)}\right]$
$\lambda = \mu \neq 0$	$\frac{c_1\lambda + 2\ln u}{\lambda u}$
$\lambda = 0$, $\mu \neq 0$	$\frac{2}{u\mu}$
$\lambda \neq 0$, $\mu = 0$	$c1 - \frac{2}{u\lambda}$
$\lambda = \mu = 0$	–

We have summarized the solutions of ordinary differential equation eq. (45) in tab. 1.

In the fifth row of tab. 1, we obtain a contradiction. Because, there is no function $g(u)$ satisfying the equation $\Delta^I\Psi = 0$ at the same time. An exception; $g(u) = c_1$, $g(u) = c_1 u + c_2$, where $c_i \in \mathbb{R}$. These cases imply that the first affine fundamental form are degenerate, that contradicts with assumption. The conditions given by the rows 1-4 do not satisfy eqs. (42) and (43).

*Definition 4***.** An affine surface in A^3 is called to be I-harmonic if it satisfies the condition $\Delta^I\Psi = 0$.

According to the tab. 1, we have proved the following theorem:

Theorem 3. There is non-*I*-harmonic affine rotation surface of elliptic type with non-degenerate given by eq. (14) in A^3.

Theorem 4. If S is a non-*I*-harmonic affine rotation surface of elliptic type with non-degenerate which is given by eq. (14) in A^3. Then, there is no affine rotation surface of elliptic type satisfying the condition $\Delta^I\Psi = \mathbf{A}\Psi$.

Affine rotation surfaces of elliptic type satisfying $\Delta^h\Psi = \mathbf{A}\Psi$

In this section, we classify affine rotation surface of elliptic type given by eq. (14) in affine 3-space satisfying the equation:

$$\Delta^h\Psi = \mathbf{A}\Psi \tag{46}$$

where $\mathbf{A} = (a_{ij})$, $i, j = 1, 2, 3$ and

$$\Delta^h\Psi = \left(\Delta^h\Psi_1,\ \Delta^h\Psi_2,\ \Delta^h\Psi_3\right) \tag{47}$$

Here:

$$\Psi_1 = u\cos v,\ \Psi_2 = u\sin v,\ \Psi_3 = g(u) \tag{48}$$

where $u \neq 0$.

Suppose that the affine rotation surface of elliptic type is with non-degenerate. By a straightforward computation, the Laplacian operator with respect to the second affine fundamental form on S with the help of eqs. (46)-(48), and (11):

$$\Delta^h\Psi = \left(\frac{u^4 g'\cos v\left[ug''^2 + g'\left(-g'' + u g'''\right)\right]}{2\left(u^3 g'g''\right)^{7/4}}, \frac{u^4 g'\sin v\left[ug''^2 + g'\left(-g'' + u g'''\right)\right]}{2\left(u^3 g'g''\right)^{7/4}}, \frac{u^4 g'^2\left[-3ug''^2 + g'\left(-g'' + u g'''\right)\right]}{2\left(u^3 g'g''\right)^{7/4}}\right) \tag{49}$$

Suppose that S satisfies eq. (36). Then from eqs. (47) and (48), we have:

$$\begin{aligned}
a_{11}u\cos v + a_{12}u\sin v + a_{13}g(u) &= \frac{u^4 g'\cos v\left[ug''^2 + g'\left(-g'' + u g'''\right)\right]}{2\left(u^3 g'g''\right)^{7/4}} \\
a_{21}u\cos v + a_{22}u\sin v + a_{23}g(u) &= \frac{u^4 g'\sin v\left[ug''^2 + g'\left(-g'' + u g'''\right)\right]}{2\left(u^3 g'g''\right)^{7/4}} \\
a_{31}u\cos v + a_{32}u\sin v + a_{33}g(u) &= \frac{u^4 g'^2\left[-3ug''^2 + g'\left(-g'' + u g'''\right)\right]}{2\left(u^3 g'g''\right)^{7/4}}
\end{aligned} \tag{50}$$

Since the functions $\cos v$ and $\sin v$ are linearly independent, by eq. (50) we get $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$, $a_{11} = a_{22} = \lambda$, $a_{33} = \mu$. Consequently the matrix **A** can be given:

$$\mathbf{A} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \tag{51}$$

and eq. (50) is re-written

$$\lambda = \frac{u^3 g'\left[ug''^2 + g'\left(-g'' + u g'''\right)\right]}{2\left(u^3 g'g''\right)^{7/4}} \tag{52}$$

$$\mu g(u) = \frac{u^4 g'^2\left[-3ug''^2 + g'\left(-g'' + u g'''\right)\right]}{2\left(u^3 g'g''\right)^{7/4}} \tag{53}$$

From eqs. (52) and (53) we obtain:

$$\frac{\left(u^3 g' g''\right)^{1/4}}{2u^2 g''^2} = \frac{\lambda u g'}{-g' g'' + u g''^2 + u g' g'''}, \quad \frac{\left(u^3 g' g''\right)^{1/4}}{2u^2 g''^2} = \frac{\mu g}{-g' g'' - 3u g''^2 + u g' g'''} \tag{54}$$

When the first and second equations of eq. (54) are combined, we attain:

$$\lambda u\left(g'^2 g'' + 3u g' g''^2 - u g'^2 g'''\right) + \mu g\left(-g' g'' + u g''^2 + u g' g'''\right) = 0 \tag{55}$$

The differential eq. (55) cannot be solved analytically, so the particular solutions are given:

$$g(u) = c_1, \ \ g(u) = c_1 u + c_2 \tag{56}$$

We have summarized the solutions of ordinary differential eq. (55) in tab. 2.

Table 2. The solutions of ordinary differential equation (55)

No.	λ, μ	$g(u)$
1	$\lambda \neq 0,\ \mu \neq 0$	$c_1,\ c_1 u + c_2$
2	$\lambda = \mu \neq 0$	$c_1,\ c_1 u + c_2$
3	$\lambda = 0,\ \mu \neq 0$	$c_1 + c_2\left(\frac{1}{2} u\sqrt{u^2 + c_3} + \frac{1}{2} c_3 \ \ln\left\lvert u + \sqrt{u^2 + c_3}\right\rvert\right)$
4	$\lambda \neq \mu \neq 0$	$c_1 + c_2\left(\ln\left\lvert u + \sqrt{u^2 - c_3}\right\rvert\right)$
5	$\lambda = 0,\ \mu = 0$	$g(u)$

In the fifth row of tab. 2, $g(u)$ can be any differentiable function. But this is a contradiction. Because, there is no the function $g(u)$ satisfying the equation $\Delta^h\Psi = 0$ at the same time. An exception: $g(u) = c_1$, $g(u) = c_1 u + c_2$, where $c_i \in \mathbb{R}$. These cases imply that the first affine fundamental form are degenerate, that contradicts with assumption. In the first and second rows of the aforementioned table, we have $L = 0$ and $N = 0$. So the first affine fundamental form in these cases are degenerate, that is a contradict with assumption. The conditions given by the rows 4 and 5 do not satisfy eqs. (52) and (53).

Definition 5. An affine surface in A^3 is called to be h-harmonic if it satisfies the condition $\Delta^h\Psi = 0$.

Therefore, we can give the following theorem:

Theorem 5. There is no h- harmonic affine rotation surface of elliptic type with non-degenerate given by eq. (14) in A^3.

Theorem 6. If S is a non-h-harmonic affine rotation surface of elliptic type with non-degenerate which is given by eq. (14) in A^3. Then, there is no affine rotation surface of elliptic type satisfying the condition $\Delta^h\Psi = \mathbf{A}\Psi$.

Using the similar calculations in this paper, almost the same results are obtained for affine rotation surface of hyperbolic type.

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