NUMERICAL SOLUTION FOR TIME PERIOD OF SIMPLE PENDULUM WITH LARGE ANGLE

by

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In this study, the numerical solution of the ordinary kind of differential equation for a simple pendulum with large-angle of oscillation was introduced to obtain the time period. The analytical solution is obtained in terms of elliptic functions, and numerical solution of the problem was achieved by using two numerical quadrature methods, namely, Simpson's 3/8 and Boole's method. The period of a simple pendulum with large angle is presented. A comparison has been carried out between the analytical solution and the numerical integration results. In the case of error analysis, absolute and relative errors of the problem have been presented. A numerical algorithm has been developed by MATLAB software 2013R and used for analyzing the result. It is established that the results of the comparison guaranty the ability and the accuracy of the present method.

Key words: *simple pendulum, time period, large angle, numerical integration, error analysis*

Introduction

Simple pendulum is one of the most popular examples of a simple mechanical system as its set-up, but difficult when someone wants to compute the factors which act on its motion, like time period, T, angle of oscillation, θ, amplitude, acting forces, and its energy [1, 2]. This simple mechanical system (simple pendulum) oscillates with a symmetric force due to gravity acting on it as a restoring force, as illustrated in fig. 1 [3]. Most of these procedures are based on the analysis of the non-linear differential equation is given [4, 5]:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+\frac{\mathrm{g}}{L}\sin\theta=0 \tag{1}$$

where g is the acceleration due to gravity, L – the length of the pendulum, and θ – the angular displacement (in radians). The time period for a simple pendulum has been derived at small angle approximation, $\sin\theta = \theta$, eq. (1) becomes linear differential equation and have a simple solution as $T=2\pi(\mathrm{g}/L)^{1/2}$. But when the angular displacement *amplitude* of the pendulum is large enough that the small-angle approximation no longer holds, and then the equation of motion

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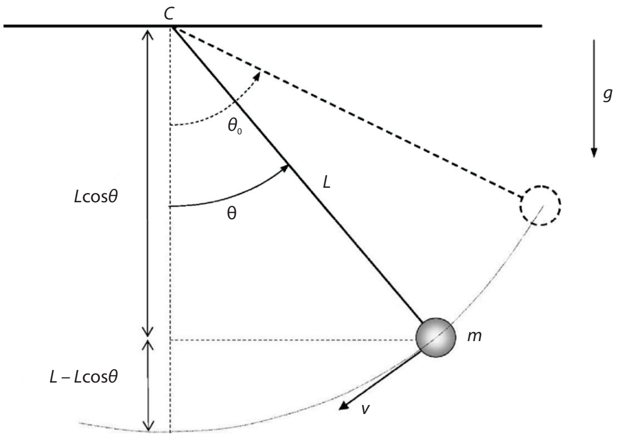

Figure 1. Scheme of a simple pendulum motion

must remain in its non-linear form. For this initial condition, the analytical solution can only be obtained numerically (with arbitrary accuracy). Conservation of energy can be used to quickly arriving the same simple form of this differential equation [3, 6]:

$$E = T + V \tag{2}$$

where E is the total energy

$$T = \frac{1}{2} m L^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}t} \right)^2$$

the kinetic energy

$$V = mgL\left(1 - \cos\theta\right)$$

the potential energy.

At the lowest point of the trajectory, the zero of potential energy was taken, also the initial conditions for the given ODE $\theta(0) = \theta_M$ and $\mathrm{d}\theta/\mathrm{d}t(0) = 0$, choose for simplicity:

$$\frac{1}{2} m L^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}t} \right)^2 + mgL\left(1 - \cos\theta\right) = mgL\left(1 - \cos\theta_M\right) \tag{3}$$

Multiplying both sides of eq. (3) by $2/mL^2$, rearranging and setting the square root of each sides of eq. (3):

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{\frac{2g}{L}} \left(\cos\theta - \cos\theta_M\right)^{1/2} \tag{4}$$

Rearranging eq. (4) and taking the integral of both sides:

$$t = \sqrt{\frac{L}{2g}} \int_0^{\theta_M} \left(\cos\theta - \cos\theta_M\right)^{-1/2} \mathrm{d}\theta \tag{5}$$

The time required for θ to increase from 0 to θ_M is ($t = T/4$), where T is a time of one period, so we can write eq. (5):

$$T = 4\sqrt{\frac{L}{2g}} \int_0^{\theta_M} \left(\cos\theta - \cos\theta_M\right)^{-1/2} \mathrm{d}\theta \tag{6}$$

For simplicity, these substitutions were used

$\cos\theta = 1 - 2\sin^2(\theta/2)$, and $\cos\theta_M = 1 - 2\sin^2(\theta_M/2)$ thus

$$T = 2\sqrt{\frac{L}{g}}\int_0^{\theta_M}\left[\sin^2\left(\frac{\theta_M}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right]^{-1/2}\mathrm{d}\theta \tag{7}$$

and

$$\sin\left(\frac{\theta}{2}\right) = \sin\left(\frac{\theta_M}{2}\right)\sin\varphi, \text{ where } \varphi = \sin^{-1}\left[\frac{\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta_M}{2}\right)}\right] \rightarrow \varphi = \begin{cases} 0 & \theta = 0 \\ \frac{\pi}{2} & \theta = \theta_M \end{cases}$$

we get

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1 - \sin^2\left(\frac{\theta_M}{2}\right)\sin^2\varphi\right]^{-1/2}\mathrm{d}\varphi \tag{8}$$

The integral in eq. (8) is the form of first kind elliptic integral [7]. The elliptic integral in eq. (8) can be expanded, by the series expansion obtain:

$$T = 2\sqrt{\frac{L}{g}}\left\{1 + \frac{1}{4}\sin^2\left(\frac{\theta_M}{2}\right) + \frac{9}{64}\sin^4\left(\frac{\theta_M}{2}\right)\right\} \tag{9}$$

Solutions of the problem

Several approximation schemes have been established to examine the large amplitude oscillations of a simple pendulum, and the different approximations have been suggested for calculating its large-angle period with precision [8-11]. In this section, we will present a numerical solution the problem. Using numerical quadrature technique for evaluating the integrations on both eqs. (7) and (8) for T as a function of θ or φ, this solution is also done with analytical approximation by literatures [12-14].

There are many numerical integration methods to evaluate composite integrals; in this paper, we use two numerical quadrature methods, Simpsons 3/8 method, and Boole's method [15-23].

If we set:

$$C_1 = 2\sqrt{\frac{L}{g}},\ f(\theta) = \left[\sin^2\left(\frac{\theta_M}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right]^{-1/2}$$

for the integral in eq. (7), and applying Simpson's 3/8 method:

$$C_1\int_0^{\theta_M} f(\theta)\mathrm{d}\theta = C\frac{3h}{8}\left[\sum_{i=1}^{n/3} f_{3i-3} + 3\left(f_{3i-2} + f_{3i-1}\right) + f_{3i}\right] + O\left(h^4\right) \tag{10}$$

where $O(h^4) = -\theta_M/80\ h^4 f^4(\xi)$, where $0 \le \xi \le \theta_M$.

Similarly, by applying Boole's method:

$$C_1\int_0^{\theta_M} f(\theta)\mathrm{d}\theta = C\frac{2h}{4}\left[\sum_{i=1}^{n/4} 7\left(f_{4i-4} + f_{4i}\right) + 32\left(f_{4i-3} + f_{4i-1}\right) + 12 f_{4i-2}\right] + O\left(h^6\right) \tag{11}$$

where $O(h^6) = -2\theta_M/945\ h^6 f^6(\xi)$, where $0 \le \xi \le \theta_M$.

By the same manner:

$$C_2 = 4\sqrt{\frac{L}{g}},\; f(\varphi) = \left[1 - \sin^2\left(\frac{\theta_M}{2}\right)\sin^2\varphi\right]^{-1/2}$$

for eq. (8) and applying Simpson's 3/8 methods:

$$C_2 \int_0^{\pi/2} f(\varphi)\mathrm{d}\varphi = C_2 \frac{3h}{8}\left[\sum_{i=1}^{n/3} f_{3i-3} + 3\left(f_{3i-2} + f_{3i-1}\right) + f_{3i}\right] + O\left(h^4\right) \tag{12}$$

where $O(h^4) = -\,\theta_M/80\; h^4 f^4(\xi)$, where $0 \le \xi \le \theta_M$.

Similarly, Boole's method gives:

$$C_2 \int_0^{\pi/2} f(\varphi)\mathrm{d}\varphi = C_2 \frac{2h}{4}\left[\sum_{i=1}^{n/4} 7\left(f_{4i-4} + f_{4i}\right) + 32\left(f_{4i-3} + f_{4i-1}\right) + 12 f_{4i-2}\right] + O\left(h^6\right) \tag{13}$$

where $O(h^6) = -\,2\theta_M/945\; h^6 f^6(\xi)$, where $0 \le \xi \le \theta_M$.

The present work focused on the time period of simple pendulum as a function of its starting amplitude at a large angle numerically. Now if we set $L = 0.1$ m, g = 9.8 m/s^2, for both integral eqs. (7) and (8), the results of Simpson's 3/8 and Boole's method will be compared with the expanded series in eq. (9) which is an analytical solution of the problem, and absolute errors, E_A, and relative errors, R_A, are calculated:

$$E_A = \left|\text{Exact value-Numerical value}\right| \quad \text{and} \quad R_A = \frac{E_A}{\text{Exact value}}$$

The obtained results of eqs. (10)-(13) are tabulated in tab. 1, where $\theta_M = \pi/2$ and the exact solution for the given angle is 0.736348344103256 (actually, this is not the totally exact value, because eq. (9) is the series expansion, but the series at five order and above do not get the solution) [22].

For making a comparison and testing accuracy of the present methods, firstly if we set $n = 600$ and $\theta_M = \pi/2$, results are given in tab. 1, one can obtain that absolute and relative errors are 10^{-2}, for Simpson's 3/8 and Boole's methods for integral eqs. (7) and (8).

Table 1. Numerical results compared with the exact solution

$n = 600$, $\theta_M = \pi/2$	Numerical value	Absolute error, E_A	Relative error, R_A
Simpson's 3/8 method in eq. (10)	0.721257812446209	$1.5091 \cdot 10^{-2}$	$2.0494 \cdot 10^{-2}$
Boole's method in eq. (11)	0.721186718860959	$1.5162 \cdot 10^{-2}$	$2.0590 \cdot 10^{-2}$
Simpson's 3/8 method in eq. (12)	0.748911905263186	$1.2564 \cdot 10^{-2}$	$2.6458 \cdot 10^{-2}$
Boole's method in eq. (13)	0.748911905263198	$1.2564 \cdot 10^{-2}$	$1.7062 \cdot 10^{-2}$

Similarly, for the mentioned equation, where $n = 600$ and $\theta_M = \pi/4$, and the exact solution for the given angle is 0.659849092173693, the numerical results and error analysis yields in tab. 2. The results for eq. (7) are approximately same as for $\theta_M = \pi/2$, but for eq. (8) the results have more accuracy with absolute and relative errors 10^{-4}.

From tabs. 1 and 2 we can conclude that both numerical quadrature methods are accurate and suitable for solving simple pendulum integra eqs. (7) and (8) and make comparison between them and equation (9) which gives the results in tabs. 1 and 2, these tables show the accuracy of the method. However, the accuracy of the results depends on increase iteration number n. We can notice that while the number of iterations n are increased, then better more accurate have been found.

Table 2. Numerical compared with the exact solution for a quarter angle

$n = 600$, $\theta_M = \pi/4$	Numerical result	Absolute error, E_A	Relative error, R_A
Simpson's 3/8 method in eq. (10)	0.636653768717357	$2.3195 \cdot 10^{-4}$	$3.5152 \cdot 10^{-4}$
Boole's method in eq. (11)	0.636593993065055	$2.3255 \cdot 10^{-4}$	$3.5243 \cdot 10^{-4}$
Simpson's 3/8 method in eq. (12)	0.66002367440478	$1.7458 \cdot 10^{-4}$	$2.6458 \cdot 10^{-4}$
Boole's method in eq. (13)	0.660023674404781	$1.7458 \cdot 10^{-4}$	$2.6458 \cdot 10^{-4}$

Conclusion

The time period of a simple pendulum with a large angle was modeled mathematically, and the problem was verified. The exact solution the problem was given by the expanded series. Results are shown for two different methods of integral equations. The result was obtained by using a numerical integration technique based on Simpson's and Boole's method. For the numerical integrations, an algorithm is constructed, and the programs are written by MATLAB software 2013Ra. Good approximations were obtained for the problem with two different large angles when compared with the exact analytical solution. It is worth mentioning that the technique can be used as a very accurate algorithm for the presented type of time period of simple pendulum converted to integral equations.

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