NEW PROMISES AND FUTURE CHALLENGES OF FRACTAL CALCULUS
From Two-Scale Thermodynamics to Fractal Variational Principle

by

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Any physical laws are scale-dependent, the same phenomenon might lead to debating theories if observed using different scales. The two-scale thermodynamics observes the same phenomenon using two different scales, one scale is generally used in the conventional continuum mechanics, and the other scale can reveal the hidden truth beyond the continuum assumption, and fractal calculus has to be adopted to establish governing equations. Here basic properties of fractal calculus are elucidated, and the relationship between the fractal calculus and traditional calculus is revealed using the two-scale transform, fractal variational principles are discussed for 1-D fluid mechanics. Additionally planet distribution in the fractal solar system, dark energy in the fractal space, and a fractal ageing model are also discussed.

Key words: *two-scale fractal dimension, two scale mathematics, fractal space, fractal variational theory, local property*

Introduction

We begin with an ancient Chinese fable called as *Blind Men and Elephant*, all blind men had no idea of an elephant, and inconsistent descriptions were given after their feeling the elephant at different parts. This fable tells us that we should not take a part for the whole. In academic experiments, it is impossible to measure each point of the studied problem, and various assumptions have to be made to predict its whole property. The most used one is the continuum assumption, which is the foundation of mechanics and thermodynamics, and differential models as well. The continuum models of course cannot study the effect of an unsmooth boundary or a porous medium.

There is also an ancient Chinese saying that seeing is believing, however, in most scientific phenomena, seeing is not always believing. The same phenomenon can lead to debating theories when measured on different scales. To elucidate this important fact, we consider a weight lifter holding silently a weight as shown in fig. 1. Everyone knows that:

$$\text{Work} = \text{force} \times \text{displacement}$$

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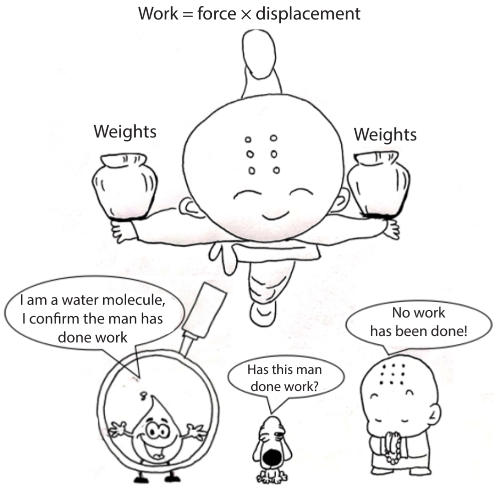

Figure 1. Two-scale cartoon showing that observations with different scales result in opposite results for the same phenomenon

Because the weight lifter does move the weight, the displacement is zero, so the work done by the man should be zero, however this is not the fact, the man must have done work, and we have to measure the done work on a much smaller scale, saying on a molecule scale. Under such a small scale, all fibers in hand muscles have done work.

In biology, there are debating laws for the metabolic law [1, 2], if a cell is considered as a continuum, we have Rubner's 2/3 law; if we consider the cell surface is not smooth, we obtain the Kleiber's 3/4 law. All disputes stopped when a fractal cell is adopted [3]. The well-known wave-particle dualism arises also in different scale observations. To reveal the hidden truth or to eliminate the inconsistency arising in different scale observations, two-scale thermodynamics is needed to obverse a same phenomenon using two different scales [4, 5]. To further understand the two-scale concept, we consider a drop of red ink in water [6] as illustrated in fig. 2.

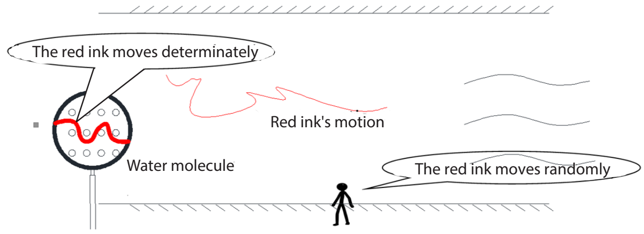

Figure 2. A drop of red ink in a moving reviver; the motion of the red ink due to the moving river and the red ink's diffusion in water seems to be random for all observers, however, its motion on a molecule's scale is determinate

Water is a continuum if a large-scale is used, and the motion of the red ink follows laws in fluid mechanics, however, fluid mechanics cannot elucidate the mechanism of the red ink's diffusion. We, therefore, need a smaller scale, saying a molecule's size, which results in a discontinuous medium of water, and fractal calculus has to be adopted [7].

A brief introduction continuous space and fractal space

As everyone knows differential equations, which are actually derived based on smooth space. Newton's mechanics is established on a smooth 3-D spatial space, Einstein's theory assumes a smooth 4-D spacetime. The smooth space or spacetime assumption can never predict any properties arising in unsmooth space or spacetime. El Naschie's E-infinity theory [8-10], on the other hand, considers a fractal and discontinuous spacetime with an average fractal dimensions of 4.236.

Distinction among El Naschie's fractal spacetime [8-10] with Newton 3 spatial world and Einstein's 4 space-time is in the dimensions, see fig. 3:

$$\text{Dimensions} = \underbrace{\underbrace{\underbrace{3}_{\text{Newton}} + 1}_{\text{Einstein}} + \phi^3}_{\text{El Naschie}} \approx 4.236 \tag{1}$$

where ϕ is the golden mean given:

$$\phi = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}} = \frac{\sqrt{5}-1}{2} \tag{2}$$

We call the number, 4.236, as the El Naschie number:

$$D = 4+\phi^3 = \frac{1}{\phi^3} \approx 4.236 \tag{3}$$

which is a number of harmony and appears everywhere from mathematics to high energy physics, for example, the absolute zero temperature can be theoretically predicted using El Naschie number:

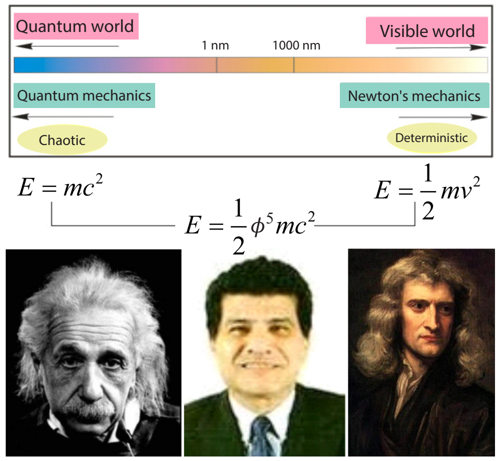

Figure 3. El Naschie's theory bridges Newton mechanics and quantum mechanics

$$T_0 = 1-(4)(10)D^{4/3} = -273.15\ ^\circ\mathrm{C} \tag{4}$$

El Naschie number is widely appeared in from nature to arts (*e. g.* architecture and painting). Figure 4 shows the golden mean and El Naschie number in a hand of human being, arranged by Fibonacci sequence.

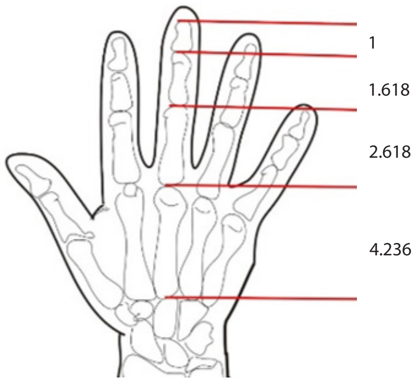

Figure 4. Golden mean and El Naschie number in a hand $1/\phi = 1.618$, $1/\phi^2 = 2.618$, $1/\phi^3 = 4.236$; the number can also be obtained from Fibonacci numbers 1, ϕ, $1+\phi$, $2+\phi$, $3+2\phi = 4+\phi^3$

Planet distribution in the fractal solar system

Fibonacci sequence appears everywhere in nature and has wide applications, it is inconceivably embodied in a variety of wildlife (*e. g.* sunflower) and modern physics as well, for example, the average fractal dimensions of our spacetime can be also obtained through Fibonacci numbers:

$$\phi^2, \phi, 1,\ 1+\phi,\ \ 2+\phi,\ \ 4+\phi^3 \tag{5}$$

Nature always gives astonishing similarity as that in the planetary sequence of our solar system:

$$0.386,\ 0.723,\ 1.00,\ 1.60,\ 2.80,... \tag{6}$$

where the number is the distance between the Sun and planet or asteroid belt in unit of AU. This sequence is very much close to the following Fibonacci sequence:

$$\phi^2, \phi, 1, 1+\phi, \phi^2 = 0.382, 0.618, 1.00, 1.618, 2.618 \tag{7}$$

Figure 5 shows an ideal planet distribution by Fibonacci numbers ϕ^2, ϕ, 1, $1+\phi$, $2+\phi$, $4+\phi^3$, where the radius of the Earth's orbit is taken as 1 AU. We, therefore, guess that planet distribution follows the golden mean law based on Fibonacci sequence.

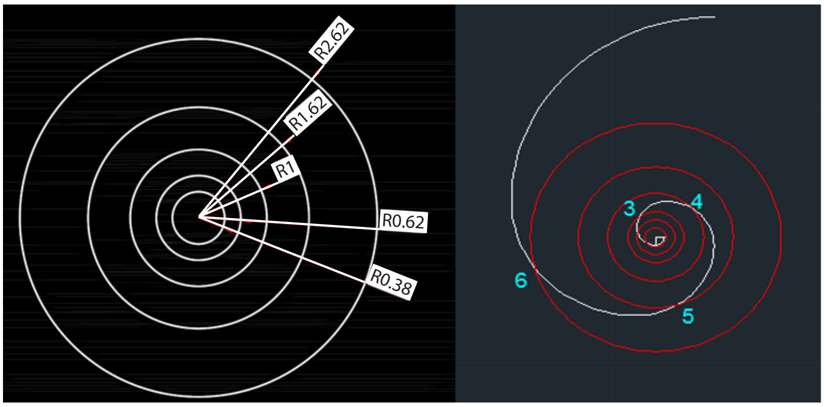

Figure 5. Ideal planet distribution by Fibonacci numbers

We argue the following unproved laws.

Spiral Law:

The solar system has a Fibonacci spiral, which is called in this paper as the gravitational spiral. A periodic and stable body cuts the spiral once per period, and at the cut point, the body has equal gravitational energy and kinetic energy. All members of solar system accelerate due to gravitation and have a tendency to spiral downward toward the Sun.

Golden mean Law:

Planet distribution on the gravitational spiral follows the Fibonacci sequence.

Further discussion on gravitational spiral will be given in a separate paper.

Fractal boundary and fluctuation dimensions

According to El Naschie's theory [8, 9], the average Hausdorff dimensions of our fractal spacetime:

$$D = 4 + \phi^3 = 4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \ddots}}} = \frac{1}{\phi^3} \approx 4.236 = (3+1) + 0.236 \tag{8}$$

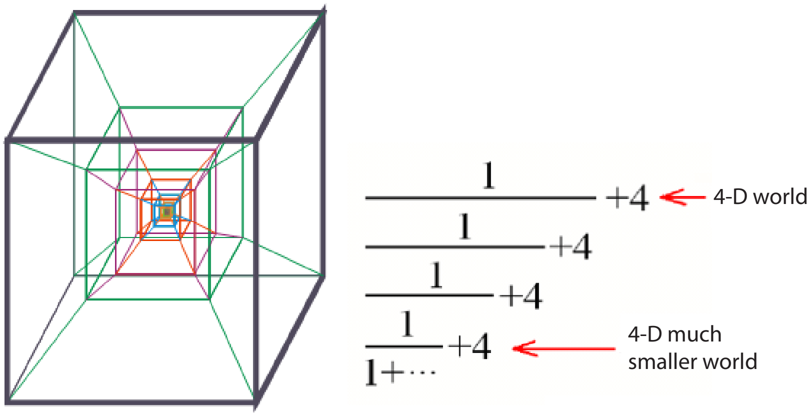

Figure 6. Fractal spacetime model with self similarity [11]

Explanation of eq. (8) can be explained by a fractal Hilbert cube spacetime model given in fig. 6, each cascade in the hierarchical spacetime is a mini 4-D spacetime, the quantum work is the inner small cascade, our solar system might be in some a cascade in the middle, some a distant galaxy might be in a cascade larger than our world. All theories established on some a cascade becomes invalid in its adjacent cascades except El Naschie's fractal E-infinity theory, which combines Newton's mechanics with quantum mechanics. A mini-symposium on fractal spacetime and dark energy in 4th International Symposium on Non-Linear Dynamics was held in Shanghai, China on October 30, 2012, for celebrating El Naschie's greatest finding, see fig. 7.

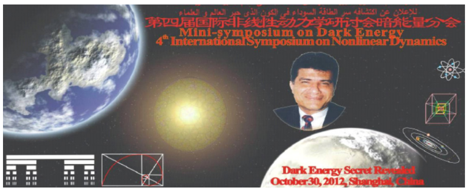

Figure 7. El Naschie's photo appearing in a historical conference poster with Cantor set, the symbolic golden mean and the fractal Hilbert cube spacetime model

All of our previous theories except El Naschie's fractal spacetime theory were established either on smooth 3-D space or smooth (3+1)-D spacetime, where time is 1-D. Now a question arises, why does our spacetime have dimensions of 4.236? Everyone can feel 3-D spatial space plus 1-D time. We have now additional dimensions of 0.236, which is the dimension fluctuation of our fractal spacetime.

To understand dimension fluctuation, we consider an extremely large surface with a fractal boundary at an extremely small scale, see fig. 8. The Hausdorff dimensions of its boundary:

$$\frac{\ln 4}{\ln 3} = 1.2618 \tag{9}$$

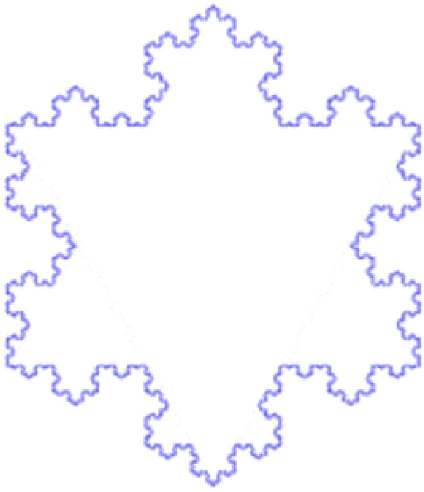

Figure 8. Fractal boundary with fractal dimensions of ln4/ln3

What is the dimension of a 3-D cube boundary? This is a trivial question since it is clearly an area, *i. e.* a surface which is 2-D. That means:

$$\text{2-D (boundary)} + 1 = \text{3-D (cube)} \tag{10}$$

Next we ask a second trivial question, namely what is the dimension of the boundary of a 2-D surface? It is obviously a 1-D line:

$$\text{1-D (boundary)} + 1 = \text{2-D (surface)} \tag{11}$$

Finally what is the dimension of the boundary of a line? This is evidently 0-D point. That means:

$$\text{0-D (boundary)} + 1 = \text{1-D (line)} \tag{12}$$

It seems natural that by induction one could write a general expression for the previous form:

$$\text{D (boundary)} + 1 = n \tag{13}$$

where n is the dimension of the geometrical object for which we would like to know the dimension of its boundary. This is a trivial case of induction. However what if we want to extend this formula below a point just as we usually extended it above a 3-D cube? We routinely deal in higher geometry with 4-D and n-D cubes as discussed by Coxeter and studied thoroughly in the context of E-infinity theory [8, 9]. In this case we use induction say that the boundary of a point has a dimension [8-10]:

$$\mathrm{D} = \mathrm{D}(0) - 1 = 0 - 1 = -1 \tag{14}$$

The procedure can continue and we have a negative 3-D space inside a point. To illustrate the concept, we consider a line between the Earth and the Sun, the dimensions of the terminals of the line are zero, however, inside the Sun the space is a –3 dimensional one if we observe it from the Earth.

Now we have a fractal boundary as illustrated in fig. 3, the dimensions of this surface is not 2:

$$\frac{\ln 4}{\ln 3}\,(\text{boundary}) + 1 = 2.2618 = 2 + 0.2618 \tag{15}$$

where 0.2618 is the fluctuation of plane dimension.

Now it is easy to understand the 0.236 dimension fluctuation in El Naschie's fractal spacetime. The boundary of our spacetime must be non-smooth, where about 95.5% of the energy in the cosmos is hidden on the boundary, which pull some of the cosmic boundary outward, and the cosmic expansion velocity can be as large as light velocity, this partly agrees with the Big Bang theory. While some part of the concave boundary of our space will shrink.

Dark Energy in the fractal space

As we know that Newton's 3-D space is an approximate one, we assume that the spatial dimension is relative to π. The spatial space consists of the planar section as shown in fig. 8 and the height. The Earth surface can be considered as the boundary of the spatial space:

$$\frac{\pi}{3} \times 2 = 2.0943950666667 \tag{16}$$

The height dimension:

$$\pi - 2 = 1.1415926535898 \tag{17}$$

The total spatial dimension:

$$\frac{\pi}{3} \times 2 + (\pi - 2) = 3.23598775598 \tag{18}$$

According to El Naschie's fractal Cantorian space-time theory [8, 9], the average fractal dimensions of our real space-time is 4.236, this replies a spatial world with fractal dimensions of 3.236, very closed the value above. In any observable scales, our universe resembles a continuous 4-D spacetime, however, on quantum scale, it becomes discontinuous.

El Naschie predicted that there are about 95.5% dark energy hidden in the fractal boundary of our space;

Einstein's energy-mass equation reads:

$$E = mc^2 \tag{19}$$

This equation is obtained from a smooth 4-D spacetime. For our fractal spacetime, the total energy is the sum of two basically quantum parts, namely that of the quantum particle energy [12, 13]:

$$E(O) = \frac{1}{2}\phi^5 mc^2 \approx \frac{1}{22} mc^2 \tag{20}$$

and that of the quantum wave energy:

$$E(D) \approx \frac{21}{22} mc^2 \tag{21}$$

$$\text{Dark Energy} \approx \frac{3}{\pi} mc^2 = \cfrac{3}{3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+\ddots}}}}}} mc^2 \tag{22}$$

Fractal medium and fractional calculus

God created the solids, the devil created their boundaries (Wolfgnang E. Pauli), a fractal boundary always leads to astonishing properties which will never behave in smooth boundary as shown in Majumder *et al.*'s experimental observation [14]. Majumder *et al.* [14] found that liquid-flow through a membrane composed of an array of aligned CNT is 4 to 5 orders of magnitude faster than would be predicted from conventional fluid-flow theory. The finding is of course interesting, but why? The main problem is on scale or dimension. The conventional fluid-flow theory is assumed to be continuous on observational scales with integer dimensions, while the studied problem is on nanoscales with fractal dimensions. Any theories established on larger scales become invalid for smaller cases as shown in Majumder *et al.*'s [14] experimental observation. Any porous media and nanoscale materials (*e. g.* nanofiber membrane) can be approximately considered as fractal media, where continuum assumption is prohibited.

There are many definitions on fractional derivatives. A systematical study of various fractional derivatives is given by Yang in his monograph [15], fractional calculus has seen wide applications, see for example, fractional cable [16], fractional vibration [17], fractional nanofluid [18], fractional electro-MHD [19], fractional electro-osmotic flow [20], fractional KdV equation [21, 22], fractional thermoelasticity [23], fractional MHD [24], and fractional soliton dynamics [25].

The variational iteration method was first used to solve fractional differential equations in 1998 [26]. Hereby we will introduce the basic properties of fractional derivatives by the variational iteration method [27-30].

We consider the following linear equation of n^{th} order:

$$u^{(n)} = f(t) \tag{23}$$

By the variational iteration method [27-30], we have the following variational iteration algorithms:

– variational iteration algorithm-I

$$u_{m+1}(t) = u_m(t) + (-1)^n \int_{t_0}^{t} \frac{1}{(n-1)!}(s-t)^{n-1}\left[u_m^{(n)}(s) - f_m(s)\right]\mathrm{d}s \tag{24}$$

– variational iteration algorithm-II [27-30]

$$u_{m+1}(t) = u_0(t) - (-1)^n \int_{t_0}^{t} \frac{1}{(n-1)!}(s-t)^{n-1} f_m(s)\,\mathrm{d}s = u_0(t) - \frac{(-1)^n}{\Gamma(n)}\int_{t_0}^{t}(s-t)^{n-1} f_m(s)\,\mathrm{d}s \tag{25}$$

Note: u_0 must satisfy the initial/boundary conditions.

For a linear equation, we have the following exact solution:

$$u(t) = u_0(t) + (-1)^n \int_{t_0}^{t} \frac{1}{(n-1)!}(s-t)^{n-1}\left[u_0^{(n)}(s) - f(s)\right]\mathrm{d}s \tag{26}$$

or

$$u(t) = u_0(t) - \frac{(-1)^n}{\Gamma(n)} \int_{t_0}^{t} (s-t)^{n-1} f(s)\,\mathrm{d}s \tag{27}$$

where $u_0(t)$ satisfies the boundary/initial conditions.

According to variational iteration algorithm-I, we introduce an integration operator I^n defined:

$$I^n f = \int_{t0}^{t} \frac{1}{(n-1)!}(s-t)^{n-1}\left[u_0^{(n)}(s) - f(s)\right]\mathrm{d}s = \frac{1}{\Gamma(n)}\int_{t_0}^{t}(s-t)^{n-1}\left[f_0(s) - f(s)\right]\mathrm{d}s \tag{28}$$

where $f_0(t) = u_0^{(n)}(t)$.

We can define a fractional derivative in the form:

$$\mathrm{D}_t^{\alpha} f = \mathrm{D}_t^{\alpha}\frac{\mathrm{d}^n}{\mathrm{d}t^n}(I^n f) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}(I^{n-\alpha} f) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{t_0}^{t}(s-t)^{n-\alpha-1}\left[f_0(s) - f(s)\right]\mathrm{d}s \tag{29}$$

In literature, eq. (29) was called as He's fractional derivative [31-42], and it has been applied to biomechanics [38], nanoscale thermodynamics [39], Zakharov-Kuznetsov equation [31], KP-BBM equation [32], solitary theory [33], non-linear vibration [34], coast protection [35, 36], high-order sub-diffusion broblem [37], fractional optimal control problems [38], drug release [39-41], and biomaterials [42].

Dimension is everything and two scale fractal geometry

Dimension is everything, and the dimension values depend upon the scale which we use to obverse various phenomena. To show this, we consider a *smooth* road. It is of course 2-D on scale of wheel diameters, but it becomes 1-D when the scale tends to very large one, *e. g.*, 1 km. When the scale becomes infinitely large, the road becomes zero dimension, *i. e.*, a point. On the other hand, if the scale tends to be extremely small, *i. e.*, few nanometers, the road becomes totally unsmooth, and its dimension values will be larger than 1 and smaller than 3. So with different scales, we have different dimensions for a same subject. We must choose a suitable scale or dimension to study a given problem. The 1-D model, for example, can never predict the properties of 2-D flows, and any integer dimensions can never predict properties of a fractal medium.

A 2-D model can never describe the 3-D properties, and a 4-D eye can see all things inside a house from its outside, which is considered impossible in view of the 3-D world. The dimension or the scale is everything for the all physical laws. There is a science fiction, saying there was a 4-D alien, who stole a heart from a man without dissection. This phenomenon is unbelievable in a 3-D space, but it can happen in a 4-D space [43].

Physical laws depend upon scales, different scales lead to different laws for a same phenomenon. When you observe the motion of the Moon from the Earth, its trajectory follows Newton's gravity law, but if you watch it from an infinite far star, its motion becomes stochastic and an uncertainty principle is found like that for an electron:

$$\Delta x \Delta P < \mathrm{C} \tag{30}$$

where ΔP is the momentum change across the distance Δx, C is a constant. The derivation of eq. (30) was available in [43]. As another example, we consider the water pressure under the depth of h. If we consider the water as a continuum on a large-scale, the water pressure scales with its depth:

$$p \propto h \tag{31}$$

However, if we watch the water on a molecule's size, saying 0.1 nanometers, an object with that size might have no water pressure, see fig. 9:

$$p \to 0 \tag{32}$$

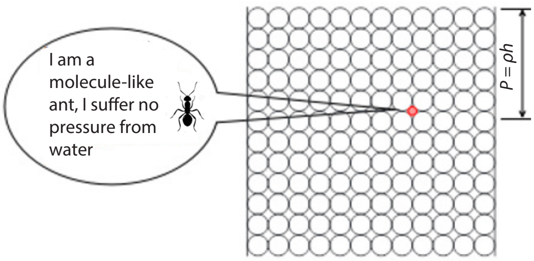

Figure 9. Water pressure on two difference scales, where the circles present water molecules, and the red point is a particle much smaller than that of the water molecule

To understand this phenomenon, we just consider a small ant under sands, the ant in the porous space is subject to no pressure from sands.

Generally on a macro-scale, we have Newton's law for continuum mechanics. On a smaller scale, for example a scale of water molecule's size, water becomes discontinuous and all laws based on continuous space or continuous time become invalid. Generally we can use Mandelbrot's fractal theory [44] to model the discontinuous phenomena [45-60]. Newton's calculus is established on an infinitesimal assumption and the function is differentiable, however, the molecule's motion in water at an infinitesimal interval of time or distance is not differentiable. For 1-D motion, a function can be expressed as $\phi(t, x)$ in a continuous space, however a function in a fractal space can be expressed as $\phi(t^\alpha, x^\beta)$ instead of $\phi(t, x)$, where α and β are fractal dimensions, which will be discussed later.

The porous medium or a space with unsmooth boundaries can be considered as a fractal space. However the fractal geometry requires self-similarity on any scales [44], which can be not found in nature, there must be a minimal level and a maximal level for a fractal-like subject, for example, a tree is a fractal-like one, however, there must be two thresholds for the minimal and maximal cascades. If a porous medium is considered as an approximate fractal pattern, *e. g.* a fractal Sierpinski carpet as illustrated in fig. 10, it implies actually the porous structure can be modelled by two adjacent levels of the fractal pattern [4, 5]. For example, pure water is continuous on a macro scale as shown in fig. 10(a), and all laws in fluid mechanics work, however, on a molecule scale, water becomes discontinuous, and it can be modelled by fig. 10(b), all phenomena arising in a molecule-scale observation can be modelled by fractal calculus [7]. So two scales are enough for description of water's porous structure.

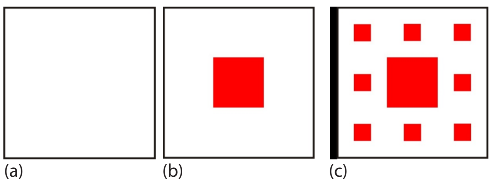

Figure 10. Fractal Sierpinski carpet; the red squares imply the porosity; (a) represents a continuum medium, (b) a porous medium, and (c) a fractal Sierpinski carpet

The two-scale dimension is defined [4, 5]:

$$\alpha = \alpha_0 \frac{A}{A_0} \tag{33}$$

where α is the two scale dimensions for the smaller scale to measure the porosity, fig. 10(b), α_0 – the dimension for the large-scale for an approximate continuity, fig. 10(a), and A and A_0 are areas for fig. 10(a) and fig. 10(b), respectively. The two-scale dimension:

$$\alpha = 2 \times \frac{8}{9} = 1.777 \tag{34}$$

The previous section, the two-scale fractal space is illustrated, any a motion in a fractal space has the fractal property. Consider a coast, along which two animals, *e. g.*, an ant and an elephant, walk from a point A to another point B with same instantaneous velocity and different steps. It can be understood that the ant with a smaller step requires walk a longer distance than that by the elephant with a larger step. So the average velocity from *A* to *B* depends upon not only its fractal patter and also the animal's step size:

$$u \propto (\Delta x)^{\alpha} \tag{35}$$

where u is the average velocity, Δx is the animal's step size.

The time needed for each step is:

$$\Delta t = \frac{\Delta x}{u} \tag{36}$$

The time scale less than Δt is meaningless, the average velocity scales:

$$u \propto (\Delta t)^{\beta} \tag{37}$$

where α and β are two-scale fractal dimensions in moving direction and time, respectively.

Before proceeding further, we first give some definitions and theorems on fractal calculus for easy understanding [7, 61-64].

Definition 1. The distance between two points x_0 and x_1 in x-direction in a fractal space is defined:

$$L(x_1, x_0) = \frac{1}{\Gamma(1+\beta)}(x_1 - x_0)^{\beta}, x_1 > x_0 \tag{38}$$

where β is the fractal dimensions in x-direction, Γ is the gamma function.

Definition 2. The time difference in a fractal time is defined:

$$T(t_1, t_0) = \frac{1}{\Gamma(1+\alpha)}(t_1 - t_0)^{\alpha}, t_1 > t_0 \tag{39}$$

where α is the fractal dimensions in time.

Definition 3. A function in a fractal space is not differentiable with respect to t and x.

Definition 4. A function in a fractal space is differentiable with respect to t^{α} and x^{β}, and can be expressed as $\phi(t^{\alpha}, x^{\beta})$, which is often called as the fractal function.

The change of ϕ across Δx or Δt can be expressed:

$$\Delta\phi \propto (\Delta x)^{\beta} \tag{40}$$

$$\Delta\phi \propto (\Delta t)^{\alpha} \tag{41}$$

where Δx and Δt are the smallest scales in space and time, respectively, any phenomena measured on a scale smaller than Δx or Δt are ignored. In fractal calculus, it always assumed that $\Delta x \neq 0$ and $\Delta t \neq 0$.

Equations (38) and (39) can be also written in the following forms, respectively:

$$L(x_1^{\beta}, x_0^{\beta}) = \frac{1}{\Gamma(1+\beta)}(x_1 - x_0)^{\beta}, \ x_1 > x_0 \tag{42}$$

$$T(t_1^{\alpha}, t_0^{\alpha}) = \frac{1}{\Gamma(1+\alpha)}(t_1 - t_0)^{\alpha}, \ t_1 > t_0 \tag{43}$$

Definition 5. A fractal function has local property. Assuming that the following inequality holds:

$$\left|\phi(t^{\alpha})-\phi(t_0^{\alpha})\right| < (\Delta t)^{\alpha} \tag{44}$$

with $|t - t_0| < \Delta t$ for α, $\Delta t > 0$, we call $\phi(t^{\alpha})$ is locally continuous a t_0 on the scale of Δt. A fractal geometry has local continuity everywhere, however, for a hierarchy, the local continuity depends upon the scale.

Definition 6. The following local approximations hold:

$$\frac{1}{\Gamma(1+\alpha)}(t_1-t_0)^{\alpha}=t_1^{\alpha}-t_0^{\alpha}+O(t_1-t_0) \tag{45}$$

$$\frac{1}{\Gamma(1+\beta)}(x_1-x_0)^{\beta}=x_1^{\beta}-x_0^{\beta}+O(x_1-x_0) \tag{46}$$

The *Definition 1* and the *Definition 2* can be also written in the forms:

$$L(x_1^{\beta},x_0^{\beta})=x_1^{\beta}-x_0^{\beta} \tag{47}$$

$$T(t_1^{\alpha},t_0^{\alpha})=t_1^{\alpha}-t_0^{\alpha} \tag{48}$$

*Definition 7***.** The two scale dimension for two adjacent levels of a hierarchy is defined as [4, 5]:

$$\frac{D}{D_0}=\frac{V}{V_0} \tag{49}$$

where D_0 and D are dimensions on a large-scale and on a small scale, respectively, V_0 and V are measured volumes or areas or lengths on a large-scale and on a small scale, respectively. Generally on a large-scale, the continuum mechanics works, but on a small scale the continuum assumption is forbidden.

A hierarchy is not an exact fractal, so the two-scale dimension is needed. We just consider two adjacent levels of a Cantor set, when we measure it using a scale of L, it is a continuous line, however, when we watch it using a scale of $L/3$, it becomes discontinuous. The two-scale dimension for the adjacent levels of the hierarchy is $D = 1 \times 2/3 = 2/3$, while its Hausdorff fractal dimension is ln2/ln3. In practical applications, the fractal order should be the value of the two-scale dimensions.

Definition 8. Fractal derivatives with respect to t^{α} and x^{β} are defined, respectively:

$$\frac{\partial\phi}{\partial t^{\alpha}}(t_0^{\alpha})=\Gamma(1+\alpha)\lim_{\substack{t-t_0\to\Delta t\\ \Delta t\neq 0}}\frac{\phi(t)-\phi(t_0)}{(t-t_0)^{\alpha}} \tag{50}$$

$$\frac{\partial\phi}{\partial x^{\beta}}(x_0^{\beta})=\Gamma(1+\beta)\lim_{\substack{x-x_0\to\Delta x\\ \Delta x\neq 0}}\frac{\phi(x)-\phi(x_0)}{(x-x_0)^{\beta}} \tag{51}$$

where Δx and Δt are the smallest scales in space and time, respectively, $\Delta x \neq 0$ and $\Delta t \neq 0$. For practical problems, the two-scale dimensions should be adopted for the values of α and β.

To elucidate the fact, we consider a man working along a coastline, which is assumed to be a Koch curve. If the man's step is Δx, and its velocity is u_0, the discontinuous property for the scale less than Δx is ignored, and the motion property depends upon its step scale. If a man's step is $3\Delta x$ with the same velocity, the discontinuity, which is measured in Δx, disappears completely. The two-scale dimension of the adjacent hierarchical levels of the Kock curve is $D = 4/3$, while its Hausdorff fractal dimension is ln4/ln3.

*Definition 9***.** The velocity and acceleration in the fractal space are defined, respectively:

$$u(t_0^\alpha)=\Gamma(1+\alpha)\lim_{\substack{t-t_0\to\Delta t\\ \Delta t\neq 0}}\frac{L(x)-L(x_0)}{(t-t_0)^\alpha}=\frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)}\lim_{\substack{t-t_0\to\Delta t\\ \Delta t\neq 0}}\frac{(x-x_0)^\beta}{(t-t_0)^\alpha}\tag{52}$$

$$a(t_0^\alpha)=\Gamma(1+\alpha)\lim_{\substack{t-t_0\to\Delta t\\ \Delta t\neq 0}}\frac{u(x^\beta)-u(x_0^\beta)}{(t-t_0)^\alpha}\tag{53}$$

*Theorem 1***.** The following chain rules hold:

$$\frac{\partial}{\partial t^\alpha}\left(\frac{\partial\phi}{\partial x^\beta}\right)=\frac{\partial}{\partial x^\beta}\left(\frac{\partial\phi}{\partial t^\alpha}\right)\tag{54}$$

$$\frac{\partial}{\partial t^\alpha}[\phi(u)]=\frac{\partial\Phi}{\partial u}\left(\frac{\partial u}{\partial t^\alpha}\right)\tag{55}$$

Theorem 2. The following differential and integration hold:

$$\frac{\partial x^m}{\partial x^\beta}=\frac{m}{\beta}x^{m-\beta}\tag{56}$$

$$\int_{x_0^\beta}^{x_1^\beta}x^m\mathrm{d}x^\beta=\frac{\beta}{m+\beta}\left[x_1^{\beta(m+\beta)}-x_0^{\beta(m+\beta)}\right]\tag{57}$$

Theorem 3. Fractal Taylor series is expressed:

$$\phi(x^\beta)=\sum_{n=0}^{N}\frac{1}{n!}\frac{\mathrm{d}^n\phi(x_0^\beta)}{\mathrm{d}x^{\alpha n}}(x^\beta-x_0^\beta)^n=\phi(x_0^\beta)+\frac{\mathrm{d}\phi(x_0^\beta)}{\mathrm{d}x^\alpha}(x^\beta-x_0^\beta)+\frac{1}{2}\frac{\mathrm{d}^2\phi(x_0^\beta)}{\mathrm{d}x^{2\alpha}}(x^\beta-x_0^\beta)^2+\cdots\tag{58}$$

When $\beta = 1$, it turns out to be the traditional Taylor series.

Theorem 4. Modified Fractal Taylor series is expressed:

$$\phi(x^\beta)=\sum_{n=0}^{N}\frac{1}{n!}\frac{\mathrm{d}^n\phi(x_0^\beta)}{\mathrm{d}x^{\alpha n}}(x^{\beta n}-x_0^{\beta n})=\phi(x_0^\beta)+\frac{\mathrm{d}\phi(x_0^\beta)}{\mathrm{d}x^\alpha}(x^\beta-x_0^\beta)+\frac{1}{2}\frac{\mathrm{d}^2\phi(x_0^\beta)}{\mathrm{d}x^{2\alpha}}(x^{2\beta}-x_0^{2\beta})+\cdots\tag{59}$$

Theorem 5. The two scale transform to convert approximately a fractal space or a fractal time into a continuous ones:

$$X=x^\beta\tag{60}$$

$$T=t^\alpha\tag{61}$$

The explanation of the two scale transform was given in [4, 5]. We just come back to the adjacent hierarchical levels of a Carton set, when we watch it using a large-scale (X), it is a continuous line, but we measure it using a small scale of x, it becomes discontinuous. So eq. (60) is to convert approximately a fractal space on a small scale of x into a continuous one on a large-scale. This transform makes the fractal calculus extremely simple in view of traditional calculus. Now the fractal calculus has been applied to non-linear vibration [62], biomechanics [63-65], electrochemical arsenic sensor [66], tsunami model [67], thermal insulation [68], fractal rate model [69], biomimic design [70, 71], fractal diffusion [72], fractal filtration [73], and nanotechnology [74-78].

A fractal ageing model

As an example, we give a detailed discuss on a fractal ageing model and its solution process. Cellulose hydrolytic degradation is a complex process, it depends upon not only the bond breaking, but also the degree of polymerization (DP). Cellulose degradation is usually characterized in terms of DP and its evolution is commonly described by the well known Ekenstam equation or its modification:

– Ekenstam equation [79-83]

$$\frac{1}{DP}-\frac{1}{DP_0}=k_1 t \tag{62}$$

– Emsley equation [79-83]

$$\frac{1}{DP}-\frac{1}{DP_0}=k_2\exp(k_3 t) \tag{63}$$

– Ding-Wang equation [84, 85]

$$1-\frac{DP}{DP_0}=k_5\left[1-\exp(k_4 t)\right] \tag{64}$$

– Calvini's multiple scale law [79-83]

$$\frac{1}{DP}-\frac{1}{DP_0}=n_6\left[1-\exp(k_6 t)\right]+n_7\left[1-\exp(k_7 t)\right]+n_8\left[1-\exp(k_8 t)\right] \tag{65}$$

– Paolo-Calvini law [79-83]

$$DP=DP_0\exp(-k_4 t) \tag{66}$$

where DP_0 and DP are, respectively, the degree of polymerization before and after the degradation, k_i (i = 1~8) are parameters, k_1 is the reaction rate, k_5 is the capacity of the DP reservoir [84, 85], n_6, n_7, n_8 are the initial scissile units in the weak links, amorphous and crystalline regions, respectively [80].

Cellulose degradation can be described by the following first-order kinetics:

$$\frac{\mathrm{d}}{\mathrm{d}t}(DP)=-k_1 DP^2,\quad DP(0)=DP_0 \tag{67}$$

The solution of eq. (67) is exactly the Ekenstam equation.

The second-order kinetics:

$$\frac{\mathrm{d}}{\mathrm{d}t}(DP)=k(t)(DP-DP^2),\quad DP(0)=DP_0,\quad k(0)=k_0 \tag{68}$$

where the reaction rate (k) is a function of time. The exact solution of eq. (68) for $k = k_0$ [78]:

$$\ln\left(1-\frac{1}{DP_0}\right)-\ln\left(1-\frac{1}{DP}\right)=k_0 t \tag{69}$$

Generally the first-order kinetic equation can be expressed [74]:

$$S=n_0(1-e^{-kt}) \tag{70}$$

where S is the number of broken bonds (*i. e.*, the number of scissions) and n_0 the initial number of scissile glycosidic linkages [79].

We assume that it requires Δt to break a bond, when $t > \Delta t$ the first-order kinetic law can be approximately modelled by eq. (70), however, when $t < \Delta t$ the cellulose degradation or the DP is unpredictable, so we need the two scales of time to describe the DP [4, 5], one is the large-scale $t > \Delta t$, and the other is a smaller scale (Δt), and a fractal model has to be adopted.

The DP is not differentiable with respect to time, it is differentiable with respect to t^α, where α is the fractal dimension. So DP depends upon not only Δt, but also t^α:

$$DP \propto (\Delta t)^\alpha \tag{71}$$

$$DP \propto t^\alpha \tag{72}$$

According to the aforementioned analysis, the first-order kinetics should be modified:

$$\frac{\mathrm{d}}{\mathrm{d}t^\alpha}(DP) = -k_1 DP^2, \quad DP(0) = DP_0 \tag{73}$$

where $\mathrm{d}DP/\mathrm{d}t^\alpha$ is the fractal derivative defined

$$\frac{\mathrm{d}DP}{\mathrm{d}t^\alpha}(t_0^\alpha) = \lim_{\substack{t-t_0\to\Delta t\\ \Delta t\neq 0}} \frac{DP(t^\alpha) - DP(t_0^\alpha)}{(t-t_0)^\alpha} \tag{74}$$

The fractal models using the aforementioned fractal derivative has been widely applied to various complex problems, eqs.(62)-(66) can be modified:

– modified Ekenstam equation

$$\frac{1}{DP} - \frac{1}{DP_0} = k_1 t^\alpha \tag{75}$$

– modified Emsley equation

$$\frac{1}{DP} - \frac{1}{DP_0} = k_2 \exp(k_3 t^\alpha) \tag{76}$$

– modified Ding-Wang equation

$$1 - \frac{DP}{DP_0} = k_5\left[1 - \exp(k_4 t^\alpha)\right] \tag{77}$$

– modified Calvini's multiple scale law

$$\frac{1}{DP} - \frac{1}{DP_0} = n_6\left[1-\exp(k_6 t^\alpha)\right] + n_7\left[1-\exp(k_7 t^\alpha)\right] + n_8\left[1-\exp(k_8 t^\alpha)\right] \tag{78}$$

– modified Paolo Calvini law

$$DP = DP_0 \exp(-k_4 t^\alpha) \tag{79}$$

Equation (79) was experimentally verified by Fan *et el.* [86].

The fractal second kinetics can be written in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t^\alpha}(DP) = k(t^\alpha)(DP - DP^2), \quad DP(0) = DP_0 \tag{80}$$

Taylor series method [87-90] is used to solve eq. (80). When the degradation evolution tends to infinity, we have the following equilibrium state:

$$DP(t \to \infty) = 1 \tag{81}$$

Equation (81) reveals that DP changes approximately exponentially from DP_0 at $t = 0$ to a final value $DP = 1$ when time tends to infinity, accordingly we can assume DP can be expressed in the following form:

$$DP = 1 + (DP_0 - 1)\exp\left\{-\sum_{n=1}^{N} a_n t^{\alpha n}\right\} \tag{82}$$

where a_n (1~ N) are unknown constants to be further determined. It is obvious that eq. (82) satisfies the initial condition at $t = 0$ and the terminal condition when t tends to infinity.

To show the solution process, we consider a simple case:

$$DP = 1 + (DP_0 - 1)\exp(-a_1 t^\alpha - a_2 t^{2\alpha}) \tag{83}$$

From eq. (80), we have:

$$\frac{\mathrm{d}DP}{\mathrm{d}t^\alpha}(0) = k_0(DP_0 - DP_0^2) \tag{84}$$

Differentiating eq. (80) with respect to t^α, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t^\alpha}\left[\frac{\mathrm{d}}{\mathrm{d}t^\alpha}(DP)\right] = k\left[\frac{\mathrm{d}}{\mathrm{d}t^\alpha}(DP) - 2DP\frac{\mathrm{d}}{\mathrm{d}t^\alpha}(DP)\right] + \frac{\mathrm{d}k}{\mathrm{d}t^\alpha}(DP - DP^2) \tag{85}$$

Setting $t^\alpha = 0$, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t^\alpha}\frac{\mathrm{d}DP}{\mathrm{d}t^\alpha}(0) = k_0^2(1 - 2DP_0)(DP_0 - DP_0^2) + \frac{\mathrm{d}k}{\mathrm{d}t^\alpha}(0)(DP_0 - DP_0^2) \tag{86}$$

On the other hand, from the trial solution, eq. (83), we have:

$$\frac{\mathrm{d}}{\mathrm{d}t^\alpha}(DP) = (DP_0 - 1)(-a_1 - 2a_2 t^\alpha)\exp(-a_1 t^\alpha - a_2 t^{2\alpha}) \tag{87}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t^\alpha}\left[\frac{\mathrm{d}}{\mathrm{d}t^\alpha}(DP)\right] = (DP_0 - 1)\left[(-a_1 - 2a_2 t^\alpha)^2 - 2a_2\right]\exp(-a_1 t^\alpha - a_2 t^{2\alpha}) \tag{88}$$

We, therefore, obtain the following relations:

$$\frac{\mathrm{d}DP}{\mathrm{d}t^\alpha}(0) = -a_1(DP_0 - 1) = k_0(DP_0 - DP_0^2) \tag{89}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t^\alpha}\left(\frac{\mathrm{d}DP}{\mathrm{d}t^\alpha}\right)(0) = (DP_0 - 1)(a_1^2 - 2a_2) = k_0^2(1 - 2DP_0)(DP_0 - DP_0^2) + \frac{\mathrm{d}k}{\mathrm{d}t^\alpha}(0)(DP_0 - DP_0^2) \tag{90}$$

Solving a_1 and a_2 from eqs. (89) and (90) results:

$$a_1 = k_0 DP_0 \tag{91}$$

$$a_2 = \frac{1}{2}\left[k_0^2(1 - DP_0) + \frac{\mathrm{d}k}{\mathrm{d}t^\alpha}(t^\alpha = 0)\right]DP_0 \tag{92}$$

We obtain the following approximate solution:

$$DP = 1 + (DP_0 - 1)\exp\left\{-k_0 DP_0 t^\alpha - \frac{1}{2}DP_0\left[k_0^2(1 - DP_0) + \frac{\mathrm{d}k}{\mathrm{d}t^\alpha}(t^\alpha = 0)\right]t^{2\alpha}\right\} \tag{93}$$

Equation (93) reveals that DP changes exponentially with a linear and square t^α dependence, hereby t^α can be understood as a scission time.

In previous derivation, we assume that $DP = 1$ at infinite time, this may be true, however, for pure hydrolysis of cellulose with sulfuric acid to the monomeric sugars, but normally cellulose degradation does not follow this path all the way to the monomer but stops at the level-off DP which is different from $DP = 1$. Equation (82) can be updated:

$$DP = DP_\infty + (DP_0 - DP_\infty)\exp\left\{-\sum_{n=1}^{N} a_n t^{\alpha n}\right\} \tag{94}$$

where DP_∞ is the level-off basic degree of polymerization.

The kinetic law, eq. (93), can be modified:

$$DP = DP_\infty + (DP_0 - DP_\infty)\exp\left\{-k_0 DP_0 t^\alpha - \frac{1}{2}DP_0\left[k_0^2(1-DP_0) + \frac{\mathrm{d}k}{\mathrm{d}t^\alpha}(t^\alpha = 0)\right]t^{2\alpha}\right\} \tag{95}$$

Equation (95) illustrates that DP changes exponentially from the initial value to the level-off basic degree of polymerization, at final version. The general kinetics can be expressed:

$$\frac{\mathrm{d}}{\mathrm{d}t^\alpha}(DP) = \frac{\sum_{n=0}^{M} c_n DP^n}{\sum_{n=0}^{N} b_n DP^n} \tag{96}$$

where b_n and c_n are constants. This model can describe the different slopes of DP at the initial stage and the terminate stage.

The 1-D unsteady compressible flow in a porous medium

As another illustrating example, we consider 1-D unsteady compressible flow in a porous medium, which is considered as two adjacent fractal levels.

Using the laws in fractal space, the governing equations for 1-D unsteady compressible flow through a porous tube can be expressed:

– mass equation

$$\frac{\partial(\rho A)}{\partial t^\alpha} + \frac{\partial(\rho u A)}{\partial x^\alpha} = 0 \tag{97}$$

– moment equation

$$\frac{\partial(u)}{\partial t^\alpha} + \frac{\partial}{\partial x^\alpha}\left(\frac{1}{2}u^2 + \frac{\gamma}{\gamma - 1}\frac{P}{\rho}\right) = 0 \tag{98}$$

– homentropic equation

$$p = cp^\gamma \tag{99}$$

where A is the tube area, ρ – the air density, u – the flow velocity, p – the pressure, γ – the homentropic index, and c – the constants.

When $\alpha = 1$, the previous problem was widely studied and its various variational formulations were established [91-95].

Before we establish a variational formulation for the previous problem in a fractal space, we give the following theorem [96-99].

The following variational formulation in a fractal space and a fractal time:

$$J(\phi) = \iint L\left\{\phi, \frac{\partial\phi}{\partial t^\alpha}, \frac{\partial\phi}{\partial x^\alpha}\right\}\mathrm{d}t^\alpha \mathrm{d}x^\alpha \tag{100}$$

admits the following Euler-Lagrange equation:

$$\frac{\partial L}{\partial\phi} - \frac{\partial}{\partial t^\alpha}\left[\frac{\partial L}{\partial\left(\frac{\partial\phi}{\partial t^\alpha}\right)}\right] - \frac{\partial}{\partial x^\alpha}\left[\frac{\partial L}{\partial\left(\frac{\partial\phi}{\partial t^\alpha}\right)}\right] = 0 \tag{101}$$

In order to obtain a variational formulation for eqs. (97)~(99), we introduce a potential function ϕ for eq. (98) in the fractal space:

$$\frac{\partial \phi}{\partial x^\alpha} = u \tag{102}$$

$$\frac{\partial \phi}{\partial t^\alpha} = -\left(\frac{1}{2}u^2 + \frac{\gamma}{\gamma - 1}\frac{P}{\rho}\right) \tag{103}$$

According to the semi-inverse method [91-95], a trial-variational formulation is defined:

$$J(\phi) = \iint L\left\{\phi, \frac{\partial \phi}{\partial t^\alpha}, \frac{\partial \phi}{\partial x^\alpha}\right\} \mathrm{d}t^\alpha \mathrm{d}x^\alpha \tag{104}$$

where F is an unknow fractal function of u, p, and ρ. It is obvious that the stationary condition with respect to ϕ is eq. (97). Taking variation with respect to u and ρ results:

$$\rho A \frac{\partial \phi}{\partial x^\alpha} + \frac{\delta F}{\delta u} = 0 \tag{105}$$

$$A \frac{\partial \phi}{\partial t^\alpha} + uA \frac{\partial \phi}{\partial x^\alpha} + \frac{\delta F}{\delta \rho} = 0 \tag{106}$$

where $\delta F/\delta u$ is the fractal variational derivative defined.

The fractal variational derivative is defined:

$$\frac{\delta F}{\delta u} = \frac{\partial F}{\partial u} - \frac{\partial}{\partial t^\alpha}\left(\frac{\partial F}{\partial \phi_{t^\alpha}}\right) - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial F}{\partial \phi_{x^\alpha}}\right) + \cdots \tag{107}$$

where the subscripts mean fractal derivatives

$$\phi_{t^\alpha} = \frac{\partial \phi}{\partial t^\alpha}, \quad \phi_{x^\alpha} = \frac{\partial \phi}{\partial x^\alpha} \tag{108}$$

In view of eqs. (102) and (103), we have:

$$\frac{\delta F}{\delta u} = -\rho A \frac{\partial \phi}{\partial x^\alpha} = -\rho A u \tag{109}$$

$$\frac{\delta F}{\delta \rho} = -A \frac{\partial \phi}{\partial t^\alpha} - uA \frac{\partial \phi}{\partial x^\alpha} = A\left(-\frac{1}{2}u^2 + \frac{\gamma}{\gamma - 1}\frac{P}{\rho}\right) \tag{110}$$

From eqs. (109) and (110), F can be calculated:

$$F = -\frac{1}{2}\rho A u^2 + \frac{\gamma}{\gamma - 1} AP \ln \rho + F_1 \tag{111}$$

where F_1 is an unknown function of p.

The variational formulation becomes:

$$J(\phi, u, p, \rho) = \iint \left\{\rho A \frac{\partial \phi}{\partial t^\alpha} + \rho u A \frac{\partial \phi}{\partial x^\alpha} - \frac{1}{2}\rho A u^2 + \frac{\gamma}{\gamma - 1} AP \ln \rho + F_1\right\} \mathrm{d}t^\alpha \mathrm{d}x^\alpha \tag{112}$$

Its Euler-Lagrange equation for δp reads:

$$\frac{\gamma}{\gamma - 1} A \ln \rho + \frac{\partial F_1}{\partial p} = 0 \tag{113}$$

By the homentropic equation, eq. (99), we have:

$$\frac{\partial F_1}{\partial P}=-\frac{\gamma}{\gamma-1}A\ln\rho=-\frac{\gamma}{\gamma-1}A\ln\left(\frac{p}{c}\right)^{1/\gamma}=-\frac{1}{\gamma-1}A(\ln p-\ln c) \tag{114}$$

We can identify F_1 as follows:

$$F_1=-\frac{1}{\gamma-1}A(p\ln p-p\ln c-1) \tag{115}$$

Now we have the following theorem.

The 1-D unsteady compressible flow in a fractal space admits the following fractal variational principle:

$$J(\phi,u,p,\rho)=$$
$$=\iint\left\{\rho A\frac{\partial\phi}{\partial t^\alpha}+\rho uA\frac{\partial\phi}{\partial x^\alpha}-\frac{1}{2}\rho Au^2+\frac{\gamma}{\gamma-1}AP\ln\rho-\frac{1}{\gamma-1}A(p\ln p-p\ln c-1)\right\}\mathrm{d}t^\alpha\mathrm{d}x^\alpha \tag{116}$$

Proof. The Euler-Lagrange equations of eq. (116):

$$-\frac{\partial}{\partial t^\alpha}(\rho A)-\frac{\partial}{\partial x^\alpha}(\rho uA)=0 \tag{117}$$

$$\rho A\frac{\partial\phi}{\partial x^\alpha}-\rho Au=0 \tag{118}$$

$$A\frac{\partial\phi}{\partial t^\alpha}+uA\frac{\partial\phi}{\partial x^\alpha}-\frac{1}{2}Au^2+\frac{\gamma}{\gamma-1}A\frac{P}{\rho}=0 \tag{119}$$

$$\frac{\gamma}{\gamma-1}A\ln\rho-\frac{1}{\gamma-1}A(\ln p+1-\ln c-1)=0 \tag{120}$$

It is obvious that eqs. (117) and (118) are equivalent to, respectively, eqs. (97) and (100). Using eqs. (100) and (119) becomes eqs. (100), and (120) can be converted to eq. (99) by a simple calculation.

If we want to establish a fractal variational formulation with a constraint of eq. (99), the trial-functional can be written in a similar way as that in eq. (112):

$$J(\phi,u,\rho)=\iint\left\{\rho A\frac{\partial\phi}{\partial t^\alpha}+\rho uA\frac{\partial\phi}{\partial x^\alpha}+F\right\}\mathrm{d}t^\alpha\mathrm{d}x^\alpha \tag{121}$$

By a similar manipulation as before, we have:

$$\rho A\frac{\partial\phi}{\partial x^\alpha}+\frac{\delta F}{\delta u}=0 \tag{122}$$

$$A\frac{\partial\phi}{\partial t^\alpha}+uA\frac{\partial\phi}{\partial x^\alpha}+\frac{\delta F}{\delta\rho}=0 \tag{123}$$

Considering eq. (99) is a constraint, from eqs. (122) and (123) we have:

$$\frac{\delta F}{\delta u}=-\rho A\frac{\partial\phi}{\partial x^\alpha}=-\rho Au \tag{124}$$

$$\frac{\delta F}{\delta\rho}=-A\frac{\partial\phi}{\partial t^\alpha}-uA\frac{\partial\phi}{\partial x^\alpha}=A\left(-\frac{1}{2}u^2+\frac{\gamma}{\gamma-1}\frac{P}{\rho}\right)=A\left(-\frac{1}{2}u^2+\frac{\gamma c}{\gamma-1}\rho^{\gamma-1}\right) \tag{125}$$

From eqs. (124) and (125), we have:

$$F = -\frac{1}{2}A\rho u^2 + \frac{cA}{\gamma - 1}\rho^\gamma \tag{126}$$

We have finally the following fractal variational principle with three independent functions of ϕ, u, and ρ.

A fractal variational formulation with three independent functions of ϕ, u, and ρ:

$$J(\phi, u, \rho) = \iint \left\{ \rho A \frac{\partial \phi}{\partial t^\alpha} + \rho u A \frac{\partial \phi}{\partial x^\alpha} - \frac{1}{2}A\rho u^2 + \frac{cA}{\gamma - 1}\rho^\gamma \right\} \mathrm{d}t^\alpha \mathrm{d}x^\alpha \tag{127}$$

Proof. Its Euler-Lagrange equations:

$$-\frac{\partial}{\partial t^\alpha}(\rho A) - \frac{\partial}{\partial x^\alpha}(\rho u A) = 0 \tag{128}$$

$$\rho A \frac{\partial \phi}{\partial x^\alpha} - \rho A u = 0 \tag{129}$$

$$A\frac{\partial \phi}{\partial t^\alpha} + uA\frac{\partial \phi}{\partial x^\alpha} - \frac{1}{2}Au^2 + \frac{c\gamma A}{\gamma - 1}\rho^{\gamma\text{-}1} = 0 \tag{130}$$

It is easy to prove that eqs. (128)-(130) are equivalent to eqs. (97), (102), and (103), respectively.

Conclusion

God created the solids, the devil created their boundaries, as commented by Wolfgnang E. Pauli, the unsmooth boundary can produce unbelievable phenomena, the two-scale mathematics is a tool to revealing the hidden truth beyond the conventional continuum mechanics. In this paper some basic properties of fractal derivatives are reviewed, and the two-scale dimension is emphasized because physical laws are scale-dependent. Fractal ageing model is suggested, and two variational formulations are established, for the first time ever, for 1-D unsteady compressible flow through a porous tube, which is considered as two adjacent hierarchical levels of a fractal pattern. This paper sheds a promising light on practical applications of fractal calculus to various engineering problems.

References

[1] Kozlowski, J., Konarzewski, M., Is West, Brown and Enquist's Model of Allometric Scaling Mathematically Correct and Biologically Relevant, *Functional Ecology*, *18* (2004), 2, pp. 283-289
[2] Riisgard, H. U., No Foundation of a 3/4 Power Scaling Law for Respiration in Biology, *Ecol. Lett. 1* (1998), 2, pp. 71-73
[3] Li, X. X., He, J. H., Along the Evolution Process Kleiber's 3/4 Law Makes Way for Rubner's Surface Law: A Fractal Approach, *Fractals*, *27* (2019), 2, 1950015
[4] Ain, Q. T., He, J. H. On Two-Scale Dimension and Its Applications, *Thermal Science*, *23* (2019), 3B, pp. 1707-1712
[5] He, J. H, Ji, F. Y., Two-Scale Mathematics and Fractional Calculus for Thermodynamics, *Thermal Science*, *23* (2019), 4, pp. 2131-2133
[6] Zhou, C. J., *et al*., Highly Selective Penetration of Red Ink in a Saline Water, *Thermal Science*, *23* (2019), 4, pp. 2265-2270
[7] He, J. H., Fractal Calculus and Its Geometrical Explanation, *Results in Physics*, *10* (2018), Sept., pp. 272-276
[8] El Naschie, M. S. The Theory of Cantorian Space Time and High Energy Particle Physics (An Informal Review), *Chaos*, *Solitons and Fractals*, *41* (2009), 5, pp. 2635-2646

[9] El Naschie, M. S., A Review of E-Infinity Theory and the Mass Spectrum of High Energy Particle Physics, *Chaos, Solitons Aand Fractals*, *19* (2004), 1, pp. 209-236
[10] Marek-Crnjac, L. A., Short History of Fractal-Cantorian Space-Time, Chaos, *Solitons and Fractals*, *41* (2009), 5, pp. 2697-2705
[11] He, J. H., Hilbert Cube Model for Fractal Spacetime, *Chaos, Solitons and Fractals*, *42* (2009), 5, pp. 2754-2759
[12] El Naschie, M. S., World Formula Interpretation of $E = mc^2$, *International Journal of Applied Science and Mathematics*, *5* (2018), 6, pp. 67-75
[13] Marek-Crnjac, L., On El Naschie's Fractal-Cantorian Space-Time and Dark Energy – A Tutorial Review, *Natural Science*, *7* (2015), 13, ID 61825
[14] Majumder, M., *et al.*, Nanoscale Hydrodynamics – Enhanced Flow in Carbon Nanotubes, *Nature*, *438* (2005), 7064, p. 44
[15] Yang, X. J., *Advanced Local Fractional Calculus and Its Applications*, World Science Publisher, New York, USA, 2012
[16] Mohebbi, A., Saffarian, M., Implicit RBF Meshless Method for the Solution of 2-D Variable Order Fractional Cable Equation, *Journal of Applied and Computational Mechanics*, *6* (2020), 2, pp. 235-247
[17] Jena, R. M., Chakraverty, S., Residual Power Series Method for Solving Time-fractional Model of Vibration Equation of Large Membranes, *Journal of Applied and Computational Mechanics*, *5* (2019), 4, pp. 603-615
[18] Ahmed, N., *et al.*, Transient MHD Convective Flow of Fractional Nanofluid between Vertical Plates, *Journal of Applied and Computational Mechanics*, *5* (2019), 4, pp. 592-602
[19] Rauf, A., Mahsud, Y., Electro-Magneto-Hydrodynamics Flows of Burgers' Fluids in Cylindrical Domains with Time Exponential Memory, *Journal of Applied and Computational Mechanics*, *5* (2019), 4, pp. 577-591
[20] Ali, N., *et al.*, Transient Electro-Osmotic Slip Flow of an Oldroyd-B Fluid with Time-Fractional Caputo-Fabrizio Derivative, *Journal of Applied and Computational Mechanics, 5* (2019), 4, pp. 779-790
[21] Jassim, H. K. Baleanu, D. A., Novel Approach for Korteweg-de Vries Equation of Fractional Order, *Journal of Applied and Computational Mechanics*, *5* (2019), 2, pp. 192-198
[22] Yang, Y. J., Wang, S. Q., A Local Fractional Homotopy Perturbation Method for Solving the Local Fractional Korteweg-de Vries Equations with Non-Homogeneous Term, *Thermal Science*, *23* (2019), 3A, pp. 1495-1501
[23] Zenkour, A., Abouelregal, A., Fractional Thermoelasticity Model of a 2-D Problem of Mode-I Crack in a Fibre-Reinforced Thermal Environment, *Journal of Applied and Computational Mechanics*, *5* (2019), 2, pp. 269-280
[24] Shah, N. A., *et al.*, Magnetohydrodynamic Free Convection Flows with Thermal Memory over a Moving Vertical Plate in Porous Medium, *Journal of Applied and Computational Mechanics*, *5* (2019), 1, pp. 150-161
[25] Zhang, S., *et al*., Bilinearization and Fractional Soliton Dynamics of Fractional Kadomtsev-Petviashvili Equation, *Thermal Science*, *23* (2019), 3A, pp. 1425-1431
[26] He, J. H., Approximate Analytical Solution for Seepage Flow with Fractional Derivatives in Porous Media, *Comput. Meth. Appl. Mech. Eng*., *167* (1998), 1-2, pp. 57-68
[27] He, J. H., *et al*., Geometrical Explanation of the Fractional Complex Transform and Derivative Chain Rule for Fractional Calculus, *Phys. Lett. A*, *376* (2012), 4, pp. 257-259
[28] He, J. H., *et al*., A New Fractional Derivative and Its Application Explanation of Polar Bear Hairs, *Journal of King Saud University Science*, *28* (2016), 2, pp. 190-192
[29] Liu, H. Y., *et a*l., Fractional Calculus for Nanoscale Flow and Heat Transfer, *International Journal of Numerical Methods for Heat and Fluid-Flow*, *24* (2014), 6, pp. 1227-1250
[30] He, J. H., A Tutorial Review on Fractal Spacetime and Fractional Calculus, *International Journal of Theoretical Physics*, *53* (2014), 11, pp. 3698-3718
[31] Wang, K. L., Yao, S. W., Numerical Method for Fractional Zakharov-Kuznetsov Equations with He's Fractional Derivative, *Thermal Science*, *23* (2019), 4, pp. 2163-2170
[32] Guner, O., Exp-Function Method and Fractional Complex Transform for Space-Time Fractional KP-BBM Equation, *Communications in Theoretical Physics*, *68* (2017), 2, pp. 149-154
[33] Wang, Y., *et al*., A Short Review on Analytical Methods for Fractional Equations with He's Fractional Derivative, *Thermal Science*, *21* (2017), 4, pp. 1567-1574

[34] Wang, Y., An, J. Y., Amplitude-Frequency Relationship to a Fractional Duffing Oscillator Arising in Microphysics and Tsunami Motion, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 1008-1012
[35] Wang, Y., *et al*., Shallow Water Waves in Porous Medium for Coast Protection, *Thermal Science*, *21* (2017), Suppl. 1, S145-S151
[36] Wang, Y., *et al*., A Fractional Whitham-Broer-Kaup Equation and Its Possible Application Tsunami Prevention, *Thermal Science*, *21* (2017), 4, pp. 1847-1855
[37] Sayevand, K., *et al*., A Reliable Implicit Difference Scheme for Treatments of Fourth Order Fractional sub-Diffusion Equation, *Scientia Iranica*, *24* (2018), 3, pp. 1100-1107
[38] Sayevand, K., Rostami, M., Fractional Optimal Control Problems: Optimality Conditions and Numerical Solution, *IMA Journal of Mathematical Control and Information*, *35* (2018), 1, pp. 123-148
[39] Lin, L., *et al*., Experimental Verification of the Fractional Model for Silver Ion Release from Hollow Fibers, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 1041-1044
[40] Lin, L., Yao, S. W., Release Oscillation in a Hollow Fiber – Part 1: Mathematical Model and Fast Estimation of Its Frequency, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 1703-1707
[41] Liu, H. Y., *et al*., A Fractional Non-Linear System for Release Oscillation of Silver Ions from Hollow Fibers, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 1, pp. 88-92
[42] Liu, F. J., *et al*., Silkworm (Bombyx Mori) Cocoon *vs*. Wild Cocoon: Multi-Layer Structure and Performance Characterization, *Thermal Science*, *23* (2019), 4, pp. 2135-2142
[43] He, J.-H., Frontier of Modern Textile Engineering and Short Remarks on Some Topics in Physics, *International Journal of Non-Linear Sciences and Numerical Simulation*, *11* (2010), June, pp. 555-563
[44] Mandelbrot, B. B., *The Fractal Geometry of Nature*, W. H. Freeman and Company, New York, USA, 1982
[45] Shen, J. F., *et al*., Fractal Study in Soil Spatial Variability and Thermal Conductivity, *Thermal Science*, *23* (2019), 5, pp. 2849-2856
[46] Cao, A. Y., *et al*., An Analytical Solution for Solving a New General Fractional Order Model for Wave in Mining Rock, *Thermal Science*, *23* (2019), Suppl. 3, pp. S983-S987
[47] Chen, J., *et al*., A Non-Linear Creep Constitutive Model for Salt Rock Based on Fractional Derivatives, *Thermal Science*, *23* (2019), Suppl. 3, pp. S773-S779
[48] Chen, Z. Q., *et al*., A New Fractional Derivative Model for the Anomalous Diffusion Problem, *Thermal Science*, *23* (2019), Suppl. 3, pp. S1005-S1011
[49] Deng, S. X., Approximate Analytical Solutions of Non-Linear Local Fractional Heat Equations, *Thermal Science*, *23* (2019), Suppl. 3, pp. S837-S841
[50] Dou, L. M., *et al*., A New General Fractional-Order Wave Model Involving Miller-Ross Kernel, *Thermal Science*, *23* (2019), Suppl. 3, pp. S953-S957
[51] Gao, Y. N., *et al*., The Mechanical Properties and Fractal Characteristics of the Coal under Temperature-Gas-Confining Pressure, *Thermal Science*, *23* (2019), Suppl. 3, pp. S789-S798
[52] Yao, S. W., Wang, K. L. A New Approximate Analytical Method for a System of Fractional Differential Equations, *Thermal Science*, *23* (2019) Suppl. 3, pp. S853-S858
[53] Wei, C. F., Application of the Homotopy Perturbation Method for Solving Fractional Lane-Emden Type Equation, *Thermal Science*, *23* (2019), 4, pp. 2237-2244
[54] Qiu, Y. Y., Numerical Approach to the Time-Fractional Reaction-Diffusion Equation, *Thermal Science*, *23* (2019), 4, pp. 2245-2251
[55] Ma, H. C., *et al*., Exact Solutions of the Space-Time Fractional Equal width Equation, *Thermal Science*, *23* (2019), 4, pp. 2307-2313
[56] Yang, X. H., *et al*., A Fractional-Order Genetic Algorithm for Parameter Optimization of the Moisture Movement in a Bioretention System, *Thermal Science*, *23* (2019), 4, pp. 2343-2350
[57] Tian, A. H., *et al*., Land Surface Temperature *vs.* Soil Spectral Reflectance Land Surface Temperature, *Thermal Science*, *23* (2019), 4, pp. 2389-2395
[58] Tian, A. H., *et al*., Lathe Tool Chatter Vibration Diagnostic Using General Regression Neural Network Based on Chua's Circuit and Fractional-Order Lorenz Master/Slave Chaotic System, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 953-966
[59] Deng, S. X., *et al*., Global Well-Posedness of a Class of Dissipative Thermoelastic Fluids Based on Fractal Theory and Thermal Science Analysis, *Thermal Science*, *23* (2019), 4, pp. 2461-2469
[60] Sun, J. C., *et al*., Analytical Study of (3+1)-Dimensional Fractional Ultralow-Frequency Dust Acoustic Waves in a Dual-Temperature Plasma, *Journal of Low Frequency Noise Vibration and Active Control*, *38* (2019), 3-4, pp. 928-952

[61] He, J. H., A New Fractal Derivation, *Thermal Science*, *15* (2011), 1, pp. S145-S147
[62] He, J. H., The Simpler, the Better: Analytical Methods for Non-Linear Oscillators and Fractional Oscillators, *Journal Low Freq. Noise V. A.*, *38* (2019), 3-4, pp. 1252-1260
[63] Wang, Q. L., *et al*., Fractal Calculus and Its Application Explanation of Biomechanism of Polar Bear Hairs, *Fractals*, *27* (2019), 5, 1992001
[64] Wang, Q. L., *et al.*, Fractal Calculus and Its Application Explanation of Biomechanism of Polar Bear Hairs, *Fractals*, *26* (2018), 6, 1850086
[65] Fan, J., *et al.*, Fractal Calculus for Analysis of Wool Fiber: Mathematical Insight of Its Biomechanism, *Journal Eng. Fiber. Fabr.*, On-line first, https://doi.org/10.1177/1558925019872200, 2019
[66] Li, X. X., *et al*., A Fractal Modification of the Surface Coverage Model for an Electrochemical Arsenic Sensor, *Electrochm. Acta*, *296* (2019), Feb., pp. 491-493
[67] Wang, Y., Deng, Q. G., Fractal Derivative Model for Tsunami Travelling, *Fractals*, *27* (2019), 1, 1950017
[68] Wang, Y., *et al*., A Fractal Derivative Model for Snow's Thermal Insulation Property, *Thermal Science*, *23* (2019), 4, pp. 2351-2354
[69] Liu, H. Y., *et al*., A Fractal Rate Model for Adsorption Kinetics at Solid/Solution Interface, Thermal Science, *23* (2019), 4, pp. 2477-2480
[70] Fan, J., He, J. H., Biomimic design of Multi-Scale Fabric with Efficient Heat Transfer Property, *Thermal Science*, *16* (2012), 5, pp. 1349-1352
[71] Fan, J., He, J. H., Fractal Derivative Model for Air Permeability in Hierarchic Porous Media, *Abstract and Applied Analysis*, *2012* (2012), ID354701
[72] Lin, L., Yao, S. W., Fractal Diffusion of Silver Ions in Hollow Cylinders with Unsmooth Inner Surface, *Journal of Engineered Fibers and Fabrics*, On-line first, https://doi.org/10.1177/1558925019895643, 2019
[73] Yang, Z. P., Filtration Efficiency of a Cigarette Filter with X-or Y-Shaped Fibers, *Thermal Science*, *23* (2019), 4, pp. 2517-2522
[74] Liu, Y. Q., *et al*., Air Permeability of Nanofiber Membrane with Hierarchical Structure, *Thermal Science*, *22* (2018), 4, pp. 1637-1643
[75] Yu, D. Y., *et al*., Wetting and Supercontraction Properties of Spider-Based Nanofibers, *Thermal Science*, *23* (2019), 4, pp. 2189-2193
[76] Tian, D., *et al*., Sea-Silk Based Nanofibers and Their Diameter Prediction, *Thermal Science*, *23* (2019), 4, pp. 2253-2256
[77] Zhou, C. J., *et al*., Highly Selective Penetration of Red Ink in a Saline Water, *Thermal Science*, *23* (2019), 4, pp. 2265-2270
[78] Li, X. X., *et al*., Thermal Property of Rock Powder-Based Nanofibers for High Temperature Filtration and Adsorption, *Thermal Science*, *23* (2019), 4, pp. 2501-2507
[79] Calvini, P., The Influence of Levelling-off Degree of Polymerisation on the Kinetics of Cellulose Degradation, *Cellulose*, *12* (2005), Aug., pp. 445-447
[80] Calvini, P., Comments on the Article ''On the Degradation Evolution Equations of Cellulose'' by Hongzhi Ding and Zhongdong Wang, *Cellulose*, *15* (2008), Sept., pp. 225-228
[81] Calvini, P., The Role of the Ekenstam Equation on the Kinetics of Cellulose Hydrolytic Degradation, *Cellulose*, *19* (2012), Jan., pp. 313-318
[82] Calvini, P., On the Meaning of the Emsley, Ding and Wang and Calvini Equations Applied to the Degradation of Cellulose, *Cellulose*, *21* (2014), Mar., pp. 1127-1134
[83] Calvini, P., *et al*., On the Kinetics of Cellulose Degradation: Looking beyond the Pseudo Zero Order Rate Equation, *Cellulose*, *15* (2008), Sept., pp. 193-203
[84] Ding, H.-Z., Wang, Z. D., Time-Temperature Superposition Method for Predicting the Permanence of Paper by Extrapolating ccelerated Ageing Data to Ambient Conditions, *Cellulose*, *14* (2007), Mar., pp. 171-181
[85] Ding, H.-Z., Wang Z. D., On the Degradation Evolution Equations of Cellulose, *Cellulose*, *15* (2008), Oct., pp. 205-224
[86] Fan, W., *et al*., Random Process Model of Mechanical Property Degradation in Carbon Fiber-Reinforced Plastics under Thermo-Oxidative Aging, *Journal of Composite Materials*, *51* (2017), 9, pp. 1253-1264
[87] He, C. H., *et al*., Taylor Series Solution for Fractal Bratu-Type Equation Arising in Electrospinning Process, *Fractals*, On-line first, https://doi.org/10.1142/S0218348X20500115, 2019
[88] He, J. H., The Simplest Approach to Non-Linear Oscillators, *Results in Physics*, *15* (2019), Dec., 102546
[89] He, J. H., A Simple Approach to 1-D Convection-Diffusion Equation and Its Fractional Modification for Ereaction Arising in Rotating Disk Electrodes, *Journal of Electroanalytical Chemistry*, *854* (2019), Dec., 113565

[90] He, J. H., Ji, F. Y., Taylor Series Solution for Lane-Emden Equation, *Journal Math. Chem*., *57* (2019), 8, pp. 1932-1934
[91] He, J. H., Generalized Variational Principles for Buckling Analysis of Circular Cylinders, *Acta Mechanica*, *231* (2020), Jan., pp. 899-906
[92] He, J.-H., Variational Principle for the Generalized KdV-Burgers Equation with Fractal Derivatives for Shallow Water Waves, *Journal Appl. Comput. Mech*., *6* (2020), 4, pp. 735-740
[93] He, J. H., A Modified Li-He's Variational Principle for Plasma, *International Journal of Numerical Methods for Heat and Fluid-Flow*, On-line first, https://doi.org/10.1108/HFF-06-2019-0523, 2019
[94] He, J. H., Lagrange Crisis and Generalized Variational Principle for 3-D Unsteady Flow, *International Journal of Numerical Methods for Heat and Fluid-Flow*, On-line first, https://doi.org/10.1108/HFF-07-2019-0577, 2019
[95] He, J. H., Sun, C., A Variational Principle for a Thin Film Equation, *Journal of Mathematical Chemistry, 57* (2019), 9, pp. 2075-2081
[96] He, J. H., A Fractal Variational Theory for 1-D Compressible Flow in a Microgravity Space, *Fractals*, On-line first, https://doi.org/10.1142/S0218348X20500243
[97] Wang, Y., *et al*., A Variational Formulation for Anisotropic Wave Traveling in a Porous Medium, *Fractals*, *27* (2019), 4, 19500476
[98] He, J.-H., Latifizadeh, H., A General Numerical Algorithm for Non-Linear Differential Equations by the Variational Iteration Method, *International Journal of Numerical Methods for Heat and Fluid-Flow*, On-line first, https://doi.org/10.1108/HFF-01-2020-0029, 2020
[99] He, J. H., *et al*. A Fractal Boussinesq Equation for Non-Linear Transverse Vibration of a Nanofiber-Reinforced Concrete Pillar, *Applied Mathematical Modelling*, *82* (2020), June, pp. 437-448

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