A NEW CLASS OF A-STABLE NUMERICAL TECHNIQUES FOR ORDINARY DIFFERENTIAL EQUATIONS
Application to Boundary-Layer Flow

by

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The present attempt is made to propose a new class of numerical techniques for finding numerical solutions of ODE. The proposed numerical techniques are based on interpolation of a polynomial. Currently constructed numerical techniques use the additional information(s) of derivative(s) on particular grid point(s). The advantage of the presently proposed numerical techniques is that these techniques are implemented in one step and can provide highly accurate solution and can be constructed on fewer amounts of grid points but has the disadvantage of finding derivative(s). It is to be noted that the high order techniques can be constructed using just two grid points. Presently proposed fourth order technique is A-stable but not L-stable. The order and maximum absolute error are found for a fourth order technique. The fourth order technique is employed to solve the Darcy-Forchheimer fluid-flow problem which is transformed further to a third-order non-linear boundary value problem on the semi-infinite domain.

Key words: *interpolation, Darch-Forchheimer flow, A-stable, derivative(s), maximum absolute error*

Introduction

Numerical methods can be considered to apply when some physical phenomena are expressed in mathematical form. Also, some analytical methods have been considered to solve the differential equations that arise from physical phenomena(s). The main benefit of using numerical methods is their speed of solving the problem when compared with analytical methods. Since the analytical methods use algebra in the computations, so due to this fact, the analytical methods may take more time to solve the problem(s). Some numerical methods are those kinds of methods that provide high order numerical solution, and some can be used to reduce oscillations in the solution. Also, one class of numerical methods is Runge-Kutta methods which may provide high order solution but these methods are applied in different steps. Some of the presently constructed numerical methods can provide high order solution and can be constructed on just two grid points.

Jones [1] applied finite difference method for solving Navier stokes equations and sea breeze model, and applied finite difference method that was developed in their/his previous work for the improvement of accuracy and computational efficiency. In [2], the comparison of homotopy perturbation method and the finite difference method is given and proved that the

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homotopy perturbation method is more accurate, effective and more stable. The work for a fourth order in space and second-order in time-based on spline in tension nine-point compact differencing technique was proposed in [3]. For solving coupled viscous Burger equations, an exact differencing technique was developed in [4] and explicit non-standard finite difference technique was extended on the basis of exact finite difference technique. In [5], new non-standard finite difference technique is implemented for solving fractional Navier stokes equations with stability and convergence. The convergence of generalized finite difference method for PDE can be seen in [6]. The work of the differencing technique for some problem can be seen in [7-11]. The third-grade fluid and flow in porous media are studies by employing Darcy Forchheimer based model in [12]. The study of Chebyshev finite difference for MHD flow of a micropolar fluid past a stretching sheet with heat transfer is given in [13]. Some more work on fluid can be seen in [14-18].

Presently constructed numerical methods are derivative-based if derivative(s) exist. Presently proposed methods may provide high accuracy for ODE either linear or non-linear. One of these numerical methods is *A*-stable but not *L*-stable. These proposed methods are based on interpolation of the polynomial(s) using two or more grid points. If one needs to construct a sixth-order numerical method using two grid points, then the first four derivatives will be calculated if derivatives exist and the interpolation of polynomial contains six unknowns will be used. Unknowns can be found using the value(s) of function and its derivative(s) on two grid points, so the resulting methods become sixth-order accurate.

Numerical techniques

Consider differential equation:

$$u'(x) = f(u) \tag{1}$$

For the sake of solving the first-order differential eq. (1), consider the cubic polynomial shifted x_i units. This polynomial function is an approximation for the solution of eq. (1):

$$p(x) = a(x - x_i)^3 + b(x - x_i)^2 + c(x - x_i) + d \tag{2}$$

and let the graph of the polynomial passing through the points

$$(x_i, f_i) \text{ and } (x_{i-1}, f_{i-1})$$

and also assume that the graph for the first-derivative of $p(x)$ passes through aforementioned two points.

Since the graph and derivative of p both pass through the points so, the eq. (1) should be satisfied by these points, and partially simplifying yields:

$$f_i = d \text{ and } f_{i-1} = -ah^3 + bh^2 - ch + d \tag{3}$$

$$f_i' = c \text{ and } f_{i-1}' = 3ah^2 - 2bh + c \tag{4}$$

The unknowns can be founded by solving eqs. (3) and (4), and these unknowns are given:

$$a = -\frac{2f_i - 2f_{i-1} - h(-f_i' + f_{i-1}')}{h^3}, \quad b = -\frac{3f_i - 3f_{i-1} - h(2f_i' + f_{i-1}')}{h^2}, \quad c = f_i', \quad d = f_i \tag{5}$$

By applying the fundamental theorem of calculus, the following equation is given:

$$u_i = u_{i-1} + \int_{x_{i-1}}^{x_i} p(x)\mathrm{d}x = u_{i-1} + \int_{x_{i-1}}^{x_i} \left[a(x-x_i)^3 + b(x-x_i)^2 + c(x-x_i) + d \right]\mathrm{d}x = \\ = u_{i-1} + \frac{h\left\{6f_i + 6f_{i-1} + h\left(f'_{i-1} - f'_i\right)\right\}}{12} \tag{6}$$

The fourth order difference technique using two points is given:

$$u_i = u_{i-1} + \frac{h}{12}\left\{6f_i + 6f_{i-1} + h\left(f'_{i-1} - f'_i\right)\right\} \tag{7}$$

Similarly, more higher-order techniques can be developed by considering more high-order derivatives on two grid points or more than two grid points can be considered.

A sixth-order technique on three grid points is given:

$$u_i = u_{i-1} + \frac{h}{240}\left(101f_i + 128f_{i-1} + 40hf'_{i-1} + 11f_{i-2} + 3hf'_{i-2} - 13hf'_i\right) \tag{8}$$

Problem formulation

Consider the laminar, steady, Newtonian, incompressible flow over the stretching sheet. The *x*-axis is considered along the flow and *y*-axis is perpendicular to the *x*-axis. Under these assumptions, the Governing equations of Darcy-Focheimer flow are given:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\nu}{k}u - Fu^2 \tag{10}$$

Subject to the following boundary conditions:

$$u = V,\ v = 0 \ \text{ when } \ y = 0 \tag{11}$$

$$u = 0 \ \text{ when } \ y \to \infty \tag{12}$$

where V is the velocity of the stretching sheet.

Consider the similarity transformations eq. (12):

$$u = axf'(\eta),\ v = -(a\nu)^{1/2} f(\eta),\ \eta = \left(\frac{a}{\nu}\right)^{1/2} y \tag{13}$$

Under the similarity transformations eq. (13), the continuity equation is satisfied, and the momentum equation is reduced:

$$f''' = \lambda f' + \left(1 + F_r\right) f'^2 - ff'' \tag{14}$$

subject to:

$$f(\eta) = 0,\ f'(\eta) = 1 \ \text{ when } \ \eta = 0 \tag{15}$$

$$f'(\eta) = 0 \ \text{ when } \ \eta \to \infty \tag{16}$$

The physical quantity skin friction coefficient is given:

$$\mathrm{Re}_x^{0.5} C_{fx} = -f''(0)$$

where $\mathrm{Re}_x = cx^2/\nu$ is the local Reynolds number.

In order to solve the eq. (14) with boundary conditions (15) and (16), first reduce the third-order eq. (14) into the system of first-order equations:

$$f' = f_1 \tag{17}$$

$$f_1' = f_2 \tag{18}$$

$$f_2' = \lambda f_1 + (1+F_r) f_1^2 - f f_2 \tag{19}$$

By implementing the fourth order technique eq. (7) using the Gauss-Seidel iterative method gives:

$$f_i^{k+1} = f_{i-1}^{k+1} + \frac{h}{12}\left\{6(f_1)_i^k + 6(f_1)_{i-1}^{k+1} + h\left[(f_2)_{i-1}^{k+1} - (f_2)_i^k\right]\right\} \tag{20}$$

$$(f_1)_i^{k+1} = (f_1)_{i-1}^{k+1} + \frac{h}{12}\left\{6(f_2)_i^k + 6(f_2)_{i-1}^{k+1} + h\left[(f_2')_{i-1}^{k+1} - (f_2')_i^k\right]\right\} \tag{21}$$

$$(f_2)_i^{k+1} = (f_2)_{i-1}^{k+1} + \frac{h}{12}\left[6(f_2')_i^k + 6(f_2')_{i-1}^{k+1} + h\left[(f_2'')_{i-1}^{k+1} - (f_2'')_i^k\right]\right] \tag{22}$$

where

$$f_2'' = \lambda f_2 - f f_2' - f_1 f_2 + 2(1+F_r) f_1 f_2$$

Stability

In order to find stability for the present technique of finite difference, consider the linear equation:

$$y' = ky \tag{23}$$

$$y'' = ky' = k^2 y \tag{24}$$

The presently proposed fourth order finite difference recurring relation for eq. (23) is given:

$$y_n = y_{n-1} + \frac{h}{12}\left[6y_n' + 6y_{n-1}' + h\left(y_{n-1}'' - y_n''\right)\right] \tag{25}$$

This implies:

$$y_n = y_{n-1} + \frac{h}{12}\left[6y_n' + 6ky_{n-1} + hk^2\left(y_{n-1} - y_n\right)\right]$$

$$y_n - \frac{hk}{2} y_n + \frac{h^2k^2}{12} y_n = y_{n-1} + \frac{hk}{2} y_{n-1} + \frac{h^2k^2}{12} y_{n-1} \tag{26}$$

$$\left(12 - 6hk + h^2k^2\right) y_n = \left(12 + 6hk + h^2k^2\right) y_{n-1}, \quad y_n = \frac{\left(12 + 6hk + h^2k^2\right) y_{n-1}}{\left(12 - 6hk + h^2k^2\right) y_n}$$

The stability function:

$$\phi(z)=\frac{12+6z+z^2}{12-6z+z^2} \tag{27}$$

and the absolute stability region:

$$\left\{z\in\mathbb{C}\left\|\frac{12+6z+z^2}{12-6z+z^2}\right|<1\right\} \tag{28}$$

Lemma. For $y'=ky$, the technique eq. (7) is fourth order accurate.

Proof. Consider the Taylor series expansion for the L.H.S. of (7) is given:

$$y_i=y_{i-1}+hy'_{i-1}+\frac{h^2}{2}y''_{i-1}+\frac{h^3}{6}y'''_{i-1}+\frac{h^4}{24}y^{iv}_{i-1}+O\left(h^5\right)+$$

$$\text{R.H.S.}\;\; y_{i-1}+\frac{h}{12}\left[6y'_i+6y'_{i-1}+h\left(y''_{i-1}-y''_i\right)\right]=y_{i-1}+\frac{h}{12}\left\{6\left(y'_{i-1}+hy''_{i-1}+\frac{h^2}{2}y'''_{i-1}+\frac{h^3}{6}y^{iv}_{i-1}\right)+\right.$$

$$\left.+6y'_{i-1}+h-\left[y''_{i-1}\left(y''_{i-1}+hy'''_{i-1}+\frac{h^2}{2}y'^{v}_{i-1}+\frac{h^3}{6}h^3y^{v}_{i-1}\right)\right]y''_{i-1}\right\}=$$

R.H.S becomes

$$=y_{i-1}+\frac{h}{12}\left[\left(12y'_{i-1}+6hy''_{i-1}+3h^2y'''_{i-1}+h^3y^{iv}_{i-1}\right)-h\left(hy'''_{i-1}+\frac{h^2}{2}y'^{v}_{i-1}+\frac{h^3}{6}h^3y^{v}_{i-1}\right)\right]=$$

$$=y_{i-1}+hy'_{i-1}+\frac{h^2}{2}y''_{i-1}+\frac{h^3}{6}y'''_{i-1}+\frac{h^4}{24}y^{iv}_{i-1}=y_i\;\;\text{which is L.H.S.}$$

Theorem. The maximum error attained by the fourth order technique eq. (7) using the Gauss-Seidel iterative method is bounded subject to the assumption of differentiability:

$$\boldsymbol{H}=\boldsymbol{H}\left(f,f_1,f_2\right)$$

and assumption for the satisfaction of the following inequality

$$3-h-\frac{h}{2}\sum_{j=1}^{3}\left|\hat{H}_j\right|>0$$

Proof. Discretize eqs. (20)-(22) using the fourth order technique eq. (7) with a Gauss-Seidel iterative method gives:

$$\frac{f_i^{k+1}-f_{i-1}^{k+1}}{h}+\frac{1}{12}\left[6\left(f_1\right)_i^k+6\left(f_1\right)_{i-1}^{k+1}\right]+h\left[\left(f_2\right)_{i-1}^{k+1}-\left(f_2\right)_i^k\right]=0 \tag{29}$$

$$\frac{\left(f_1\right)_i^{k+1}-\left(f_1\right)_{i-1}^{k+1}}{h}+\frac{1}{12}\left\{6\left(f_2\right)_i^k+6\left(f_2\right)_{i-1}^{k+1}+h\left[\left(f'_2\right)_{i-1}^{k+1}-\left(f'_2\right)_i^k\right]\right\}=0 \tag{30}$$

$$\frac{\left(f_2\right)_i^{k+1}-\left(f_2\right)_{i-1}^{k+1}}{h}+\frac{1}{12}\left\{6\left(f'_2\right)_{i-1}^{k+1}+6\left(f'_2\right)_i^k+h\left[\left(f''_2\right)_{i-1}^{k+1}-\left(f''_2\right)_i^k\right]\right\}=0 \tag{31}$$

and let the exact techniques are given:

$$\frac{f_i^E - f_{i-1}^E}{h} + \frac{1}{12}\left[6\left(f_1\right)_i^E + 6\left(f_1\right)_{i-1}^E\right] + h\left[\left(f_2\right)_{i-1}^E - \left(f_2\right)_i^E\right] = 0 \tag{32}$$

$$\frac{\left(f_1\right)_i^E - \left(f_1\right)_{i-1}^E}{h} + \frac{1}{12}\left\{6\left(f_2\right)_i^E + 6\left(f_2\right)_{i-1}^E + h\left[\left(f_2'\right)_{i-1}^E - \left(f_2'\right)_i^E\right]\right\} = 0 \tag{33}$$

$$\frac{\left(f_2\right)_i^E - \left(f_2\right)_{i-1}^E}{h} + \frac{1}{12}\left[6\left(f_2'\right)_{i-1}^E + 6\left(f_2'\right)_i^E + h\left[\left(f_2''\right)_{i-1}^E - \left(f_2''\right)_i^E\right]\right] = 0 \tag{34}$$

Let the error between exact and approximate solutions of any point on the grid is given:

$$\left(e_1\right)_i^k = f_i^k - f_i^E,\ \left(e_2\right)_i^k = \left(f_1\right)_i^k - \left(f_1\right)_i^E,\ \left(e_3\right)_i^k = \left(f_2\right)_i^k - \left(f_2\right)_i^E$$

By applying the mean value theorem on the function $\boldsymbol{H}$ gives:

$$\boldsymbol{H}\left[f_i^k, \left(f_1\right)_i^k, \left(f_2\right)_i^k\right] - \boldsymbol{H}\left[f_i^E, \left(f_1\right)_i^E, \left(f_2\right)_i^E\right] = \left(\boldsymbol{e}_1\right)_i^k \nabla\boldsymbol{H}\left(c_1, c_2, c_3\right) \tag{35}$$

where

$$c_1 = f_i^k + \varepsilon_1\left(e_3\right)_i^k,\ c_2 = \left(f_1\right)_i^k + \varepsilon_2\left(e_3\right)_i^k,\ c_3 = \left(f_2\right)_i^k + \varepsilon_3\left(e_3\right)_i^k$$

and

$$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0,1],\ \text{and}\ \left(\boldsymbol{e}_1\right)_i^k = \left[\left(e_1\right)_i^k, \left(e_2\right)_i^k, \left(e_3\right)_i^k\right]$$

The convergence error equations using Gauss-Seidel iterative method can be described:

$$\left(e_1\right)_i^{k+1} - \left(e_1\right)_{i-1}^{k+1} = \frac{h}{12}\left\{6\left(e_2\right)_i^k + 6\left(e_2\right)_{i-1}^{k+1} + h\left[\left(e_3\right)_{i-1}^{k+1} - \left(e_3\right)_i^k\right]\right\} \tag{36}$$

$$\left(e_2\right)_i^{k+1} - \left(e_2\right)_{i-1}^{k+1} = \frac{h}{12}\left[6\left(e_3\right)_i^k + 6\left(e_3\right)_{i-1}^{k+1} + h\left(e_1\right)_i^{k+1} H_1^1 + h\sum_{j=2}^{3}\left(e_j\right)_i^k H_j^1\right] \tag{37}$$

$$\left(e_2\right)_i^{k+1} - \left(e_2\right)_{i-1}^{k+1} = \frac{h}{12}\left[6\left(e_3\right)_i^k + 6\left(e_3\right)_{i-1}^{k+1} + h\sum_{j=1}^{2}\left(e_j\right)_i^{k+1} H_j^2 + h\left(e_3\right)_i^k H_3^2\right] \tag{38}$$

Followings inequalities can be derived using eqs. (36)-(38):

$$\left|\left(e_1\right)_i^{k+1}\right| \le \left|\left(e_1\right)_{i-1}^{k+1}\right| + \frac{h}{2}\left(\left|\left(e_2\right)_i^k\right| + \left|\left(e_2\right)_{i-1}^{k+1}\right|\right) + M_1\mathcal{O}\left(h^4\right) \tag{39}$$

$$\left|\left(e_2\right)_i^{k+1}\right| \le \left|\left(e_2\right)_{i-1}^{k+1}\right| + \frac{h}{2}\left(\left|\left(e_3\right)_i^k\right| + \left|\left(e_3\right)_{i-1}^{k+1}\right|\right) + M_1\mathcal{O}\left(h^2\right) \tag{40}$$

$$\left|\left(e_3\right)_i^{k+1}\right| \le \left|\left(e_3\right)_{i-1}^{k+1}\right| + \frac{h}{2}\left(\sum_{j=1}^{3}\left|\left(e_j\right)_i^k\right|\hat{H}_j + \sum_{j=1}^{3}\left|\left(e_j\right)_{i-1}^{k+1}\right|\hat{H}_j\right) + M_1\mathcal{O}\left(h^2\right) \tag{41}$$

let

$$\left(e_1\right)^k = \max_{i=1,\ldots,N}\left|\left(e_1\right)_i^k\right|,\ \left(e_2\right)^k = \max_{i=1,\ldots,N}\left|\left(e_2\right)_i^k\right|,\ \left(e_3\right)^k = \max_{i=1,\ldots,N}\left|\left(e_3\right)_i^k\right|$$

$$\overline{e}^k = \max\left[\max_{i=1,\ldots,N}\left|\left(e_1\right)_i^k\right|, \max_{i=1,\ldots,N}\left|\left(e_2\right)_i^k\right|, \max_{i=1,\ldots,N}\left|\left(e_3\right)_i^k\right|\right]$$

and

$$\nabla \boldsymbol{H} = \left[\frac{\partial \boldsymbol{H}_1}{\partial f}, \frac{\partial \boldsymbol{H}_1}{\partial f_1}, \frac{\partial \boldsymbol{H}_1}{\partial f_2} \right] = \left[\hat{H}_1^1, \hat{H}_1^2, \hat{H}_1^3 \right]$$

where N is the symbol used for the total number of nodes.

By adding the inequalities (39)-(41) gives:

$$3\overline{e}^{k+1} \le \frac{h}{2}\left(\overline{e}^k + \overline{e}^{k+1}\right) + \frac{h}{2}\left(\overline{e}^k + \overline{e}^{k+1}\right) + \frac{h}{2}\overline{e}^k \sum_{j=1}^{3}\left|\hat{H}_j\right| + \frac{h}{2}\overline{e}^{k+1}\sum_{j=1}^{3}\left|\hat{H}_j\right| \tag{42}$$

This implies:

$$3\overline{e}^{k+1} - \frac{h}{2}\overline{e}^{k+1} - \frac{h}{2}\overline{e}^{k+1} - \frac{h}{2}\overline{e}^{k+1}\sum_{j=1}^{3}\left|\hat{H}_j\right| \le \frac{h}{2}\overline{e}^k + \frac{h}{2}\overline{e}^k + \frac{h}{2}\overline{e}^k\sum_{j=1}^{3}\left|\hat{H}_j\right|$$

$$\left(3 - h - \frac{h}{2}\sum_{j=1}^{3}\left|\hat{H}_j\right|\right)\overline{e}^{k+1} \le \left(h + \frac{h}{2}\sum_{j=1}^{3}\left|\hat{H}_j\right|\right)\overline{e}^k$$

According to the assumption given in the hypothesis, the last inequality implies:

$$\overline{e}^{k+1} \le \frac{\beta}{\alpha}\overline{e}^k \tag{43}$$

with the assumption

$$\alpha = \left(3 - h - \frac{h}{2}\sum_{j=1}^{3}\left|\hat{H}_j\right|\right) > 0$$

Then, the inequality (43) becomes for $k = 0$:

$$\overline{e}^1 \le \alpha\overline{e}^0 + \hat{M}\Theta\left(h^2\right)$$

For $k = 1$:

$$\overline{e}^2 \le \alpha\overline{e}^1 + \hat{M}\Theta\left(h^2\right) \le \alpha^2\overline{e}^0 + (\alpha + 1)\hat{M}\Theta\left(h^2\right)$$

If this is continued in the same manner, the following inequality can be formed:

$$\overline{e}^k \le \alpha^k\overline{e}^0 + \left(\alpha^{k-1} + \ldots + \alpha + 1\right)\hat{M}\Theta\left(h^2\right)$$

this implies

$$\overline{e}^k \le \alpha^k\overline{e}^0 + \frac{\alpha^k - 1}{\alpha - 1}\hat{M}\Theta\left(h^2\right)$$

This ends the proof.

The condition of convergence is given:

$$\left|3 - h - \frac{h}{2}\sum_{j=1}^{3}\left|\hat{H}_j\right|\right| < 1$$

Because the geometric series converges for $|\alpha| < 1$.

Comparison

Comparison of the present method with the mixed finite difference is made for the computation of numerical values of physical quantity skin friction coefficient is given in tab. 1.

Table 1. Comparison of three different numerical methods with N (No. of grid points) = 30, length of domain = 7 for the values of skin friction coefficient

λ	Fr	$-f''(0)$		
		Second-order mixed finite difference method	Present 4th order method	MATLAB solver bvp4c
0.1	0.4	1.1307	1.1646	1.1646
0.3		1.2080	1.2482	1.2482
0.5		1.2798	1.3263	1.3263
0.4	0.1	1.1795	1.2102	1.2102
	0.5	1.2652	1.3128	1.3128
	0.7	1.3050	1.3615	1.3615

By looking at tab. 1, the high accuracy of the computed results can be seen. Keller-Box method and standard central second-order methods are used to discretized the reduced ordinary differential eq. (14). Since the eq. (14) is a third-order ODE which is converted into a system of first and second-order differential equations given:

$$f' = f_1 \tag{44}$$

$$f_1'' = \lambda f_1 + \left(1+F_r\right) f_1^2 - f f_1'' \tag{45}$$

Discretize the eq. (44) using Keller-Box method and eq. (45) using standard second-order finite difference methods given:

$$\frac{f_i - f_{i-1}}{h} = \frac{1}{2}\left(f_{1,i} + f_{1,i-1}\right) \tag{46}$$

$$\frac{f_{1,i-1} - 2f_{1,i} + f_{1,i+1}}{h^2} = \lambda f_{1,i} + \left(1+F_r\right) f_{1,i}^2 - f_i\left(\frac{f_{1,i+1} - f_{1,i-1}}{2h}\right) \tag{47}$$

By using the Gauss-Seidel method for eqs. (46) and (47), the resulting equations are given:

$$\frac{f_i^{k+1} - f_{i-1}^{k+1}}{h} = \frac{1}{2}\left(f_{1,i}^k + f_{1,i-1}^{k+1}\right)$$

$$\frac{f_{1,i-1}^{k+1} - 2f_{1,i}^{k+1} + f_{1,i+1}^k}{h^2} = \lambda f_{1,i}^k + \left(1+F_r\right)\left(f_{1,i}^k\right)^2 - f_i^{k+1}\left(\frac{f_{1,i+1}^k - f_{1,i-1}^{k+1}}{2h}\right)$$

In order to find the numerical values of the skin friction coefficient, second-order forward difference formula is implemented on the first derivative f_1.

Results and discussions

The present contribution is made to develop a class of numerical techniques for solving ODE. The mentioned class of numerical techniques deals with interpolation of the different curve on some grid points with the use of derivative on one or more grid points. Any high order technique can be constructed using the idea of the present contribution. One of the constructed techniques in this work has consisted of interpolation of a curve of a function and its derivative using two grid points. Since Adams Moulton proposed the numerical methods with the interpolation of the curves on different grid points but the present method uses the information of two grid points along with derivative(s) on same grid point(s). If the function and its derivatives are evaluated on two grid points, then the resulting technique becomes fourth order accurate. If the second derivative is possible, then use the second derivative on two grid points and the technique becomes sixth-order. So, one can find high order techniques by using higher-order derivatives. So if the task is to find sixth-order accurate numerical method on two grid points from the present class of numeral method, then one can use the function and its first four derivatives if they exist on two grid points.

The other way to get the high-order technique is to use three or more grid points and for finding the unknowns in the interpolated curve, some information of the function with the use of its derivatives can be used. So, if one uses the three grid points and the interpolated curve uses six points, then the function and its first derivate evaluated on three grid points give the technique which will be sixth-order accurate. The order of the presently proposed techniques can be checked by using the Taylor series expansions for any linear problem and the present scheme are consistent because these are more than first-order accurate, so the consistency criteria is fulfilled. The error made by any present technique can be checked by finding the difference between applied technique and MATLAB built-in solver bvp4c or exact solution if it is available. Figures 1 and 2 show the velocity profile and residual when the parameters vary. Figure 1(a) shows that the velocity decreases with the enhancement of the parameter F_r. The increment can be seen in another figure (zoomed part) for a more closer look.

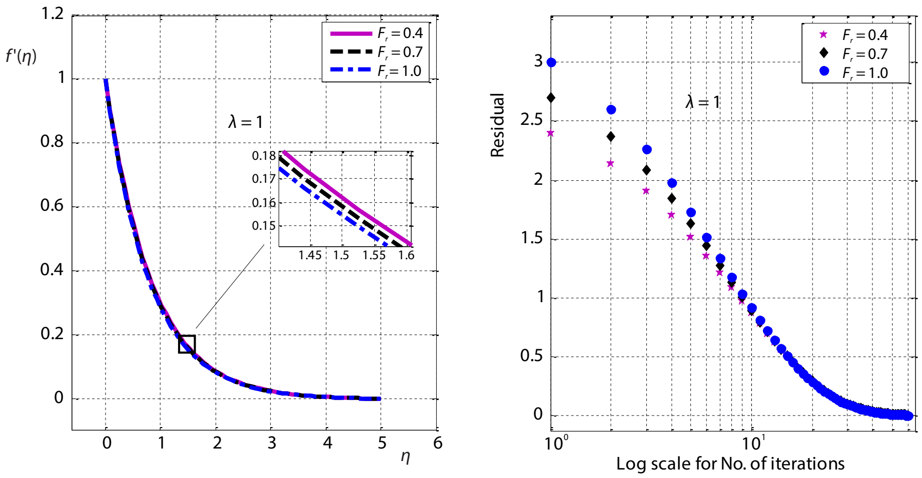

Figure 1. Velocity profile and residual

Figure 2(a) shows the decrease in velocity with the increment of the porous parameter λ, and it can be seen that the thickness of the momentum boundary-layer also decreases in both of the cases.

The residual is found by finding one extra derivative of a third eq. (19), and in the present contribution second order forward, central and backward formulas are implemented to find one extra derivative, and this residual is given in [16]. Since the considered problem is third-order but the obtained results do not contain the third-order formula, so extra derivative numerically found and when the third-order derivative is found, then numerical residual can be

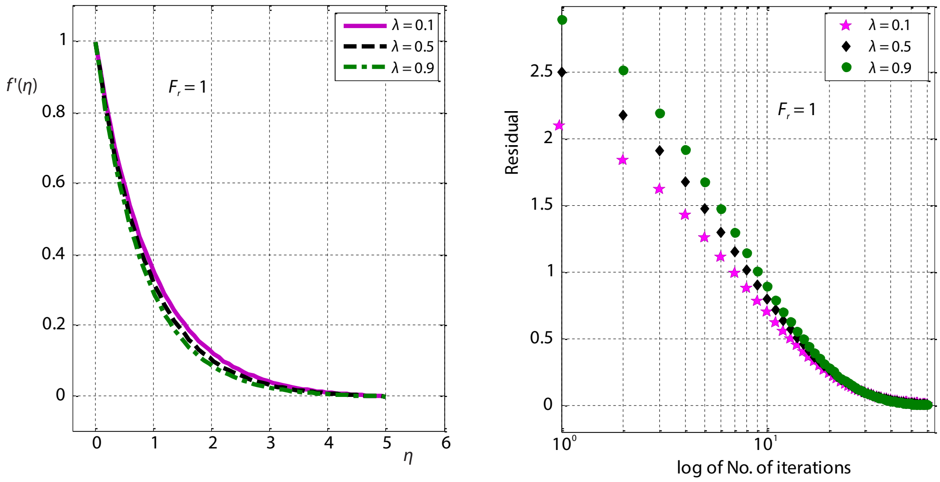

Figure 2. Velocity profile and residual with different λ

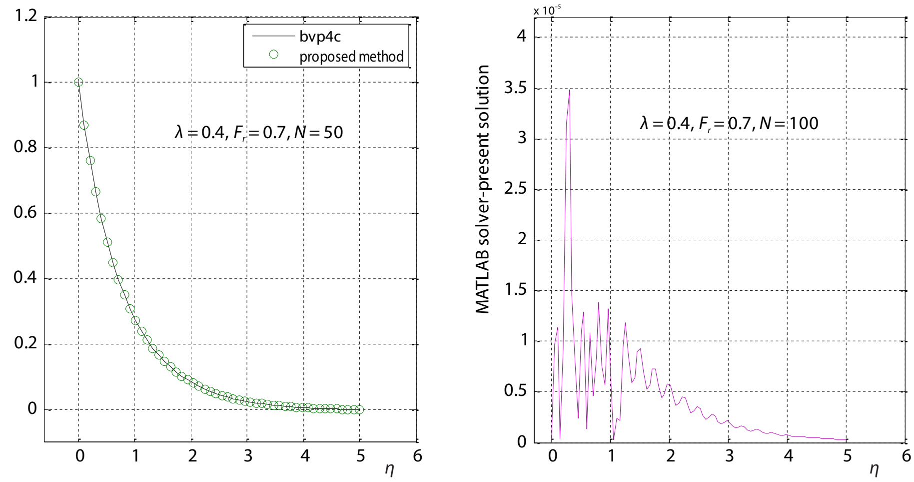

Figure 3. Verification of present technique's with the MATLAB solver bvp4c

found by plugging in the required information into the considered differential equation. Figure 3 shows the comparison of MATLAB solver bvp4c with fourth order proposed technique derived earlier in the present work.

Conclusion

An idea of a new class of numerical methods is given with fourth order technique is constructed using interpolation of a polynomial and sixth-order technique is given. Based on the idea of two techniques, higher-order techniques can be developed. The order of a technique is proved by using Taylor expansion and it can be noted that the present constructed fourth order technique is A-stability and a theorem for finding maximum error bound is given and this can be considered as the main ingredient for convergence of the scheme. In addition, comparison of the technique is made with MATLAB built-in solver bvp4c. Present numerical techniques can be used further to solve different linear and non-linear problems in applied sciences.

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