# NEW NON-CONVENTIONAL METHODS FOR QUANTITATIVE CONCEPTS OF ANOMALOUS RHEOLOGY 

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This paper addresses the general calculus operators with respect to another functions containg the power-law and exponential functions. The Boltzmann-type superposition principles for the anomalous linear viscoelasticity are considered for the first time. The new technologies are as non-conventional tools proposed to extend the quantitative concepts of anomalous rheology for solid mechanics.
Key words: heat transfer, fractional derivative of constant, fractional derivative of variable order, anomalous relaxation, fractional differential equation

## Introduction

The general calculi with respect to monotone functions containing the general derivative which are derived from the Newton-Leibniz derivatives for the composite functions (that is to say, the chain rules for the Newton-Leibniz derivatives), and the general integrals with respect to another function, which are derivated from the antidifferentiation of a composite function and the rule for the change of variables for definite integrals, were considered to describe the anomalous linear viscoelasticity with use of the general calculi involving the positive scaling law function and exponential functions [1-3].

In 1878, Boltzmann [4] proposed the superposition principle for the linear viscoelasticity. The problems for the superposition principle for the anomalous linear viscoelasticity involving the functions of the power law [5], fractional exponential [6] and Kohlrausch-WilliamsWatts [7] decay laws were discussed in detail.

Inspired by the ideas and due to the different monotone functions, which are inclusive of the power-law function (the positive scaling law function [8]) and exponential function, the aim of the report is to propose the Boltzmann type superposition principles for the anomalous linear viscoelasticity with the aid of the general calculi involving the positive scaling law function and exponential functions.

## The geometric interpretations of the general calculi with respect to monotone functions

In this section, we give the comparison of the geometric interpretations of the NewtonLeibniz calculus and general calculus with respect to monotone function [1-3].

Let $\mathbb{N}, \mathbb{R}$, and $\mathbb{R}_{+}$be the sets of the natural numbers, real numbers and positive real numbers, respectively.

## The Newton-Leibniz calculus

Let us recall the Newton's and Leibniz's results as follows.
The Newton-Leibniz derivative with respect to the variable $t$ is defined [1,2]:

$$
\begin{equation*}
\mathrm{D}^{(1)} \phi(t)=\frac{\mathrm{d} \phi(t)}{\mathrm{d} t}=\lim _{\Delta t \rightarrow 0} \frac{\phi(t+\Delta t)-\phi(t)}{\Delta t} \tag{1}
\end{equation*}
$$

The Leibniz integral with respect to the variable $t$ is defined [2]:

$$
\begin{equation*}
{ }_{a} I_{t}^{(1)} \psi(t)=\int_{a}^{t} \psi(t) \mathrm{d} t=\lim _{n \rightarrow \infty} \sum_{k=1}^{n} \psi\left(t_{k}\right)\left(\frac{b-a}{n}\right) \tag{2}
\end{equation*}
$$

where $a \in \mathbb{R}$ and $t_{k}=a+k(b-a) / n$.

## General calculus with respect to monotone function

The general derivative with respect to another function is defined [1]:

$$
\begin{equation*}
\mathrm{D}_{t, g(\cdot)}^{(t)} \varphi(t)=\frac{\mathrm{d} \phi[g(t)]}{\mathrm{d} g(t)}=\frac{1}{g^{(1)}(t)} \lim _{\Delta t \rightarrow 0} \frac{\varphi(t+\Delta t)-\varphi(t)}{\Delta t}=\frac{1}{g^{(1)}(t)} \frac{\mathrm{d} \varphi(t)}{\mathrm{d} t} r \tag{3}
\end{equation*}
$$

where $\varphi(t)=\phi[g(t)]$ and $g^{(1)}(t)>0$.
The theorems for general derivative with respect to another function are given as follows.

Theorem 1. (The sum and difference rules for general derivative with respect to another function).

If $\mathrm{D}_{t, g(\cdot)}^{(1)} \varphi_{1}(t)$ and $\mathrm{D}_{t, g(\cdot)}^{(1)} \varphi_{2}(t)$ exist and $g^{(1)}(t)>0$, then we have:

$$
\begin{equation*}
\mathrm{D}_{t, g(\cdot)}^{(1)}\left[\varphi_{1}(t) \pm \varphi_{2}(t)\right]=\mathrm{D}_{t, g(\cdot)}^{(1)} \varphi_{1}(t) \pm \mathrm{D}_{t, g(\cdot)}^{(1)} \varphi_{2}(t) \tag{4}
\end{equation*}
$$

Theorem 2. (The constant multiple rule for general derivative with respect to another function).

If $\mathrm{D}_{t, g(\cdot)}^{(1)} \varphi(t)$ exists, $g^{(1)}(t)>0$ and $l$ is a constant, then we have:

$$
\begin{equation*}
\mathrm{D}_{t, g(\cdot)}^{(1)}[l \varphi(t)]=l \mathrm{D}_{t, g(\cdot)}^{(1)} \varphi(t) \tag{5}
\end{equation*}
$$

Theorem 3. (The product rule for general derivative with respect to another function). If $\mathrm{D}_{t, g(\cdot)}^{(1)} \varphi_{1}(t)$ and $\mathrm{D}_{t, g(\cdot)}^{(1)} \varphi_{2}(t)$ exist and $g^{(1)}(t)>0$, then we have:

$$
\begin{equation*}
\mathrm{D}_{t, g(\cdot)}^{(1)}\left[\varphi_{1}(t) \cdot \varphi_{2}(t)\right]=\varphi_{2}(t) \mathrm{D}_{t, g(\cdot)}^{(1)} \varphi_{1}(t)+\varphi_{1}(t) \mathrm{D}_{t, g(\cdot)}^{(1)} \varphi_{2}(t) \tag{6}
\end{equation*}
$$

Theorem 4. (The quotient rule for general derivative with respect to another function). If $\mathrm{D}_{t, g(\cdot)}^{(1)} \varphi_{1}(t)$ and $\mathrm{D}_{t, g(\cdot)}^{(1)} \varphi_{2}(t)$ exist, $g^{(1)}(t)>0$ and $\varphi_{2}(t) \neq 0$, then we have:

$$
\begin{equation*}
\mathrm{D}_{t, g(\cdot)}^{(1)}\left[\frac{\varphi_{1}(t)}{\varphi_{2}(t)}\right]=\frac{\varphi_{2}(t) \mathrm{D}_{t, g(\cdot)}^{(1)} \varphi_{1}(t)-\varphi_{1}(t) \mathrm{D}_{t, g(\cdot)}^{(1)} \varphi_{2}(t)}{\varphi_{2}(t) \varphi_{2}(t)} \tag{7}
\end{equation*}
$$

Theorem 5. (The chain rule for general derivative with respect to another function). If $\mathrm{d} \varphi(w) / \mathrm{d} w=\varphi^{(1)}(w)$ and $\mathrm{D}_{t, g(\cdot)}^{(1)} w(t)$ exist and $g^{(1)}(t)>0$, then we have:

$$
\begin{equation*}
\mathrm{D}_{t, g(\cdot)}^{(1)}\{\varphi[w(t)]\}=\varphi^{(1)}(w) \mathrm{D}_{t, g(\cdot)}^{(1)} w(t) \tag{8}
\end{equation*}
$$

For more details of the analogous proofs, we refer to the results [11].
The general definite integral with respect to another function is defined [1]:
${ }_{a} I_{t, g(\cdot)}^{(1)} \psi(t)=\lim _{n \rightarrow \infty} \sum_{k=1}^{n} \Omega\left[g_{k}(t)\right]\left[\frac{g(b)-g(a)}{n}\right]=\int_{a}^{t} \Omega[g(t)] \mathrm{d} g(t)=\int_{a}^{t} \psi(t) g^{(1)}(t) \mathrm{d} t$
where $a \in \mathbb{R}, \psi(t)=\Omega[g(t)]$ and $g_{k}(t)=g(a)+k[g(b)-g(a)] / n$.
The theorems for general integral with respect to another function are given as follows.
Theorem 6. (The sum and difference rules for general definite integral with respect to another function).

If ${ }_{a} I_{t, g(\cdot)}^{(1)} \varphi_{1}(t)$ and ${ }_{a} I_{t, g(\cdot)}^{(1)} \varphi_{2}(t)$ exist and $g^{(1)}(t)>0$, then we have:

$$
\begin{equation*}
{ }_{a} I_{t, g(\cdot)}^{(1)}\left[\varphi_{1}(t) \pm \varphi_{2}(t)\right]={ }_{a} I_{t, g(\cdot)}^{(1)} \varphi_{1}(t) \pm{ }_{a} I_{t, g(\cdot)}^{(1)} \varphi_{2}(t) \tag{10}
\end{equation*}
$$

Theorem 7. (The constant multiple rule for general definite integral with respect to another function).

If ${ }_{a} I_{t, g(\cdot)}^{(1)} \varphi(t)$ exists, $g^{(1)}(t)>0$ and $l$ is a constant, then we have:

$$
\begin{equation*}
{ }_{a} I_{t, g(\cdot)}^{(1)}[l \varphi(t)]=l_{a} I_{t, g(\cdot)}^{(1)} \varphi(t) \tag{11}
\end{equation*}
$$

Theorem 8. If ${ }_{a}^{S L} I_{t}^{(1)} \varphi(t)$ exists and $g^{(1)}(t)>0$, then we have:

$$
\begin{equation*}
{ }_{a} I_{t, g(\cdot)}^{(1)} \varphi(t)=-{ }_{t} I_{a, g(\cdot)}^{(1)} \varphi(t) \tag{12}
\end{equation*}
$$

Theorem 9. (The first fundamental theorem of general definite integral with respect to another function).

If $\mathrm{D}_{t, g(\cdot)}^{(1)} \psi(t)$ exists and $g^{(1)}(t)>0$, then we have:

$$
\begin{equation*}
\psi(t)-\psi(a)={ }_{a} I_{t, g(\cdot)}^{(1)}\left[\mathrm{D}_{t, g(\cdot)}^{(1)} \psi(t)\right] \tag{13}
\end{equation*}
$$

Theorem 10. (The mean value theorem for general definite integral with respect to another function).

If $\mathrm{D}_{t, g(\cdot)}^{(1)} \psi(t)$ exists and $g^{(1)}(t)>0$, then we have:

$$
\begin{equation*}
{ }_{a} I_{t, g(\cdot)}^{(1)} \psi(t)=\psi(l)[g(t)-g(a)] \tag{14}
\end{equation*}
$$

Theorem 11. (The second fundamental theorem of general definite integral with respect to another function).

If $\mathrm{D}_{t, g(\cdot)}^{(1)} \psi(t)$ exists and $g^{(1)}(t)>0$, then we have:

$$
\begin{equation*}
\psi(t)=\mathrm{D}_{t, g(\cdot)}^{(1)}\left[{ }_{a} I_{t, g(\cdot)}^{(1)} \psi(t)\right] \tag{15}
\end{equation*}
$$

Theorem 12. (The net change theorem for general definite integral with respect to another function).

If $\mathrm{D}_{t, g(\cdot)}^{(1)} \psi(t)$ exists and $g^{(1)}(t)>0$, then we have:

$$
\begin{equation*}
\psi(b)-\psi(a)={ }_{a} I_{b, g(\cdot)}^{(1)}\left[\mathrm{D}_{t, g(\cdot)}^{(1)} \psi(t)\right] \tag{16}
\end{equation*}
$$

Theorem 13. (The integration by parts for general definite integral with respect to another function).

$$
\begin{align*}
& \text { If } \mathrm{D}_{t, g(\cdot)}^{(1)} \varphi_{1}(t) \text { and } \mathrm{D}_{t, g(\cdot)}^{(1)} \varphi_{2}(t) \text { exist and } g^{(1)}(t)>0 \text {, then we have: } \\
& { }_{a}^{(1)} I_{t, g(\cdot)}^{(1)}\left[\varphi_{2}(t) \mathrm{D}_{t, g(\cdot)}^{(1)} \varphi_{1}(t)\right]=\varphi_{1}(t) \varphi_{2}(t)-\varphi_{1}(a) \varphi_{2}(a)-{ }_{a} I_{t, g(\cdot)}^{(1)}\left[\varphi_{1}(t) \mathrm{D}_{t, g(\cdot)}^{(1)} \varphi_{2}(t)\right] \tag{17}
\end{align*}
$$

For more details of the analogous proofs, we refer to the results [11].
The comparison of the geometric interpretations of the Newton-Leibniz derivative and general derivative with respect to another function is illustrated in fig. 1.


Figure 1. The geometric interpretations of the Newton-Leibniz derivative and general derivative with respect to another function

The comparison of the geometric interpretations of the Newton-Leibniz integral and general integral with respect to another function is illustrated in fig. 2.

## General calculus with respect to positive scaling law function

To determine the behavior of the positive scaling law function in nature [8], we introduce the following general calculus with respect to positive scaling law function.

The general derivative with respect to positive scaling law function is defined [2]:

$$
\begin{equation*}
{ }_{S L} \mathrm{D}_{t}^{(1)} \varphi(t)=\frac{1}{\beta \kappa t^{\beta-1}} \frac{\mathrm{~d} \varphi(t)}{\mathrm{d} t} \tag{18}
\end{equation*}
$$

where $\kappa t^{\beta}$ is the positive scaling law function with the normalization constant $\kappa$ and the scaling exponent $\beta$ for $\kappa \in \mathbb{R}_{+}$and $\beta \in \mathbb{R}_{+}$.



Figure 2. The geometric interpretations of the Newton-Leibniz integral and general integral with respect to another function

The general integral with respect to positive scaling law function is defined:

$$
\begin{equation*}
{ }_{a}^{S L} I_{t}^{(1)} \psi(t)=\beta \kappa \int_{a}^{\tau} \psi(t) t^{\beta-1} \mathrm{~d} t \tag{19}
\end{equation*}
$$

with $\psi(t)={ }_{S L} \mathrm{D}_{t}^{(1)}\left[{ }_{a}^{S L} I_{t}^{(1)} \psi(t)\right]$ and $\psi(t)-\psi(a)={ }_{a}^{S L} I_{t}^{(1)}\left[{ }_{S L} \mathrm{D}_{t}^{(1)} \psi(t)\right]$.
Taking the different values of the parameters, we have the following results:
(M1) When $\kappa=1$, one has the general derivative with respect to power-law function defined by [1-3]:

$$
\begin{equation*}
{ }_{P} \mathrm{D}_{t}^{(1)} \varphi(t)=\frac{1}{\beta t^{\beta-1}} \frac{\mathrm{~d} \varphi(t)}{\mathrm{d} t} \tag{20}
\end{equation*}
$$

and the general integral with respect to power-law function defined by [1-3]:

$$
\begin{equation*}
{ }_{a}^{P} I_{t}^{(1)} \psi(t)=\beta \int_{a}^{\tau} \psi(t) t^{\beta-1} \mathrm{~d} t \tag{21}
\end{equation*}
$$

with $\psi(t)={ }_{P} \mathrm{D}_{t}^{(1)}\left[{ }_{a}^{P} I_{t}^{(1)} \psi(t)\right]$ and $\psi(t)-\psi(a)={ }_{a}^{P} I_{t}^{(1)}\left[{ }_{P} \mathrm{D}_{t}^{(1)} \psi(t)\right]$.
(M2) When $\beta=1$, one has the general derivative with respect to the monotone function defined:

$$
\begin{equation*}
{ }_{L} \mathrm{D}_{t}^{(1)} \varphi(t)=\frac{1}{\kappa} \frac{\mathrm{~d} \varphi(t)}{\mathrm{d} t} \tag{22}
\end{equation*}
$$

and the general integral with respect to power-law function defined:

$$
\begin{equation*}
{ }_{a}^{L} I_{t}^{(1)} \psi(t)=\kappa \int_{a}^{\tau} \psi(t) \mathrm{d} t \tag{23}
\end{equation*}
$$

with $\psi(t)={ }_{L} \mathrm{D}_{t}^{(1)}\left[{ }_{a}^{L} I_{t}^{(1)} \psi(t)\right]$ and $\psi(t)-\psi(a)={ }_{a}^{L} I_{t}^{(1)}\left[{ }_{L} \mathrm{D}_{t}^{(1)} \psi(t)\right]$.
(M3) When $\kappa=1$ and $\beta=1$, one has the Newton-Leibniz calculus [9-11].
The basic results of the general calculus with respect to positive scaling law function are presented as follows:

If $\varphi(t)=\kappa^{n} t^{n \beta}$ for $n \in \mathbb{N}$, then there exists:

$$
\begin{equation*}
{ }_{S L} \mathrm{D}_{t}^{(1)} \varphi(t)=n \kappa^{n-1} t^{(n-1) \beta} \tag{24}
\end{equation*}
$$

If $\psi(t)=n \kappa^{n-1} t^{(n-1) \beta}$ for $n \in \mathbb{N}$, then there exists:

$$
\begin{equation*}
{ }_{0}^{S L} I_{t}^{(1)} \psi(t)=\kappa^{n} t^{n \beta} \tag{25}
\end{equation*}
$$

If $\varphi(t)=\mathrm{e}^{\lambda \kappa t^{\beta}}$, where $\mathrm{e}^{t^{\beta}}$ is the Kohlrausch-Williams-Watts function [7, 12-14] and $\lambda \in \mathbb{R}$, then there exists:

$$
\begin{equation*}
{ }_{S L} \mathrm{D}_{t}^{(1)} \varphi(t)=\lambda \varphi(t) \text { and }{ }_{S L} \mathrm{D}_{t}^{(1)} \varphi(t)=\lambda^{2} \varphi(t) \tag{26a,b}
\end{equation*}
$$

If $\psi(t)=\lambda \mathrm{e}^{\lambda \kappa t^{\beta}}$ for $\lambda \in \mathbb{R}$, then there exists:

$$
\begin{equation*}
{ }_{0}^{S L} I_{t}^{(1)} \psi(t)=\mathrm{e}^{\lambda \kappa t^{\beta}} \tag{27}
\end{equation*}
$$

If $\psi(t)=\eta \mathrm{e}^{-\lambda \kappa t^{\beta}}+\kappa t^{\beta} \mathrm{e}^{-\lambda \kappa t^{\beta}}$ for $\lambda \in \mathbb{R}$, then there exists:

$$
\begin{equation*}
{ }_{S L} \mathrm{D}_{t}^{(1)} \varphi(t)=-\lambda \psi(t)+\mathrm{e}^{-\lambda \kappa t^{\beta}} \tag{28}
\end{equation*}
$$

If $\varphi(t)=\eta \mathrm{e}^{i \lambda \kappa t^{\beta}}$, where $i=(-1)^{1 / 2}$ and $\lambda \in \mathbb{R}$, then there exists:

$$
\begin{equation*}
{ }_{S L} \mathrm{D}_{t}^{(1)} \varphi(t)=i \lambda \varphi(t) \text { and }{ }_{S L} \mathrm{D}_{t}^{(2)} \varphi(t)=-\lambda^{2} \varphi(t) \tag{29a,b}
\end{equation*}
$$

where the subcurve is given:

$$
\begin{equation*}
\eta \mathrm{e}^{i \lambda \kappa t^{\beta}}=\eta\left[\operatorname{sub} \cos \left(\lambda \kappa t^{\beta}\right)+i \operatorname{subsin}\left(\lambda \kappa t^{\beta}\right)\right] \tag{30}
\end{equation*}
$$

with the subsine function defined:

$$
\begin{equation*}
\operatorname{subsin}\left(\lambda \kappa t^{\beta}\right)=\frac{1}{2 i}\left(\mathrm{e}^{i \lambda \kappa t^{\beta}}-\mathrm{e}^{-i \lambda \kappa t^{\beta}}\right) \tag{31}
\end{equation*}
$$

and the subcosine function defined:

$$
\begin{equation*}
\operatorname{sub} \cos \left(\lambda \kappa t^{\beta}\right)=\frac{1}{2}\left(\mathrm{e}^{i \lambda \kappa t^{\beta}}+\mathrm{e}^{-i \lambda \kappa t^{\beta}}\right) \tag{32}
\end{equation*}
$$

If $\varphi(t)=\operatorname{subcos}\left(\lambda \kappa t^{\beta}\right)$, where $\lambda \in \mathbb{R}$, then there exist:

$$
\begin{equation*}
{ }_{S L} \mathrm{D}_{t}^{(1)} \varphi(t)=-\lambda \operatorname{subsin}\left(\lambda \kappa t^{\beta}\right) \text { and }{ }_{S L} \mathrm{D}_{t}^{(2)} \varphi(t)=-\lambda^{2} \operatorname{subcos}\left(\lambda \kappa t^{\beta}\right)=-\lambda^{2} \varphi(t) \tag{33a,b}
\end{equation*}
$$

If $\varphi(t)=\operatorname{subsin}\left(\lambda \kappa t^{\beta}\right)$, where $i=(-1)^{1 / 2}$ and $\lambda \in \mathbb{R}$, then there exists:

$$
\begin{equation*}
{ }_{S L} \mathrm{D}_{t}^{(1)} \varphi(t)=\lambda \operatorname{subcos}\left(\lambda \kappa t^{\beta}\right) \text { and }{ }_{S L} \mathrm{D}_{t}^{(2)} \varphi(t)=-\lambda^{2} \operatorname{subsin}\left(\lambda \kappa t^{\beta}\right)=-\lambda^{2} \varphi(t) \tag{34a,b}
\end{equation*}
$$

If $\varphi(t)=\operatorname{subcosh}\left(\lambda \kappa t^{\beta}\right)$, where $\lambda \in \mathbb{R}$, then there exist:

$$
{ }_{S L} \mathrm{D}_{t}^{(1)} \varphi(t)=\lambda \operatorname{subsinh}\left(\lambda \kappa t^{\beta}\right) \text { and }{ }_{S L} \mathrm{D}_{t}^{(2)} \varphi(t)=\lambda^{2} \operatorname{subcosh}\left(\lambda \kappa t^{\beta}\right)=\lambda^{2} \varphi(t) \quad(35 \mathrm{a}, \mathrm{~b})
$$

where the hyperbolic subcosine function defined:

$$
\begin{equation*}
\operatorname{sub} \cosh \left(\lambda \kappa t^{\beta}\right)=\frac{1}{2}\left(\mathrm{e}^{\lambda \kappa t^{\beta}}+\mathrm{e}^{-\lambda \kappa t^{\beta}}\right) \tag{36}
\end{equation*}
$$

and the hyperbolic subsine function defined:

$$
\begin{equation*}
\operatorname{subsinh}\left(\lambda \kappa t^{\beta}\right)=\frac{1}{2}\left(\mathrm{e}^{\lambda \kappa t^{\beta}}-\mathrm{e}^{-\lambda \kappa t^{\beta}}\right) \tag{37}
\end{equation*}
$$

If $\varphi(t)=\operatorname{subsinh}\left(\lambda \kappa t^{\beta}\right)$, where $\lambda \in \mathbb{R}$, then there exist:

$$
\begin{equation*}
{ }_{S L} \mathrm{D}_{t}^{(1)} \varphi(t)=\lambda \operatorname{subcosh}\left(\lambda \kappa t^{\beta}\right) \text { and }{ }_{S L} \mathrm{D}_{t}^{(2)} \varphi(t)=\lambda^{2} \operatorname{subsinh}\left(\lambda \kappa t^{\beta}\right)=\lambda^{2} \varphi(t) \tag{38a,b}
\end{equation*}
$$

If $\psi(t)=\lambda \mathrm{e}^{\lambda t^{\beta}}$ for $\lambda \in \mathbb{R}$, then there exists:

$$
\begin{equation*}
{ }_{0}^{P} I_{t}^{(1)} \psi(t)=\mathrm{e}^{\lambda \kappa t^{\beta}} \tag{39}
\end{equation*}
$$

If $\varphi(t)=\eta \mathrm{e}^{i \lambda t^{\beta}}$, where $i=(-1)^{1 / 2}$ and $\lambda \in \mathbb{R}$, then there exists:

$$
\begin{equation*}
{ }_{P} \mathrm{D}_{t}^{(1)} \varphi(t)=i \lambda \varphi(t) \text { and }{ }_{P} \mathrm{D}_{t}^{(2)} \varphi(t)=-\lambda^{2} \varphi(t) \tag{40a,b}
\end{equation*}
$$

where the subcurve is presented:

$$
\begin{equation*}
\eta \mathrm{e}^{i \lambda t^{\beta}}=\eta\left[\operatorname{subcos}\left(\lambda t^{\beta}\right)+i \operatorname{subsin}\left(\lambda t^{\beta}\right)\right] \tag{41}
\end{equation*}
$$

with $\operatorname{subcos}\left(\lambda t^{\beta}\right)=\left(\mathrm{e}^{i \lambda t^{\beta}}+\mathrm{e}^{-i \lambda t^{\beta}}\right) / 2$ and $\operatorname{subsin}\left(\lambda t^{\beta}\right)=(1 / 2 i)\left(\mathrm{e}^{i \lambda t^{\beta}}-\mathrm{e}^{-i \lambda t^{\beta}}\right)$.
If $\varphi(t)=\operatorname{subcos}\left(\lambda t^{\beta}\right)$, where $\lambda \in \mathbb{R}$, then there exist:

$$
\begin{equation*}
{ }_{P} \mathrm{D}_{t}^{(1)} \varphi(t)=-\lambda \operatorname{subsin}\left(\lambda t^{\beta}\right) \text { and }{ }_{P} \mathrm{D}_{t}^{(2)} \varphi(t)=-\lambda^{2} \operatorname{subcos}\left(\lambda t^{\beta}\right)=-\lambda^{2} \varphi(t) \tag{42a,b}
\end{equation*}
$$

If $\varphi(t)=\operatorname{subsin}\left(\lambda t^{\beta}\right)$, where $i=(-1)^{1 / 2}$ and $\lambda \in \mathbb{R}$, then there exists:

$$
\begin{equation*}
{ }_{P} \mathrm{D}_{t}^{(1)} \varphi(t)=\lambda \operatorname{subcos}\left(\lambda t^{\beta}\right) \text { and }{ }_{S L} \mathrm{D}_{t}^{(2)} \varphi(t)=-\lambda^{2} \operatorname{subsin}\left(\lambda \kappa t^{\beta}\right)=-\lambda^{2} \varphi(t) \tag{43a,b}
\end{equation*}
$$

If $\varphi(t)=\operatorname{sub} \cosh \left(\lambda t^{\beta}\right)$, where $\lambda \in \mathbb{R}$, then there exist:

$$
\begin{equation*}
{ }_{P} \mathrm{D}_{t}^{(1)} \varphi(t)=\lambda \operatorname{subsinh}\left(\lambda t^{\beta}\right) \text { and }{ }_{P} \mathrm{D}_{t}^{(2)} \varphi(t)=\lambda^{2} \operatorname{subcosh}\left(\lambda t^{\beta}\right)=\lambda^{2} \varphi(t) \tag{44a,b}
\end{equation*}
$$

where

$$
\begin{equation*}
\operatorname{subcosh}\left(\lambda t^{\beta}\right)=\frac{1}{2}\left(\mathrm{e}^{\lambda t^{\beta}}+\mathrm{e}^{-\lambda t^{\beta}}\right) \text { and } \operatorname{subsinh}\left(\lambda t^{\beta}\right)=\frac{1}{2}\left(\mathrm{e}^{\lambda t^{\beta}}-\mathrm{e}^{-\lambda t^{\beta}}\right) \tag{45a,b}
\end{equation*}
$$

Suppose that $\varphi(t)=\operatorname{subsinh}\left(\lambda t^{\beta}\right)$, where $\lambda \in \mathbb{R}$, then there exist:

$$
\begin{equation*}
{ }_{P} \mathrm{D}_{t}^{(1)} \varphi(t)=\lambda \operatorname{subcosh}\left(\lambda t^{\beta}\right) \text { and }{ }_{P} \mathrm{D}_{t}^{(2)} \varphi(t)=\lambda^{2} \operatorname{subsinh}\left(\lambda t^{\beta}\right)=\lambda^{2} \varphi(t) \tag{46a,b}
\end{equation*}
$$

It is seen that:

$$
\operatorname{subcos}^{2}\left(\lambda \kappa t^{\beta}\right)+\operatorname{subsin}^{2}\left(\lambda \kappa t^{\beta}\right)=1 \text { and } \operatorname{subcos}^{2}\left(\lambda \kappa t^{\beta}\right)+\operatorname{subsin}^{2}\left(\lambda \kappa t^{\beta}\right)=1(47 \mathrm{a}, \mathrm{~b})
$$

## General calculus with respect to exponential function

To decribe the behavior of the complex materials, we introduce the following general calculus with respect to exponential function.

The general derivative with respect to exponential function is defined [1-3]:

$$
\begin{equation*}
{ }_{E} \mathrm{D}_{t}^{(1)} \varphi(t)=\frac{1}{\lambda \rho \mathrm{e}^{\lambda t}} \frac{\mathrm{~d} \varphi(t)}{\mathrm{d} t} \tag{48}
\end{equation*}
$$

where $\lambda, \rho \in \mathbb{R}$.
The general definite integral with respect to exponential function is defined [1-3]:

$$
\begin{equation*}
{ }_{a}^{E} I_{t}^{(1)} \psi(t)=\lambda \rho \int_{a}^{t} \psi(t) \mathrm{e}^{\lambda t} \mathrm{~d} t \tag{49}
\end{equation*}
$$

where $a \in \mathbb{R}$.
Their relationships between eqs. (48) and (49) can be written [1-5]:

$$
\begin{equation*}
\psi(t)=\left(\frac{1}{\lambda \rho \mathrm{e}^{\lambda t}} \frac{\mathrm{~d}}{\mathrm{~d} t}\right){ }_{a}^{E} I_{t}^{(1)} \psi(t)=\left(\mathrm{e}^{-\lambda t} \frac{\mathrm{~d}}{\mathrm{~d} t}\right) \int_{a}^{t} \psi(t) \mathrm{e}^{\lambda t} \mathrm{~d} t \tag{50}
\end{equation*}
$$

and

$$
\begin{equation*}
\psi(t)=\int_{0}^{t}\left(\mathrm{e}^{-\lambda t} \frac{\mathrm{~d} \psi(t)}{\mathrm{d} t}\right) \mathrm{e}^{\lambda t} \mathrm{~d} t+\psi(a) \tag{51}
\end{equation*}
$$

## The Boltzmann type superposition principles for the anomalous linear viscoelasticity

Let $\varepsilon(t)$ be the strain, $\sigma(t)$ be the stress, $t$ be the time, $G(t)$ be the relaxation modulus, $J(t)$ be the creep compliance, $\Delta \sigma_{i}$ be the discrete stress increment applied at time $t=\tau_{i}$, $\Delta \varepsilon_{i}$ be the discrete strain increment applied at time $t=\tau_{i}, \sigma_{0}$ be the constant strain, and $\varepsilon_{0}$ be the constant strain.

## The Boltzmann superposition principle for the linear

 viscoelasticityWith the Boltzmann superposition principle [4], one has:

$$
\begin{equation*}
\varepsilon(t)=\sum_{i=1}^{n} \Delta \sigma_{i} J\left(t-\tau_{i}\right) \tag{52}
\end{equation*}
$$

which can be written:

$$
\begin{equation*}
\varepsilon(t)=\sum_{i=1}^{n} J\left(t-\tau_{i}\right) \frac{\Delta \sigma_{i}}{\Delta \tau_{i}} \Delta \tau_{i} \tag{53}
\end{equation*}
$$

In this case, one has from eq. (53) that:

$$
\begin{equation*}
\varepsilon(t)=\int_{-\infty}^{t} J(t-\tau) \frac{\mathrm{d} \sigma(\tau)}{\mathrm{d} \tau} \mathrm{~d} \tau \tag{54}
\end{equation*}
$$

which leads to:

$$
\begin{equation*}
\varepsilon(t)=\sigma_{0} J(t)+\int_{0}^{\infty} \sigma(t-\tau) \frac{\mathrm{d} J(\tau)}{\mathrm{d} \tau} \mathrm{~d} \tau \tag{55}
\end{equation*}
$$

In an analogous manner, one has:

$$
\begin{equation*}
\sigma(t)=\sum_{i=1}^{n} \Delta \varepsilon_{i} G\left(t-\tau_{i}\right) \tag{56}
\end{equation*}
$$

which leads to:

$$
\begin{equation*}
\sigma(t)=\sum_{i=1}^{n} G\left(t-\tau_{i}\right) \frac{\Delta \varepsilon_{i}}{\Delta \tau_{i}} \Delta \tau_{i} \tag{57}
\end{equation*}
$$

Here, one has from eq. (57) that:

$$
\begin{equation*}
\sigma(t)=\int_{-\infty}^{t} G(t-\tau) \frac{\mathrm{d} \varepsilon(\tau)}{\mathrm{d} \tau} \mathrm{~d} \tau \tag{58}
\end{equation*}
$$

which can be given:

$$
\begin{equation*}
\sigma(t)=\varepsilon_{0} G(t)+\int_{0}^{\infty} \varepsilon(t-\tau) \frac{\mathrm{d} G(\tau)}{\mathrm{d} \tau} \mathrm{~d} \tau \tag{59}
\end{equation*}
$$

The Boltzmann type superposition principle for the anomalous
linear viscoelasticity within the power-law function
To decribe the behavior of the complex materials, we present the Boltzmann type superposition principle within the general calculus with respect to power-law function.

From eq. (55) one has:

$$
\begin{equation*}
\varepsilon(t)=\sigma_{0} J(t)+\beta \kappa \int_{0}^{\infty} \sigma(t-\tau) \tau^{\beta-1}{ }_{S L} \mathrm{D}_{t}^{(1)} J(\tau) \mathrm{d} \tau \tag{60}
\end{equation*}
$$

which can be written:

$$
\begin{equation*}
\varepsilon(t)=\beta \kappa \int_{-\infty}^{t} J(t-\tau) \tau^{\beta-1}{ }_{S L} \mathrm{D}_{t}^{(1)} \sigma(\tau) \mathrm{d} \tau \tag{61}
\end{equation*}
$$

which leads to:

$$
\begin{equation*}
\varepsilon(t)=\sum_{i=1}^{n} \Delta \sigma_{i} J\left(t-\tau_{i}\right) \tag{62}
\end{equation*}
$$

In an analogous manner, one has:

$$
\begin{equation*}
\sigma(t)=\varepsilon_{0} G(t)+\beta \kappa \int_{0}^{\infty} \varepsilon(t-\tau) \tau^{\beta-1}{ }_{S L} D_{t}^{(1)} G(\tau) \mathrm{d} \tau \tag{63}
\end{equation*}
$$

which can be written:

$$
\begin{equation*}
\sigma(t)=\beta \kappa \int_{-\infty}^{t} G(t-\tau) \tau^{\beta-1}{ }_{S L} \mathrm{D}_{t}^{(1)} \varepsilon(\tau) \mathrm{d} \tau \tag{64}
\end{equation*}
$$

which leads to:

$$
\begin{equation*}
\sigma(t)=\sum_{i=1}^{n} \Delta \varepsilon_{i} G\left(t-\tau_{i}\right) \tag{65}
\end{equation*}
$$

## The Boltzmann type superposition principle for the anomalous

 linear viscoelasticity within the exponential functionTo describe the behavior of the complex materials, we introduce the Boltzmann type superposition principle within the general calculus with respect to exponential function.

From eq. (55) one has:

$$
\begin{equation*}
\varepsilon(t)=\sigma_{0} J(t)+\lambda \rho \int_{0}^{\infty} \sigma(t-\tau) \mathrm{e}^{\lambda t}{ }_{E} \mathrm{D}_{t}^{(1)} J(\tau) \mathrm{d} \tau \tag{66}
\end{equation*}
$$

which can be given:

$$
\begin{equation*}
\varepsilon(t)=\lambda \rho \int_{-\infty}^{t} J(t-\tau) \mathrm{e}_{E}^{\lambda t}{ }_{E}{ }_{t}^{(1)} \sigma(\tau) \mathrm{d} \tau \tag{67}
\end{equation*}
$$

which implies that:

$$
\begin{equation*}
\varepsilon(t)=\sum_{i=1}^{n} \Delta \sigma_{i} J\left(t-\tau_{i}\right) \tag{68}
\end{equation*}
$$

In an analogous manner, one has:

$$
\begin{equation*}
\sigma(t)=\varepsilon_{0} G(t)+\lambda \rho \int_{0}^{\infty} \varepsilon(t-\tau) \mathrm{e}^{\lambda t}{ }_{E} \mathrm{D}_{t}^{(1)} G(\tau) \mathrm{d} \tau \tag{69}
\end{equation*}
$$

which can be presented:

$$
\begin{equation*}
\sigma(t)=\lambda \rho \int_{-\infty}^{t} G(t-\tau) \mathrm{e}_{E}^{\lambda t} \mathrm{D}_{t}^{(1)} \varepsilon(\tau) \mathrm{d} \tau \tag{70}
\end{equation*}
$$

which can be written:

$$
\begin{equation*}
\sigma(t)=\sum_{i=1}^{n} \Delta \varepsilon_{i} G\left(t-\tau_{i}\right) \tag{71}
\end{equation*}
$$

## Conclusion

In our work the theorems for the general calculus operators with respect to another function were considered. The sub-trigonometric functions via Kohlrausch-Williams-Watts function are proposed and their characteristic equations were discussed in detail. The discrete forms of the Boltzmann type superposition principles are same but their integral forms are different due to the different quantitative concepts of anomalous rheology for solid mechanics. The results can be used to explain the Kohlrausch-Williams-Watts decay law in complex materials.

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## Nomenclature

```
t -time, [s] }|\mp@subsup{\sigma}{i}{}\mathrm{ -discrete stress increment, [Pa]
Greek symbols
\varepsilon(t) -strain, [-]
\sigma(t) -stress, [Pa]
\(\Delta \varepsilon_{i} \quad\)-discrete strain increment, [-]
```


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