APPLICATION OF HE'S FRACTIONAL DERIVATIVE AND FRACTIONAL COMPLEX TRANSFORM FOR TIME FRACTIONAL CAMASSA-HOLM EQUATION

by

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In this article He's fractional derivative is studied for time fractional Camassa-Holm equation. To transform the considered fractional model into a differential equation, the fractional complex transform is used and He's homotopy perturbation method is adopted to solve the equation. Physical understanding of the fractional complex transform is elucidated by the two-scale fractal theory.

Key words: *He's fractional derivative, time fractional Camassa-Holm, equation, fractional complex transform, homotopy perturbation method*

Introduction

In this paper, we deliberate the time fractional Camassa-Holm (CH) equation in the following manner:

$$\frac{\partial^\beta u}{\partial t^\beta}-\frac{\partial^{2+\beta}u}{\partial x^2\partial t^\beta}+2\frac{\partial u}{\partial x}+3u\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+u\frac{\partial^3 u}{\partial x^3} \tag{1}$$

subject to initial condition:

$$u(x,0)=\mathrm{e}^{-x}-1 \tag{2}$$

where β is the fractal dimensions of the fractal medium and $\partial^\beta/\partial t^\beta$ is He's fractional derivative defined [1-3]:

$$\frac{\partial^\beta u}{\partial t^\beta}=\frac{1}{\Gamma(n-\beta)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int\limits_{t_0}^{t}(s-t)^{n-\beta-1}[u_0(s)-u(s)]\mathrm{d}s \tag{3}$$

where $u_0(x, t)$ is the solution of its continuous partner as well as the same starting point (initial guess) of the fractal partner of the model.

When $\beta = 1$, eq. (1) becomes to be the original well-known CH equation and frequently used in fluid mechanics and many other fields. This equation was presented by Camassa and Holm [4] in 1993 which describes shallow water waves with peakon solutions. The peakon solution is a special solitary wave solution which is peaked in the limiting case and the first derivatives are discontinuous in the peaks [4]. In order to model the peak properties, eq. (1) is adopted. When time tends to infinitively small, the classic CH equation will not be continuous, while eq. (3) can describe the motion well.

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The generalized form of the classical calculus is the fractional calculus. The literature has witnessed that the fractal/fractional derivatives have attained much consideration of physicists, mathematicians and engineers in the past three decades because of their realistic behaviors. In the field of engineering and other sciences, different kinds of interdisciplinary problems had been modeled using fractional/fractal derivatives [5-20] but still it is not easy to find the exact solutions of fractional differential equations. Many analytical and approximation methods [21-27] have recently been presented to solve linear and non-linear fractional differential equations. The homotopy perturbation method (HPM) [28-30] is widely applied to various science and engineering problems. This method was first proposed by He [28] and now it is considered as a powerful tool to finding fairly accurate analytic solutions of both linear as well as non-linear problems. Fractional complex transform, which was suggested also by He [31-35], converts a fractional differential equation into its differential partner, so that the HPM can be effectively used.

In this paper, we will employ the fractional complex transform to convert the time fractional CH differential equation into its PDE and then use HPM to find a fairly accurate solution of eq. (1) subject to eq. (2).

Preliminaries

Basic idea of HPM

The hybridization of perturbation method and homotopy method is called HPM [29], which eliminates the drawbacks of the traditional perturbation methods. To explain the simple idea of HPM [28], we consider a non-linear differential equation:

$$\mathrm{D}(u) - g(v) = 0, \qquad v \in \Psi \tag{4}$$

with boundary condition:

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, \qquad v \in \Lambda \tag{5}$$

where D is a general differential operator, B – the boundary operator, $g(v)$ – the known analytical function, and Λ – the boundary of the domain Ψ. The operator D can be divided into linear L and non-linear N parts thus eq. (4) can be expressed:

$$L(u) + N(u) - g(v) = 0 \tag{6}$$

According to homotopy technique, we construct following homotopy:

$$H(\xi, q) = (1-q)[L(\xi) - L(\xi_0)] + q[D(\xi) - g(v)] = 0, \qquad q \in [0,1] \tag{7}$$

where q is the embedding parameter and ξ_0 – the initial solution of eq. (4) which fulfills the boundary conditions generally.

Obviously, from eq. (7):

$$H(\xi, 0) = L(\xi) - L(\xi_0) = 0 \tag{8}$$

$$H(\xi, 1) = A(\xi) - g(v) = 0 \tag{9}$$

The HPM uses embedding parameter q as an expanding parameter [29], and basic assumption is that the solution of eq. (7) can be specified as a power series in q:

$$\xi = \xi_0 + q\xi_1 + q^2\xi_2 + q^3\xi_3 + q^4\xi_4 + \cdots \tag{10}$$

Setting $q = 1$ results the approximate analytic solution of eq. (7):

$$u = \lim_{q\to 1}\xi = \xi_0 + \xi_1 + \xi_2 + \xi_3 + \xi_4 + \cdots \tag{11}$$

The series in eq. (10) may converge in the whole solution domain as q tends to unit.

Fractional complex transform

The fractional complex transform is [31-35]:

$$\Delta y = \frac{\Delta t^\beta}{\Gamma(1+\beta)} \tag{12a}$$

where Δt is a small scale and Δy – a larger scale. In a small scale, Δt, the CH equation behaves discontinuously, especially at the peak of the solitary wave. While on a larger scale, Δy, a smooth solitary wave is predicted. So eq. (12a) is also called as the two-scale transform [34, 35]. A same phenomenon, when observed by different scales, leads to different laws. For example, a flow is a continuum, and its motion follows the fluid mechanics laws when it is observed on any an observable scale, but when we observe the flow in a molecule's size, the flow becomes discontinuous. So the two-scale transform is to convert a fractal space in a small scale to an approximate smooth space in a larger scale.

Numerical application

To solve eq. (1) with the help of HPM, the first step is to transform the fractional model equation into its equivalent differential model by means of eq. (12a):

$$y = \frac{t^\beta}{\Gamma(1+\beta)} \tag{12b}$$

Equation (1) can be written into its equivalent differential equation form:

$$\frac{\partial u}{\partial y} - \frac{\partial^3 u}{\partial x^2 \partial y} + 2\frac{\partial u}{\partial x} + 3u\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - u\frac{\partial^3 u}{\partial x^3} = 0 \tag{13}$$

According to HPM, substituting eq. (13) into eq. (7), we have a system of $m + 1$ equations that is to be simultaneously solved, m is the order of perturbation parameter in eq. (10). Assuming $m = 4$ the system is:

$$\frac{\partial u_0}{\partial y} - \mathrm{e}^{-x} + 1 = 0, \quad u_0(x,0) = \mathrm{e}^{-x} - 1 \tag{14a}$$

$$\frac{\partial u_1}{\partial y} - \frac{\partial^3 u_0}{\partial x^2 \partial y} + 2\frac{\partial u_0}{\partial x} + 3u_0\frac{\partial u_0}{\partial x} - 2\frac{\partial u_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2} - u_0\frac{\partial^3 u_0}{\partial x^3} = 0, \quad u_1(x,0) = 0 \tag{14b}$$

$$\begin{gathered}\frac{\partial u_2}{\partial y} - \frac{\partial^3 u_1}{\partial x^2 \partial y} + 2\frac{\partial u_1}{\partial x} + 3u_0\frac{\partial u_1}{\partial x} + 3u_1\frac{\partial u_0}{\partial x} - 2\frac{\partial u_0}{\partial x}\frac{\partial^2 u_1}{\partial x^2} - 2\frac{\partial u_1}{\partial x}\frac{\partial^2 u_0}{\partial x^2} - \\ - u_0\frac{\partial^3 u_1}{\partial x^3} - u_1\frac{\partial^3 u_0}{\partial x^3} = 0, \quad u_2(x,0) = 0\end{gathered} \tag{14c}$$

$$\frac{\partial u_3}{\partial y}-\frac{\partial^3 u_2}{\partial x^2\partial y}+2\frac{\partial u_2}{\partial x}+3u_0\frac{\partial u_2}{\partial x}+3u_1\frac{\partial u_1}{\partial x}+3u_2\frac{\partial u_0}{\partial x}-2\frac{\partial u_0}{\partial x}\frac{\partial^2 u_2}{\partial x^2}-2\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2}-$$

$$-2\frac{\partial u_2}{\partial x}\frac{\partial^2 u_0}{\partial x^2}-u_0\frac{\partial^3 u_2}{\partial x^3}-u_1\frac{\partial^3 u_1}{\partial x^3}-u_2\frac{\partial^3 u_0}{\partial x^3}=0,\quad u_3(x,0)=0 \tag{14d}$$

$$\frac{\partial u_4}{\partial y}-\frac{\partial^3 u_3}{\partial x^2\partial y}+2\frac{\partial u_3}{\partial x}+3u_0\frac{\partial u_3}{\partial x}+3u_1\frac{\partial u_2}{\partial x}+3u_2\frac{\partial u_1}{\partial x}+3u_3\frac{\partial u_0}{\partial x}-2\frac{\partial u_0}{\partial x}\frac{\partial^2 u_3}{\partial x^2}-2\frac{\partial u_1}{\partial x}\frac{\partial^2 u_2}{\partial x^2}-$$

$$-2\frac{\partial u_2}{\partial x}\frac{\partial^2 u_1}{\partial x^2}-2\frac{\partial u_3}{\partial x}\frac{\partial^2 u_0}{\partial x^2}-u_0\frac{\partial^3 u_3}{\partial x^3}-u_1\frac{\partial^3 u_2}{\partial x^3}-u_2\frac{\partial^3 u_1}{\partial x^3}-u_3\frac{\partial^3 u_0}{\partial x^3}=0,\quad u_3(x,0)=0 \tag{14e}$$

Consequently one can get a solution for the eq. (14a) to eq. (14e) in the form:

$$u_0(x,y)=\mathrm{e}^{-x}(y+1)-(y+1) \tag{15a}$$

$$u_1(x,y)=-\mathrm{e}^{-x}\left(y^2+\frac{2}{3}y^3\right)+y \tag{15b}$$

$$u_2(x,y)=\mathrm{e}^{-x}\left(\frac{1}{2}y^4+\frac{4}{15}y^5\right) \tag{15c}$$

$$u_3(x,y)=-\mathrm{e}^{-x}\left(\frac{1}{6}y^6+\frac{8}{105}y^7\right) \tag{15d}$$

$$u_4(x,y)=\mathrm{e}^{-x}\left(\frac{1}{24}y^8+\frac{16}{945}y^9\right) \tag{15e}$$

The rest of the components can also be obtained in a similar manner. Thus our obtained fairly accurate solution according to HPM has the following form:

$$u_{\text{approx}}(x,y)=\mathrm{e}^{-x}\begin{pmatrix}1+y-y^2-\frac{2}{3}y^3+\frac{1}{2}y^4+\frac{4}{15}y^5-\frac{1}{6}y^6-\\-\frac{8}{105}y^7+\frac{1}{24}y^8+\frac{16}{945}y^9-\cdots\end{pmatrix}-1 \tag{16}$$

Substituting eq. (12) in eq. (16) yields:

$$u_{\text{approx}}(x,t)=\mathrm{e}^{-x}\begin{pmatrix}1+\frac{t^\beta}{\Gamma(1+\beta)}-\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^2-\frac{2}{3}\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^3+\frac{1}{2}\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^4+\\+\frac{4}{15}\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^5-\frac{1}{6}\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^6-\frac{8}{105}\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^7+\\+\frac{1}{24}\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^8+\frac{16}{945}\left[\frac{t^\beta}{\Gamma(1+\beta)}\right]^9-\cdots\cdots\end{pmatrix}-1 \tag{17}$$

Results and discussion

When $\beta = 1$, eq. (1) has exact solution:

$$u(x,t)=\exp\left(-x+\frac{1}{2}\right)-1 \tag{18}$$

Figure 1 shows the surface plot of the approximate solution obtained by fractional complex transform and HPM whereas fig. 2 displays the surface plot of exact solitary wave solution of time fractional CH equation for value of order $\beta = 1$. Both diagrams indicate the fair resemblance with the approximate solution and solitary wave solution.

Moreover, the surface plot of the solution for the time fractional CH equation against x and t for different values of β is displayed in figs 3 and 4. From figs. 1-4, the solution is a single soliton wave which displays that the balancing scenario between dispersion and non-linearity is valid.

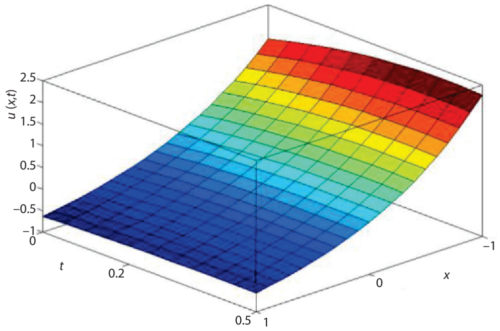

Figure 1. Approximate results of eq. (1) as a 3-D graph for $\beta = 1$

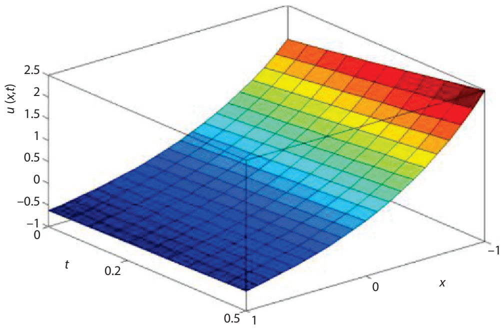

Figure 2. Exact results of eq. (1) as a 3-D graph for $\beta = 1$

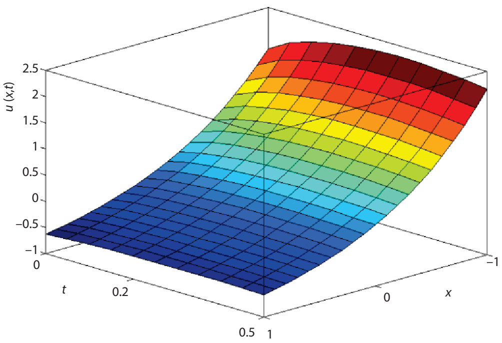

Figure 3. Approximate results of eq. (1) as a 3-D graph for $\beta = 0.8$

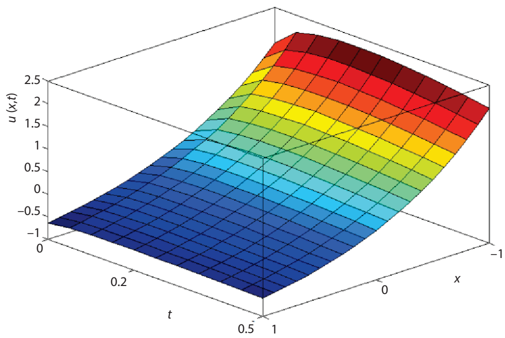

Figure 4. Approximate results of eq. (1) as a 3-D graph for $\beta = 0.6$

Figures 5 and 6 shows the variation in width and amplitude of the soliton because of change in fractal order β. This behavior point outs that there is directly relationship between both the width and the height of the solitary wave and the value of β *i. e.*, increase in the value of order β results in increase both the width and hight of the wave. Thus we can change the shape of solitary wave without disturbing non-linearity with the help of increasing or decreasing the value of fractal order β.

Although the surface plot of the exact solution shown in fig. 2 indicates good agreement with the surface plot of the approximate solution fo CH equation shown in fig. 1 but the numerical results illustrated by follwing figs. 7 and 8 confirms the high accuracy of the method.

Conclusion

In this manuscript, we couple fractional complex transform with HPM to obtain the fairly accurate analytic solution of the non-linear time fractional CH equation. The results shows

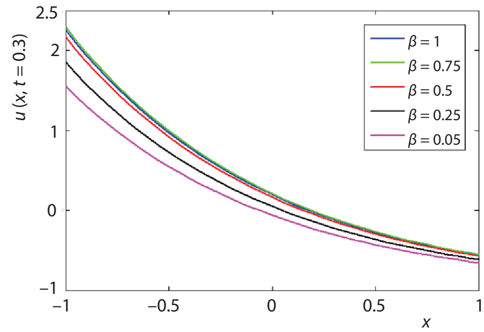

Figure 5. Relationship between β and amplitude of the soliton at $t = 0.3$

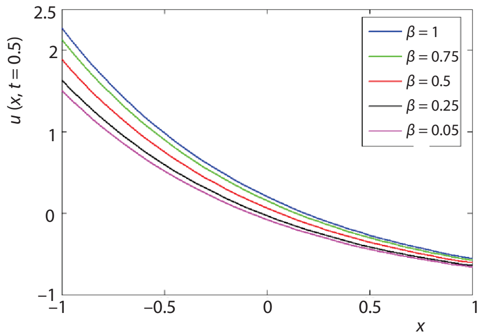

Figure 6. Relationship between β and amplitude of the soliton at $t = 0.5$

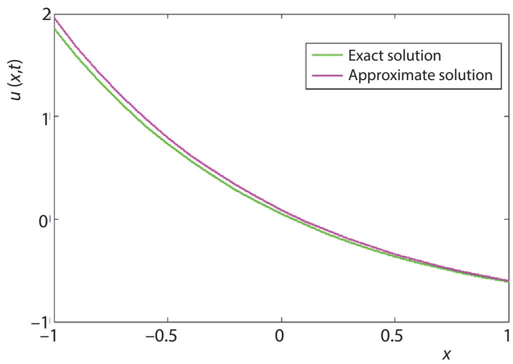

Figure 7. Exact solution *vs.* approximate solution of eq. (1) at $\beta = 1$ and $t = 0.1$

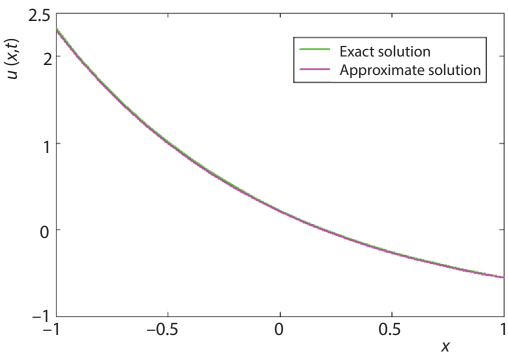

Figure 8. Exact solution *vs.* approximate solution of eq. (1) at $\beta = 1$ and $t = 0.4$

the high accuracy and efficiency of this proposed coupling for the solution. This mathematical technology is easy and can be applied to other time dependent non-linear fractional differential models in science and engineering. The present method can be easily extended to differential equations with fractal derivatives [36-41].

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