SINGULAR HOMOLOGY ALGORITHM FOR MA-SPACES

by

Emel UNVER DEMIR *

Department of Mathematics, Faculty of Art and Sciences,
Manisa Celal Bayar University, Manisa, Turkey

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The work on digitizing subspaces of the 2-D Euclidean space with a certain digital approach is an important discipline in both digital geometry and topology. The present work considers Marcus-Wyse topological approach which was established for studying 2-D digital spaces, $\mathbb{Z}^2$. *We introduce the digital singular homology groups of MA-spaces (M-topological space with an M-adjacency), and we compute singular homology groups of some certain MA-spaces, we give a formula for singular homology groups of 2-D simple closed MA-curves, and an algorithm for determining homology groups of an arbitrary MA-space.*

Key words: *Marcus-Wyse topology, MA-map, simple closed MA-curve, digital singular homology*

Introduction

Homology is a mathematical method that assigns each topological space to a sequence of groups. It has applications that involves computer algorithms in order to determine homology groups of physical objects, also this is the idea of defining homology groups that came forward to distinguish the similar shapes by counting their holes. Besides, homology find uses in medical imaging, computer graphics, image analysis, and geology to determine properties such as connectivity and porosity.

There are several homology theories like singular homology, cellular homology, simplicial homology, Floer homology, *etc.* However singular homology theory is a general and useful structure since we can consider an arbitrary topological space and maps of simplexes into the space itself. Arslan *et al*., [1] introduced the digital simplicial homology groups. Vergili and Karaca [2] investigated some properties of homology groups of spaces associated with Khalimsky topology.

A way for digitizing 2-D spaces is to use Marcus-Wyse topological approach since this topology is improved to study digital spaces in the set of lattice points of $\mathbb{R}^2$, *i. e.* $\mathbb{Z}^2$. In order to classify subspaces of $\mathbb{R}^2$ equipped with the M-topology, Han uses the fundamental group and an MA-fundamental group [3], also he introduce the category of MA-spaces shown with MAC in [4].

Preliminaries and definitions

Let $\mathbb{Z}$ and $\mathbb{Z}^2$ reveal, respectively, the set of integers and lattice points in 2-D Euclidean spaces. Generalizations of the adjacency relations of 2- and 3-D digital spaces [5] which

* Author's e-mail: emel.unver@cbu.edu.tr

is the κ-adjacency relations of $\mathbb{Z}^n$ can be given [6], (see also [5]), as follows: let r be a natural number where $1 \leq r \leq n$, two distinct points $p = (p_i)_{i\in[1,n]_\mathbb{Z}}$ and $q = (q_i)_{i\in[1,n]_\mathbb{Z}}$ in $\mathbb{Z}^n$ are called $\kappa(r, n)$-(for short, κ-)adjacent if:
– there are at most r indices i such that $|p_i - q_i| = 1$ and
– for all other indices i, $p_i = q_i$.

The κ-adjacency relations for $\mathbb{Z}^n$ [7] can be given with the formula:

$$\kappa := \kappa(r,n) = \sum_{i=n-r}^{n-1} 2^{n-i} C_i^n \quad \text{where} \quad C_i^n = \frac{n!}{(n-i)!i!} \tag{1}$$

For instance:

$$(n,r,\kappa) \in \begin{cases} (2,2,8),(2,1,4); \\ (3,3,26),(3,2,18),(3,1,6); \\ (4,4,80),(4,3,64),(4,2,32),(4,1,8) \end{cases}$$

Rosenfeld [8] called (X, κ) pair a digital image where $X \subset \mathbb{Z}^n$ with a κ-adjacency. By using the κ-adjacency of $\mathbb{Z}^2$, $\kappa \in \{4,8\}$, the set [8] $N_\kappa(p) := \{q \in \mathbb{Z}^2 : p$ is κ-adjacent to $q\}$ is called a digital κ-neighborhood of p in $\mathbb{Z}^2$. Also we use the notation [5] $N_\kappa^*(p) := N_\kappa(p) \cup \{p\}$; $\kappa \in \{4,8\}$.

For each point $p = (x, y) \in \mathbb{Z}^2$, let:

$$SN_M(p) = \begin{cases} N_4^*(p), & \text{if } x+y \text{ is even} \\ \{p\}, & \text{else} \end{cases} \tag{2}$$

The topology on $\mathbb{Z}^2$ induced by the set $\{SN_M(p)\}$ is called Marcus-Wyse topology (M-topology) and denoted by $(\mathbb{Z}^2, \gamma)$ [9].

A point $p = (x_1, x_2) \in \mathbb{Z}^2$ is called double even if $x_1 + x_2$ is an even number such that each x_i is even, $i \in \{1, 2\}$; even if $x_1 + x_2$ is an even number such that each x_i is odd, $i \in \{1, 2\}$; and odd if $x_1 + x_2$ is an odd number [10] in this paper.

The ● and ♦ symbols mean an odd point and a double even point, respectively, and the big black spot means an even point and further, in all subspaces of $(\mathbb{Z}^2, \gamma)$ of fig. 1. The following comment will be valid according to the eq. (2): the one-point set with either an even point or a double even point is a closed set under $(\mathbb{Z}^2, \gamma)$. Besides, the one-point set which have an odd point is an open set.

For two M-topological spaces, a function between these two M-topological spaces is called M-continuous at a point from the perspective of M-topology. An M-continuous map and an M-homeomorphism are very useful notions, however they are very strict and cause problems, see [11] Remark 3.2. To overcome these restrictions in M-continuity, Han [11] defines concepts of an *MA-neighborhood* of a point in $\mathbb{Z}^2$ and an *MA-relation* for arbitrary two points in $\mathbb{Z}^2$.

Let x and y be two distinct points in $(\mathbb{Z}^2, \gamma)$. If $y \in SN_M(x)$ or $x \in SN_M(y)$ where $SN_M(p)$ means the smallest open set containing the point $p \in \mathbb{Z}^2$, x and y are called M-adjacent [11].

Let us consider $MA(p) := \{q \in \mathbb{Z}^2 | p$ and q are M-adjacent to each other$\}$ for a point $p \in \mathbb{Z}^2$, and take $MA_X(p) := MA(p) \cap X$ for a space $(X, \gamma_X) := X$. If $q \in MA_X(p)$ or $p \in MA_X(q)$ where p and q are two distinct points in X, these two points are called M-adjacent [11] to each other and the set $MA_X(p) \cup \{p\} := MN_X(p)$ is called MA-neighborhood of p [11] in X .

Hereafter, to indicate an M-topological space (X, γ_X) with an M-adjacency, we use the notation (X, γ_X) and call the space an MA-space.

Let $(X, \gamma_X) := X$ and $(Y, \gamma_Y) := Y$ be two MA-spaces. A function $f : X \to Y$ is called an MA-map at a point $x \in X$ [11] if:

$$f[MN(x)] \subset MN[f(x)] \tag{3}$$

Moreover, if the map $f: X \to Y$ is an MA-map at every point $x \in X$, then f is called an *MA-map*.

Remark 1. [11]

- An MA-map preserves the M-connectedness. However, the converse does not hold.
- Inverse map of a bijective MA-map does not need to be an MA-map.

Consider two MA-spaces $(X, \gamma_X) := X$ and $(Y, \gamma_Y) := Y$. If $f: X \to Y$ is a bijective, continuous MA-map that has a continuous inverse $f^{-1}: Y \to X$ (which is also an MA-map), then f is called an MA-isomorphism [11].

If there is a path $(x_i)_{i\in[0,m]_\mathbb{Z}}$ on X between two distinct points x, $y \in X$ with $\{x_0 = x, x_1,...,x_m = y\}$ such that $\{x_i, x_{i+1}\}$ is MA-connected where $i \in [0, m-1]_\mathbb{Z}$, $m > 1$, we call x and y are MA-path connected. Also, the number m is called the length of this MA-path. Moreover, if $x_0 = x_m$, then the MA-path is called a closed MA-curve. A simple MA-path in X is the finite sequence $(x_i)_{i\in[0,\, m]_\mathbb{Z}}$ such that x_i and x_j are M-adjacent to each other if and only if $|i - j| = 1$. A simple closed MA-curve [11] with l elements $(x_i)_{i\in[0,\, l]_\mathbb{Z}}$ in $\mathbb{Z}^n$ is a simple MA-path with $x_0 = x_l$ (or MA-loop) and that x_i and x_j are M-adjacent if and only if $|i - j| = 1$ $(mod\ l)$ [11], also it is denoted by SC_{MA}^l.

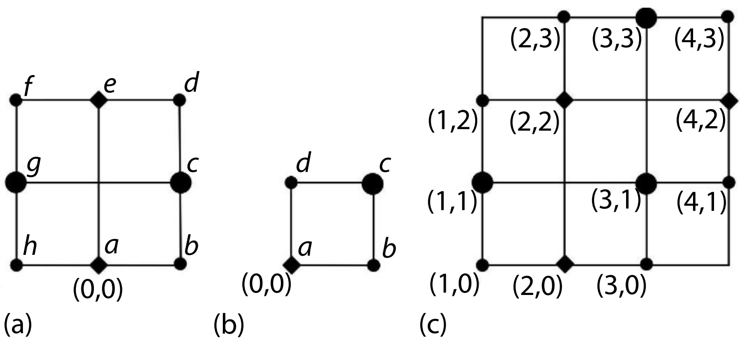

Figure 1. (a) SC_{MA}^8, (b) SC_{MA}^4, and (c) SC_{MA}^{12} [12]

The (a), (b), and (c) given in fig. 1 are sorts of SC_{MA}^8, SC_{MA}^4, and SC_{MA}^{12}, respectively. Besides, we see that $SC_{MA}^{l_1}$ is MA-isomorphic to $SC_{MA}^{l_2}$ if and only if $l_1 = l_2$ [11].

Let (W, κ) be a digital image and S be a set of non-empty subsets of W. The elements of S which hold the followings are called simplices [1] of (W, κ):

- If x and y are distinct points of $s \in S$, then x and y are κ-adjacent.
- If $s \in S$ and $\phi \neq t \subset s$, then $t \in S$ (note this implies every point x that belongs to a simplex determines a simplex $\{x\}$).

An m-simplex consists of $m + 1$ elements. Every non-empty proper subset of a digital m-simplex is called a face [1] of m-simplex.

Let (W, κ) be a finite collection of digital m-simplices, and d be a non-negative integer such that $0 \leq m \leq d$. We say that (W, κ) is a finite digital simplicial complex [1], if the followings hold:

- If S belongs to W, then every face of S also belongs to W.
- If S, $T \in W$, then $S \cap T$ is either empty or a common face of S and T.

The dimension of W is the biggest integer m such that W has an m-simplex.

Singular homology groups of MA-spaces

Homology groups of MA-spaces

Let Δ^n denote the standard n-simplex in $\mathbb{Z}^n$ having the vertices:

$$e_0 = (0,0,...,0),\ e_1 = (1,0,...,0),\ e_2 = (0,1,...,0),...,\ e_n = (0,0,...,1)$$

Under this definition, Δ^{n-1} can be considered as a *face* of Δ^n.

Example 1.

– if $n = 0$, $\Delta^0 = [e_0]$ and the Marcus-Wyse topology on Δ^0 is $\gamma_{\Delta_0} = \{\varnothing, \Delta^0\}$ and
– if $n = 1$, $\Delta^1 = [e_0, e_1]$ and the Marcus-Wyse topology on Δ^1 is $\gamma_{\Delta_1} = \{\varnothing, \Delta^1, \{e_0\}\}$.

Remark 2. According to the definition of a simplex and the structure of MA-spaces, n cannot be greater or equal than 2. Thus we only have 0 and 1-simplices in MA-spaces, fig. 2.

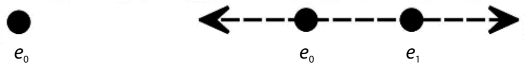

Figure 2. The Δ^0 and Δ^1

An orientation of $\Delta^n = [e_0\ e_1,...,e_n]$ is made by ordering of its vertices linearly. The induced orientation of faces of a singular n-simplex can be defined by orienting the i^{th} face via $(-1)^i[e_0,...,\hat{e}_i,...,e_n]$ where minus means opposite orientation of the vertices ordered as displayed (*i. e.* $e_0 < e_1 <...< e_n$) and a vertex with the hat means deleting that vertex.

The boundary of Δ^n is $\cup_{i=0}^{p}\ [e_0,..., \hat{e}_i,...,e_n]$ and the oriented boundary of Δ^n is $\cup_{i=0}^{p}(-1)^i$ $[e_0,...,\hat{e}_i,...,e_n]$.

Definition 1. An MA-map $\sigma^n : \Delta^n \to X$ where $(X, \gamma_X) := X$ is an MA-space is called a digital singular n-simplex in X. The $S_n(X)$ is defined as the free abelian group with basis all singular n-simplexes in X for each $n \geq 0$; besides $S_{-1}(X)$ is defined as trivial group. The elements of $S_n(X)$ are called the digital singular n-chains in X.

The i^{th} face map $\varepsilon_i = \varepsilon_i^n : \Delta^{n-1} \to \Delta^n$ for each n and i is defined to be a map which takes the vertices $\{e_0,...,e_{n-1}\}$ to $\{e_0,...,\hat{e}_i,...,e_n\}$ by keeping the displayed orderings.

Definition 2. Let $(X, \gamma_X) := X$ be an MA-space. The boundary of a digital singular n-simplex $\sigma^n : \Delta^n \to X$ is $\partial_0\sigma = \sum_{i=0}^{n}(-1)^i\sigma\varepsilon_i^n \in S_{n-1}(X)$. Define $\partial_0\sigma = 0$ when $n = 0$.

The homomorphisms $\partial_n : S_n(X) \to S_{n-1}(X)$ are called *boundary operators*. Since boundary operators are linear, there is a unique homomorphism $\partial_n : S_n(X) \to S_{n-1}(X)$ for each $n \geq 0$ with $\partial_n(\sigma^n) = \sum_{i=0}^{n}(-1)^i\sigma\varepsilon_i$ for every digital singular n-simplex σ^n in X.

Definition 3. A digital singular complex of the MA-space (X, γ_X) which is denoted by $S_*(X)$ is the sequence of homomorphisms and free abelian groups:

$$...\xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} ... \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0$$

Lemma 1. The face maps satisfy: $\varepsilon_j^{n+1}\varepsilon_k^n = \varepsilon_k^{n+1}\varepsilon_{j-1}^n : \Delta^{n-1} \to \Delta^n$ where $k < j$.

For the proof of the following theorem, see [13, Theorem 4.6]:

Theorem 1. The $\partial_n\partial_{n+1} = 0$ for all $n \geq 0$.

Definition 4. In an MA-space $(X, \gamma_X) := X$, the kernel of the boundary operator ∂_n is called the group of the singular n-cycles $Z_n(X) :=$ Kernel ∂_n, and the image of the boundary operator ∂_{n+1} in X is called the group of the singular n-boundaries $B_n(X) :=$ Image ∂_{n+1}.

For every MA-space $(X, \gamma_X) := X$ and $n \geq 0$, we have $B_n(X) \subset Z_n(X) \subset S_n(X)$ since $\partial_n\partial_{n+1} = 0$.

Definition 5. Let X be an MA-space. The n^{th} singular homology group of X for each $n \geq 0$ is:

$$H_n(X) := \frac{Z_n(X)}{B_n(X)}$$

Theorem 2. Let $(X, \gamma_X) := X \subset \mathbb{Z}^2$ be the MA-space such that

$$X = \{a = (0,0), b = (1,0), c = (1,1)\}$$

then $H_0(X) \cong \mathbb{Z}$, $H_1(X) \cong \mathbb{Z}^3$, and $H_n(X) \cong 0$ for all $n \geq 2$.

Proof.

– $S_0(X)$ has for the basis $\{\sigma_i^0 : i \in \{1, 2, 3\}\}$ where $\sigma_1^0 : e_0 \mapsto a$, $\sigma_2^0 : e_0 \mapsto b$, and $\sigma_3^0 : e_0 \mapsto c$
– $S_1(X)$ has for the basis $\{\sigma_i^1 : i \in \{1, 2, 3, 4, 5\}\}$ where:

$$\begin{array}{lllll} \sigma_1^1 : e_0 \mapsto a & \sigma_2^1 : e_0 \mapsto b & \sigma_3^1 : e_0 \mapsto c & \sigma_4^1 : e_0 \mapsto a & \sigma_5^1 : e_0 \mapsto b \\ \quad e_1 \mapsto a & \quad e_1 \mapsto b & \quad e_1 \mapsto c & \quad e_1 \mapsto b & \quad e_1 \mapsto c \end{array}$$

Thus we have $S_0(X) \cong \mathbb{Z}^3$ and $S_1(X) \cong \mathbb{Z}^5$. Now we will determine the boundaries and cycles of each singular n-chains.

For $\sigma_i^1 \in S_1(X)$, we have the differential map $\partial_1 : S_1(X) \to S_0(X)$ such that:

$$\partial_1\left(\sigma_i^1\right) = \sigma_i^1\left(e_1\right) - \sigma_i^1\left(e_0\right) \text{ for } i = 1,2,3,4,5$$

thus

$$\begin{array}{ll} \partial_1(\sigma_1^1) = 0 & \partial_1(\sigma_4^1) = \sigma_4^1(e_1) - \sigma_4^1(e_0) = b - a = \sigma_2^0 - \sigma_1^0 \\ \partial_1(\sigma_2^1) = 0 & \partial_1(\sigma_5^1) = \sigma_5^1(e_1) - \sigma_5^1(e_0) = c - b = \sigma_3^0 - \sigma_2^0 \\ \partial_1(\sigma_3^1) = 0 & \end{array}$$

Then we get $Im\ \partial_1 \cong \mathbb{Z}^2$. In order to find the kernel, let $\partial_1(\sum_{i=1}^5 t_i\, \sigma_i^1) = 0$ where $t_i \in \mathbb{Z}$, $i \in \{1, 2, 3, 4, 5\}$. Since ∂_1 is linear, $\sum_{i=1}^5 t_i\, \partial_1(\sigma_i^1) = 0$. By solving the equation:

$$-t_4\sigma_1^0 + (t_4 - t_5)\sigma_2^0 + t_5\sigma_3^0 = 0$$

we obtain $t_4 = t_5$, so $Ker\ \partial_1 \cong \mathbb{Z}^3$.

Thus the singular homology groups of the MA-space $(X,\gamma_X) := X$ are $H_0(X) \cong \mathbb{Z}$, $H_1(X) \cong \mathbb{Z}^3$, and $H_n(X) = 0$; $n \geq 2$.

Homology of simple closed curves

Theorem 3. Let $(SC_M^4, \gamma_{SC_M^4}) := SC_M^4 \subset \mathbb{Z}^2$ be the MA-space

$$SC_M^4 \{a = (0,0), b = (1, 0), d = (0,1)\}$$

see in fig. 1(b), then $H_0(SC_M^4) \cong \mathbb{Z}$, $H_1(SC_M^4) \cong \mathbb{Z}^5$, and $H_n(SC_M^4) \cong 0$ for all $n \geq 2$.

Proof.

– $S_0(SC_M^4) \cong \mathbb{Z}^4$, since $S_0(SC_M^4)$ has for the basis $\{\sigma_i^0 : i \in \{1, 2, 3, 4, 5\}\}$ where:

$$\sigma_1^0 : e_0 \mapsto a,\ \sigma_2^0 : e_0 \mapsto b,\ \sigma_3^0 : e_0 \mapsto c, \text{ and } \sigma_4^0 : e_0 \mapsto d$$

– $S_1(SC_M^4) \cong \mathbb{Z}^8$, since $S_1(SC_M^4)$ has for the basis $\{\sigma_i^1 : i \in \{1, 2, 3, 4, 5, 6, 7, 8\}\}$ where:

$$\begin{array}{llll} \sigma_1^1 : e_0 \mapsto a & \sigma_2^1 : e_0 \mapsto b & \sigma_3^1 : e_0 \mapsto c & \sigma_4^1 : e_0 \mapsto d \\ \quad e_1 \mapsto a & \quad e_1 \mapsto b & \quad e_1 \mapsto c & \quad e_1 \mapsto d \\ \sigma_5^1 : e_0 \mapsto a & \sigma_6^1 : e_0 \mapsto a & \sigma_7^1 : e_0 \mapsto b & \sigma_8^1 : e_0 \mapsto c \\ \quad e_1 \mapsto b & \quad e_1 \mapsto d & \quad e_1 \mapsto c & \quad e_1 \mapsto d \end{array}$$

– $S_n(SC_M^4) = 0$ for all $n \geq 2$.

Consider $\partial_1 : S_1(SC_M^4) \to S_0(SC_M^4)$. For $\sigma_i^1 \in S_1(SC_M^4)$, we have the differential map:

$$\partial_1(\sigma_i^1) = \sigma_i^1(e_1) - \sigma_i^1(e_1) \text{ for } i \in [1, 8]_{\mathbb{Z}}.$$

Then

$$\begin{array}{ll} \partial_1(\sigma_1^1)=0 & \partial_1(\sigma_5^1)=\sigma_5^1(e_1)-\sigma_5^1(e_0)=b-a=\sigma_2^0-\sigma_1^0 \\ \partial_1(\sigma_2^1)=0 & \partial_1(\sigma_6^1)=\sigma_6^1(e_1)-\sigma_6^1(e_0)=d-a=\sigma_4^0-\sigma_1^0 \\ \partial_1(\sigma_3^1)=0 & \partial_1(\sigma_7^1)=\sigma_7^1(e_1)-\sigma_7^1(e_0)=c-b=\sigma_3^0-\sigma_2^0 \\ \partial_1(\sigma_4^1)=0 & \partial_1(\sigma_8^1)=\sigma_8^1(e_1)-\sigma_8^1(e_0)=d-c=\sigma_4^0-\sigma_3^0 \end{array}$$

By the linearity of ∂_1:

$$\partial_1(\textstyle\sum_{i=1}^8 t_i\sigma_i^1)=\sum_{i=1}^8 t_i\partial_1(\sigma_i^1)=\sum_{j=1}^4 s_j\sigma_j^0$$

$$t_5(\sigma_2^0-\sigma_1^0)+t_6(\sigma_4^0-\sigma_1^0)+t_7(\sigma_3^0-\sigma_2^0)+t_8(\sigma_4^0-\sigma_3^0)=\textstyle\sum_{j=1}^4 s_j\sigma_j^0$$

$$(-t_5-t_6)\sigma_1^0+(t_5-t_7)\sigma_2^0+(t_7-t_8)\sigma_3^0+(t_6+t_8)\sigma_4^0=\textstyle\sum_{j=1}^4 s_j\sigma_j^0$$

where t_i, $s_j \in \mathbb{Z}$, we have $Im\ \partial_1 \cong \mathbb{Z}^3$. When we solve the equation:

$$\partial_1(\textstyle\sum_{i=1}^8 t_i\sigma_i^1)=\sum_{i=1}^8 t_i\partial_1(\sigma_i^1)=0$$

we obtain $t_5=-t_6=t_7=t_8$, so $Ker\ \partial_1 \cong \mathbb{Z}^5$.

Hence the singular homology groups of the MA-space SC_M^4 are $H_0(SC_M^4)\cong\mathbb{Z}$, $H_1(SC_M^4)\cong\mathbb{Z}^5$, $H_n(SC_M^4)=0$, $n\geq 2$.

Theorem 4. Let $(SC_M^8, \gamma_{SC_M^8}) := SC_M^8 \subset \mathbb{Z}^2$ be the MA-space such that:

$SC_M^8=\{a=(0,0), b=(1,0), c=(1,1), d=(1,2), e=(0,2), f=(-1,2), g=(-1,1), h=(-1,0)\}$

see in fig. 1(a), then $H_0(SC_M^8)\cong\mathbb{Z}$, $H_1(SC_M^8)\cong\mathbb{Z}^9$, and $H_n(SC_M^8)\cong 0$ for all $n\geq 2$.

Proof.

– $S_0(SC_M^8)\cong\mathbb{Z}^8$, since $S_0(SC_M^8)$ has for the basis $\{\sigma_i^0: e_0\to x : x\in SC_M^8, i\in[1,8]_{\mathbb{Z}}\}$.
– $S_1(SC_M^8)\cong\mathbb{Z}^{16}$, since $S_1(SC_M^8)$ has for the basis $\{\sigma_i^1: i\in[1,16]_{\mathbb{Z}}\}$ where:

$$\begin{array}{llllll} \sigma_1^1:e_0\mapsto a, & \sigma_2^1:e_0\mapsto b, & \sigma_3^1:e_0\mapsto c, & \sigma_4^1:e_0\mapsto d, & \sigma_5^1:e_0\mapsto e, & \sigma_6^1:e_0\mapsto f \\ \quad e_1\mapsto a, & \quad e_1\mapsto b, & \quad e_1\mapsto c, & \quad e_1\mapsto d, & \quad e_1\mapsto e, & \quad e_1\mapsto f \\ \sigma_7^1:e_0\mapsto g, & \sigma_8^1:e_0\mapsto h, & \sigma_9^1:e_0\mapsto a, & \sigma_{10}^1:e_0\mapsto a, & \sigma_{11}^1:e_0\mapsto b, & \sigma_{12}^1:e_0\mapsto c \\ \quad e_1\mapsto g, & \quad e_1\mapsto h, & \quad e_1\mapsto b, & \quad e_1\mapsto h, & \quad e_1\mapsto c, & \quad e_1\mapsto d \\ \sigma_{13}^1:e_0\mapsto d, & \sigma_{14}^1:e_0\mapsto e, & \sigma_{15}^1:e_0\mapsto f, & \sigma_{16}^1:e_0\mapsto g, & & \\ \quad e_1\mapsto e, & \quad e_1\mapsto f, & \quad e_1\mapsto g, & \quad e_1\mapsto h & & \end{array}$$

– $S_n(SC_M^8)=0$ for all $n\geq 2$.

Consider $\partial_1 : S_1(SC_M^8)\to S_0(SC_M^8)$. For $\sigma_i^1\in S_1(SC_M^8)$, we have the following differential map:

$$\partial_1(\sigma_i^1)=\sigma_i^1(e_1)-\sigma_i^1(e_0) \text{ for } i\in[1,16]_{\mathbb{Z}}.$$

Then:

$$\begin{array}{ll} \partial_1(\sigma_1^1)=0 & \partial_1(\sigma_9^1)=\sigma_9^1(e_1)-\sigma_9^1(e_0)=b-a=\sigma_2^0-\sigma_1^0 \\ \partial_1(\sigma_2^1)=0 & \partial_1(\sigma_{10}^1)=\sigma_{10}^1(e_1)-\sigma_{10}^1(e_0)=h-a=\sigma_8^0-\sigma_1^0 \\ \partial_1(\sigma_3^1)=0 & \partial_1(\sigma_{11}^1)=\sigma_{11}^1(e_1)-\sigma_{11}^1(e_0)=c-b=\sigma_3^0-\sigma_2^0 \\ \partial_1(\sigma_4^1)=0 & \partial_1(\sigma_{12}^1)=\sigma_{12}^1(e_1)-\sigma_{12}^1(e_0)=d-c=\sigma_4^0-\sigma_3^0 \end{array}$$

$$\begin{aligned}
&\partial_1(\sigma_5^1)=0 && \partial_1(\sigma_{13}^1)=\sigma_{13}^1(e_1)-\sigma_{13}^1(e_0)=e-d=\sigma_5^0-\sigma_4^0\\
&\partial_1(\sigma_6^1)=0 && \partial_1(\sigma_{14}^1)=\sigma_{14}^1(e_1)-\sigma_{14}^1(e_0)=f-e=\sigma_6^0-\sigma_5^0\\
&\partial_1(\sigma_7^1)=0 && \partial_1(\sigma_{15}^1)=\sigma_{15}^1(e_1)-\sigma_{15}^1(e_0)=g-f=\sigma_7^0-\sigma_6^0\\
&\partial_1(\sigma_8^1)=0 && \partial_1(\sigma_{16}^1)=\sigma_{16}^1(e_1)-\sigma_{16}^1(e_0)=h-g=\sigma_8^0-\sigma_7^0.
\end{aligned}$$

By the linearity of ∂_1:

$$\partial_1\left(\sum_{i=1}^{16} t_i\sigma_i^1\right)=\sum_{i=1}^{16} t_i\partial_1(\sigma_i^1)=\sum_{j=1}^{8} s_j\sigma_j^0$$

$$t_9(\sigma_2^0-\sigma_1^0)+t_{10}(\sigma_8^0-\sigma_1^0)+t_{11}(\sigma_3^0-\sigma_2^0)+t_{12}(\sigma_4^0-\sigma_3^0)+t_{13}(\sigma_5^0-\sigma_4^0)+$$
$$+t_{14}(\sigma_6^0-\sigma_5^0)+t_{15}(\sigma_7^0-\sigma_6^0)+t_{16}(\sigma_8^0-\sigma_7^0)=\sum_{j=1}^{8} s_j\sigma_j^0$$

$$(-t_9-t_{10})\sigma_1^0+(t_9-t_{11})\sigma_2^0+(t_{11}-t_{12})\sigma_3^0+(t_{12}-t_{13})\sigma_4^0+(t_{13}-t_{14})\sigma_5^0+$$
$$+(t_{14}-t_{15})\sigma_6^0+(t_{15}-t_{16})\sigma_7^0+(t_{10}+t_{16})\sigma_8^0=\sum_{j=1}^{8} s_j\sigma_j^0$$

where $t_i, s_j \in \mathbb{Z}$, we have $Im\ \partial_1 \cong \mathbb{Z}^7$.

When we solve the equation:

$$\partial_1\left(\sum_{i=1}^{16} t_i\sigma_i^1\right)=\sum_{i=1}^{16} t_i\partial_1\left(\sigma_i^1\right)=0$$

we obtain $t_9=-t_{10}=t_{11}=t_{12}=t_{13}=t_{14}=t_{15}=t_{16}$ so $Ker\ \partial_1 \cong \mathbb{Z}^9$.

Hence the singular homology groups of the MA-space SC_M^8 are $H_0(SC_M^8) \cong \mathbb{Z}$, $H_1(SC_M^8) \cong \mathbb{Z}^9$, and $H_n(SC_M^8) = 0$; $n \geq 2$.

If we try to determine the singular homology groups of any 2-D simple closed curve with l points, we have the following formula:

Corollary 1. Let $SC_M^l = \{x_0, x_1, \dots, x_{l-1}\}$ be any simple closed MA-curve with l elements. Then the singular homology groups of SC_M^l are:

$$H_n(SC_M^l)=\begin{cases}\mathbb{Z}, & n=0\\ \mathbb{Z}^{l+1}, & n=1\\ 0, & n\geq 2\end{cases}$$

Proof. Let x_i be any element of SC_M^l. If x_i is odd, then $SN(x_i) = \{x_i\}$. If x_i is even or double even, then $SN(x_i) = \{x_{i-1}, x_i, x_{i+1}\}$ (*mod l*). There is an ordering of elements of any simple closed curve such that between two odd points there is an even or a double even point. According to this ordering, consecutive elements of a simple closed curve become an MA-neighbor of each other since every odd point is an element of the smallest neighborhood of even or double even point. Hence $MA(x_i) = \{x_{i-1}, x_i, x_{i+1}\}$(*mod l*) for every $x_i \in SC_M^l$. Via the singular chain maps which are the base elements satisfying the expression (3), we always get the singular chain groups of SC_M^l isomorphic to $\mathbb{Z}^l$ in dimension 0, isomorphic to $\mathbb{Z}^{2l}$ in dimension 1, and isomorphic to trivial group in dimensions greater or equal to 2 because of *Remark 2*. Also we have $Ker\ \partial_0 \cong \mathbb{Z}^l$, $Ker\ \partial_1 \cong \mathbb{Z}^{l+1}$, $Im\ \partial_1 \cong \mathbb{Z}^{l-1}$, and $Im\ \partial_2 \cong 0$. As a result, it is easy to obtain $H_0(SC_M^l) \cong \mathbb{Z}$, $H_1(SC_M^l) \cong \mathbb{Z}^{l+1}$, and $H_n(SC_M^l) \cong 0$ for $n \geq 2$.

An algorithm for computing homology groups of an MA-space

Although we have a formula for determining homology groups of a simple closed MA-curve by the *Corollary 1*, if one would like to compute singular homology groups of any

MA-space no matter it is a simple closed curve or not, following algorithm can be used for this purpose.

```
Input: An MA-space with m points, X⊂ℤ².
Output: Homology groups of the given MA-space.

BEGIN

      Take the coordinates of m points of MA-space into
 an integer array A[m][2].
(c0 = (c01, c02),c1=(c11, c12), c2 = (c21,c22),...cm-1 =(c(m-1)1,c(m-1)2))

 Order the points with respect to dictionary order.

 FOR i ← 0 TO m DO
  if(i <= 1){
  detect Si(X)
       }

  else {Si(X)=0
       }
REPEAT

  //While constructing ∂1: S1(X) → S0(X) use Definition 2.
  //Define ∂1 as zero homomorphism and ∂0
     as trivial homomorphism.
  //While constructing Zi(X),Bi(X) use Definition 4.
  //While constructing Hi(X), use Definition 5.

 FOR i ← 0 TO m DO
  if (i <= 1){
  detect Zi(X)
         Bi(X)
         Hi(X)= Zi(X)/ Bi(X)

           }

  else {Hi(X) = 0
       }
 REPEAT

END
```

Conclusion

The main aim of this study is to introduce the singular homology groups based on MA-spaces. To achieve this, we have modified the classical definitions and compute the homology groups of some basic MA-spaces by using the properties of M-adjacency relation and MA-

maps which give us interesting results for 1-D homology groups. Besides any simple closed MA-curve coincides with a digital image with 4-adjacency, thus zero dimensional homology groups for these spaces are $\mathbb{Z}$ as usual. Finally, a general algorithm is given for computing digital singular homology groups of 2-D MA-spaces.

References

[1] Arslan, H., *et al*., Homology Groups of n-Dimensional Digital Images, *Proceedings*, 21th Turkish National Mathematics Symposium, Istanbul, Turkey, B, 2008, pp. 1-13
[2] Vergili, T., Karaca, I., Some Properties of Homology Groups of Khalimsky Spaces, *Mathematical Sciences Letters*, *4* (2015), 2, pp. 131-140
[3] Han, S. E., Yao, W., An MA-Digitization of Hausdorff Spaces By Using a Connectedness Graphof the Marcus-Wyse Topology, *Discrete Applied Mathematics*, *276* (2017), Part 2, pp. 335-347
[4] Han, S. E., Almost Fixed Point Property for Digital Spaces Associated with Marcus-Wyse Topological Spaces, *Journal of Non-linear Science and Applications*, *10* (2017), 1, pp. 34-47
[5] Kong, T. Y., Rosenfeld, A., *Topological Algorithms for Digital Image Processing*, Elsevier Science, Amsterdam, The Netherlands, 1996
[6] Han, S. E., Non-Product Property of the Digital Fundamental Group, *Information Sciences*, *171* (2005), 1, pp. 73-91
[7] Han, S. E., The k-Homotopic Thinning and a Torus-Like Digital Image in Zn, *Journal of Mathematical Imaging and Vision*, *31* (2008), 1, pp. 1-16
[8] Rosenfeld, A., Digital Topology, *American Mathematical Monthly*, *86* (1979), 8, pp. 76-87
[9] Wyse, F., Marcus, D., Solution Problem 5712, *American Mathematical Monthly*, *77* (1970), 1119
[10] Slapal, J., Topological Structuring of the Digital Plane, *Discrete Mathematics and Theoretical Computer Science*, *15* (2013), 2, pp. 165-176
[11] Han, S. E., Generalizations of Continuity of Maps and Homeomorphisms for Studying 2-D DigitalTopological Spaces and Their Applications, *Topology and its Applications*, *196* (2015), Part B, pp. 468-482
[12] Han, S. E., Yao, W., Homotopy Based on Marcus-Wyse, *Topology and its Applications*, *201* (2016), Mar., pp. 358-371
[13] Rotman, J. J., *An Introduction Algebraic Topology*, Springer-Verlag, New York, USA, 1998

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