NUMERICAL STUDY FOR FRACTIONAL MODEL OF NON-LINEAR PREDATOR-PREY BIOLOGICAL POPULATION DYNAMICAL SYSTEM

by

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The key objective of the present paper is to propose a numerical scheme based on the homotopy analysis transform technique to analyze a time-fractional non-linear predator-prey population model. The population model are coupled fractional order non-linear PDE often employed to narrate the dynamics of biological systems in which two species interact, first is a predator and the second is a prey. The proposed scheme provides the series solution with a great freedom and flexibility by choosing appropriate parameters. The convergence of the results is free from small or large parameters. Three examples are discussed to demonstrate the correctness and efficiency of the used computational approach.

Key words: *homotopy analysis transform technique, biological systems, fractional non-linear predator-prey population model*

Introduction

In recent years, ample interest in differential equations of fractional order has been triggered due to their large number of utilities in modelling of many scientific processes such as fluid-flow, heat and mass transfers, *etc.* [1-11]. In this work, we put up the combined form of homotopy analysis method and classical Laplace transform to produce a modified algorithm termed as the homotopy analysis transform technique (HATT) [12-18], to analyze the time-fractional non-linear predator-prey population model.

The dynamical connection between predator and prey is the main issue in ecology and mathematical sciences. The considerable development in population dynamic was presented independently by many research workers [19-27].

We appraise the two-species competitive model by taking into consideration population of prey u and population of predator v. For population of prey $u \to 2u$ (population of prey tends to twice of present population of prey), having the rate λ, $\lambda > 0$ indicates the rate of natural birth. For population of predator $v \to 0$, having the rate μ, $\mu > 0$ stands for the rate of natural death. The connected term between predator and prey populations is $u + v \to 2v$, at rate σ, parameter σ designate the competitive rate. In view of an extensively accepted theory of

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fractional biological population models, the mathematical model of a predator-prey system of fractional order can be illustrated as:

$$\begin{aligned}&\frac{\partial^\rho \xi}{\partial t^\rho}=\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial^2 \xi}{\partial y^2}+\lambda\xi-\sigma\xi\eta, \quad \xi(x,y,0)=\varphi(x,y)\\ &\frac{\partial^\beta \eta}{\partial t^\beta}=\frac{\partial^2 \eta}{\partial x^2}+\frac{\partial^2 \eta}{\partial y^2}+\sigma\xi\eta-\mu\eta, \quad \eta(x,y,0)=\phi(x,y)\end{aligned} \tag{1}$$

In the previous eq. (1) $t > 0$, $x, y \in R$, $\lambda, \sigma, \mu > 0$, the function ξ represents the density of prey population and the function η indicates the density of predator population, $\varphi(x, y)$, $\phi(x, y)$ indicates initial conditions for the growth of population.

Basic concept of HATT

We assume a non-linear PDE involving fractional derivative of the form [12-15]:

$$D_t^\rho \xi + R\xi + N\xi = a(t), \quad n-1<\rho\le n \tag{2}$$

In eq. (2) $D_t^\rho\xi$ is representing the fractional derivative of ξ understood in Caputo sense [1], R – the standing for the linear differential operator, N – the indicating the general non-linear differential operator, and $a(t)$ – the representing the term due to source.

By putting the LT operator on eq. (2) and simplifying [2], we get:

$$L[\xi]-\frac{1}{s^\rho}\sum_{k=0}^{n-1} s^{\rho-k-1}\xi^{(k)}(0)+\frac{1}{s^\rho}\{L[R\xi]+L[N\xi]-L[a(t)]\}=0 \tag{3}$$

We interpret the non-linear operator:

$$\Omega[\vartheta(t;z)]=L[\vartheta(t;z)]-\frac{1}{s^\rho}\sum_{k=0}^{n-1} s^{\rho-k-1}\vartheta^{(k)}(t;z)(0^+)+\frac{1}{s^\rho}\big[L[R\vartheta(t;z)]+L[N\vartheta(t;z)]-L[a(t)]\big], \tag{4}$$

where $z \in [0, 1]$ and $\vartheta(t; z)$ is a real function of t and z. We develop a homotopy as:

$$(1-z)L[\vartheta(t;z)-\xi_0(t)]=\hbar z\Omega[\xi(t)] \tag{5}$$

In the expression (5), L is indicating the LT operator, $z \in [0, 1]$ – the denoting the embedding parameter, $\hbar \neq 0$ – the standing for an auxiliary parameter, ξ_0 – the denoting an initial guess of ξ, and $\vartheta(t; z)$ – the indicating an unknown function. Clearly, when the embedding parameter $z = 0$ and $z = 1$ it yields:

$$\vartheta(t;0)=\xi_0(t), \quad \vartheta(t;1)=\xi(t) \tag{6}$$

respectively. It is to be noticed that as z increases form 0 to 1, the solution $\vartheta(t; z)$ converts from the initial guess ξ_0 to the solution ξ. Now using the Taylor theorem we write the $\vartheta(t; z)$ in series form given below:

$$\vartheta(t;z)=\xi_0(t)+\sum_{m=1}^{\infty}\xi_m(t)z^m \tag{7}$$

where

$$\xi_m(t)=\frac{1}{m!}\frac{\partial^m\vartheta(t;z)}{\partial z^m}\Big|_{z=0} \tag{8}$$

If the initial approximation of the solution and the parameter $\hbar$ are taken in well manner, the series (7) converges at $z = 1$, then we have the subsequent series solution of the given non-linear equation of fractional order:

$$\xi = \xi_0 + \sum_{m=1}^{\infty} \xi_m \tag{9}$$

Making use of eq. (9), the governing equation can be derived from the eq. (5). Now we define the following vectors:

$$\overline{\xi}_m = \{\xi_0, \xi_1, ..., \xi_m\} \tag{10}$$

On differentiation of the eq. (5) m-times w. r. t. z and then division them by $m!$ and then putting $z = 0$, we arrive at the following result:

$$L[\xi_m - \chi_m \xi_{m-1}] = \hbar \Re_m(\overline{\xi}_{m-1}) \tag{11}$$

Applying the inverse LT, we get the following result:

$$\xi_m = \chi_m \xi_{m-1} + \hbar L^{-1}[\Re_m(\overline{\xi}_{m-1})] \tag{12}$$

where

$$\Re_m(\overline{\xi}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \Omega[\vartheta(t;z)]}{\partial z^{m-1}}\Big|_{z=0}, \tag{13}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases} \tag{14}$$

Implementation of the method

Here we apply the HATT to examine the fractional predator-prey system given in eq. (1) with initial gauss $\xi_0(x, y, t) = \varphi(x, y)$ and $\eta_0(x, y, t) = \phi(x, y)$. We express the non-linear operator:

$$\Omega_1[\vartheta_1(x,y,t;z), \vartheta_2(x,y,t;z)] = \\ = L[\vartheta_1(x,y,t;z)] - (1-\chi_m)\frac{1}{s}\phi(x,y) - \frac{1}{s^\rho} L\left[\frac{\partial^2 \vartheta_1}{\partial x^2} + \frac{\partial^2 \vartheta_1}{\partial y^2} + \lambda \vartheta_1 - \sigma \vartheta_1 \vartheta_2\right] \tag{15}$$

and

$$\Omega_2[\vartheta_1(x,y,t;z), \vartheta_2(x,y,t;z)] \\ = L[\vartheta_2(x,y,t;z)] - (1-\chi_m)\frac{1}{s}\varphi(x,y) - \frac{1}{s^\beta} L\left[\frac{\partial^2 \vartheta_2}{\partial x^2} + \frac{\partial^2 \vartheta_2}{\partial y^2} + \sigma \vartheta_1 \vartheta_2 - \mu \vartheta_2\right] \tag{16}$$

Further the LT operators are written:

$$L\left[\xi_m(x,y,t) - \chi_m \xi_{m-1}(x,y,t)\right] = \hbar R_{1,m}[\vec{\xi}_{m-1}, \vec{\eta}_{m-1}] \tag{17}$$

and

$$L\left[\eta_m(x,y,t) - \chi_m \eta_{m-1}(x,y,t)\right] = \hbar R_{2,m}[\vec{\xi}_{m-1}, \vec{\eta}_{m-1}] \tag{18}$$

In the previous eqs. (17) and (18), the $R_{1,\mathrm{m}}[\vec{\xi}_{m-1}, \vec{\eta}_{m-1}]$ and $R_{2,\mathrm{m}}[\vec{\xi}_{m-1}, \vec{\eta}_{m-1}]$ are expressed:

$$R_{1,m}[\vec{\xi}_{m-1},\vec{\eta}_{m-1}] = L\{\xi_{m-1}(x,y,t)\} - (1-\chi_m)\frac{1}{s}\xi_0 - \\ -\frac{1}{s^{\rho}}L\left\{\frac{\partial^2\xi_{m-1}}{\partial x^2} + \frac{\partial^2\xi_{m-1}}{\partial y^2} + \lambda\xi_{m-1} - \sigma\sum_{i=0}^{m-1}\xi_i\eta_{m-1-i}\right\} \tag{19}$$

and

$$R_{2,m}[\vec{\xi}_{m-1},\vec{\eta}_{m-1}] = L\{\eta_{m-1}(x,y,t)\} - (1-\chi_m)\frac{1}{s}\eta_0 - \\ -\frac{1}{s^{\beta}}L\left\{\frac{\partial^2\eta_{m-1}}{\partial x^2} + \frac{\partial^2\eta_{m-1}}{\partial y^2} + \sigma\sum_{i=0}^{m-1}\xi_i\eta_{m-1-i} - \mu\eta_{m-1}\right\} \tag{20}$$

It is obvious from eqs. (17) and (18), we get:

$$\xi_m = \chi_m\xi_{m-1} + \hbar L^{-1}\{R_{1,m}[\vec{\xi}_{m-1},\vec{\eta}_{m-1}]\} \tag{21}$$

and

$$\eta_m = \chi_m\eta_{m-1} + \hbar L^{-1}\{R_{2,m}[\vec{\xi}_{m-1},\vec{\eta}_{m-1}]\} \tag{22}$$

The approximate solutions of the original eq. (1) can be expressed:

$$\xi = \sum_{r=0}^{\infty}\xi_r, \;\; \eta = \sum_{r=0}^{\infty}\eta_r \tag{23}$$

Numerical simulation for fractional model of predator-prey system

In order to show and effectiveness and correctness of the HATT for analyzing the fractional model of non-linear PDE, we apply it to the subsequent several initial conditions problems.

Case 1. Firstly, we take the predator-prey system of fractional order along with the constant initial condition:

$$\xi(x,y,0) = \xi_0, \;\; \eta(x,y,0) = \eta_0 \tag{24}$$

By appealing to the HATT to solve the eqs. (21) and (22) with initial condition (24) we get the following results:

$$\xi(x,y,0) = \xi_0, \;\; \eta(x,y,0) = \eta_0$$

$$\xi_1 = -\frac{\hbar(\lambda\xi_0 - \sigma\xi_0\eta_0)t^{\rho}}{\Gamma(\rho+1)}, \;\; \eta_1 = -\frac{\hbar(\sigma\xi_0\eta_0 - \mu\eta_0)t^{\beta}}{\Gamma(\beta+1)} \tag{25}$$

$$\vdots$$

In the same way the remaining terms can be obtained. The HATT solutions of the original eq. (1) with the initial condition (24) obtained in series form:

$$\xi = \xi_0 + \xi_1 + \xi_2 + \cdots, \qquad \eta = \eta_0 + \eta_1 + \eta_2 + \cdots \tag{26}$$

In figs. 1-4, the effect of time and order of fractional derivative is shown on prey and predator populations. The numerical results found by the HATT are very near to the anomalous biological diffusion nature seen in the real world, with high accuracy at the third-term approximations.

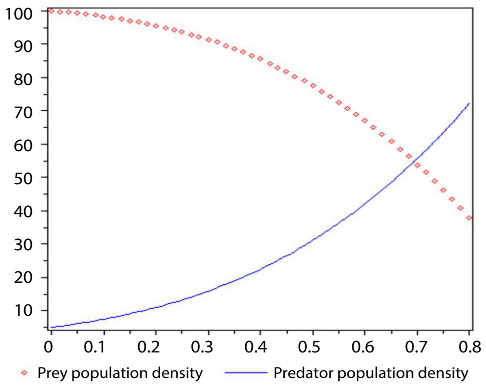

Figure 1. Time evaluation of prey ξ and predator η population density for *Case 1* when $\xi_0 = 100$, $\eta_0 = 5$, $\lambda = 0.08$, $\sigma = 0.04$, $\mu = 0.02$, and $\hbar = -1$, with $\rho = \beta = 1$

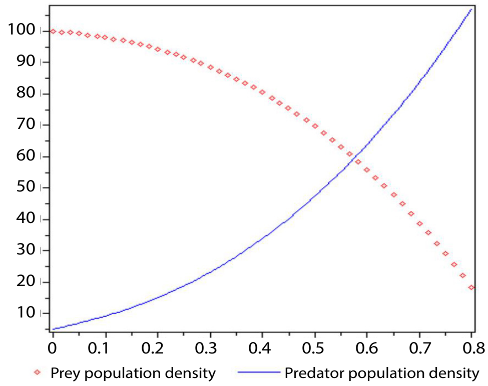

Figure 2. Time evaluation of prey ξ and predator η population density for *Case 1* when $\xi_0 = 100$, $\eta_0 = 5$, $\lambda = 0.08$, $\sigma = 0.04$, $\mu = 0.02$, $\hbar = -1$, $\rho = 1$, and $\beta = 0.85$

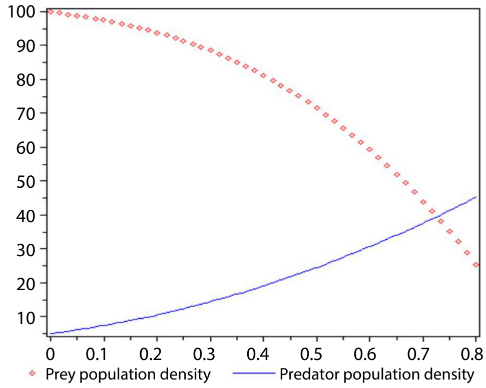

Figure 3. Time evaluation of prey ξ and predator η population density for *Case 1* when $\xi_0 = 100$, $\eta_0 = 5$, $\lambda = 0.08$, $\sigma = 0.04$, $\mu = 0.02$, $\hbar = -1$, $\rho = 0.85$, and $\beta = 1$

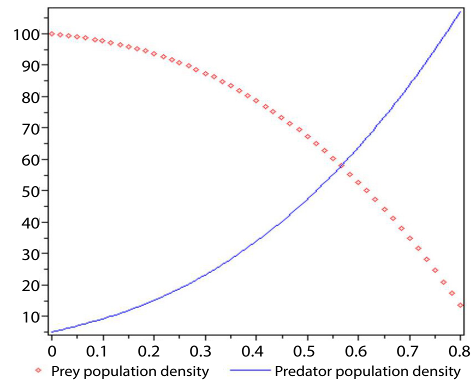

Figure 4. Time evaluation of prey ξ and predator η population density for *Case 1* when $\xi_0 = 100$, $\eta_0 = 5$, $\lambda = 0.08$, $\sigma = 0.04$, $\mu = 0.02$, $\hbar = -1$, $\rho = 0.95$, and $\beta = 0.60$

Case 2. Secondly, we assume the predator-prey equation of fractional order having the initial conditions:

$$\xi(x,y,0) = \mathrm{e}^{x+y},\ \ \eta(x,y,0) = \mathrm{e}^{x+y} \tag{27}$$

By using the HATT to solve the eqs. (21) and (22) with initial conditions (27), we get:

$$\xi_0 = \mathrm{e}^{x+y},\ \ \eta_0 = \mathrm{e}^{x+y}$$

$$\xi_1 = -\frac{\hbar \mathrm{e}^{x+y}(2+\lambda-\sigma \mathrm{e}^{x+y})t^{\rho}}{\Gamma(\rho+1)},\ \ \eta_1 = -\frac{\hbar \mathrm{e}^{x+y}(2-\mu+\sigma \mathrm{e}^{x+y})t^{\beta}}{\Gamma(\beta+1)} \tag{28}$$

$$\vdots$$

In this manner the remaining terms can be achieved.

The HATT solutions of the eq. (1) having the initial condition (27) obtained in series form:

$$\xi = \xi_0 + \xi_1 + \xi_2 + \cdots, \qquad \eta = \eta_0 + \eta_1 + \eta_2 + \cdots \tag{29}$$

In figs. 5-8, the effect of order of fractional derivative is shown on prey and predator populations with respect to time and space variables. It is clear to see the figs. 5-8 that the fractional predator-prey model exist and continuous with space and time variables.

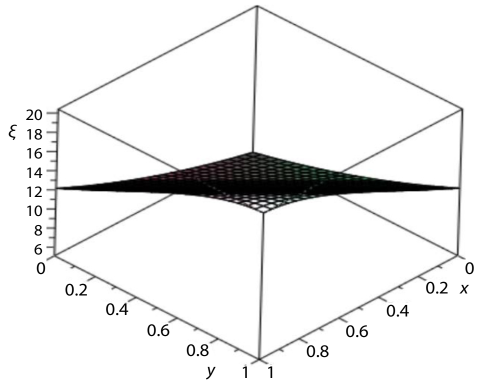

Figure 5. The shape presents the prey $\xi(x, y, t)$ population density with appropriate parameter for *Case 2* when $\lambda = 0.8$, $\sigma = 0.04$, $\mu = 0.3$, $\hbar = -1$, and $t = 0.65$, with $\rho = \beta = 1$

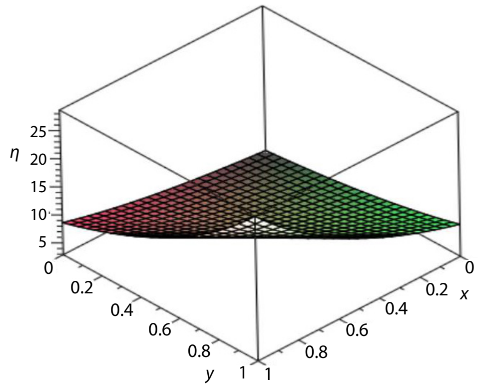

Figure 6. The shape presents predator $\eta(x, y, t)$ population density with appropriate parameter for *Case 2* when $\lambda = 0.8$, $\sigma = 0.04$, $\mu = 0.3$, $\hbar = -1$, and $t = 0.65$, with $\rho = \beta = 1$

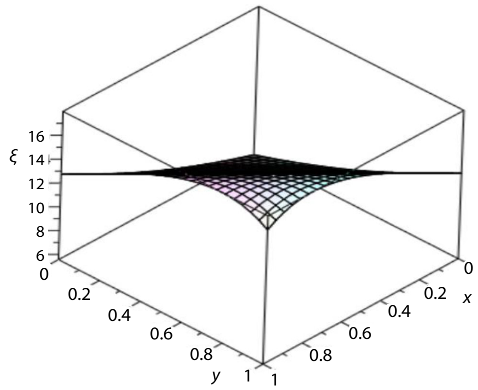

Figure 7. The shape presents the prey $\xi(x, y, t)$ population density with appropriate parameter for *Case 2* when $\lambda = 0.8$, $\sigma = 0.04$, $\mu = 0.3$, $\hbar = -1$, and $t = 0.65$, with $\rho = 0.95$ and $\beta = 0.60$

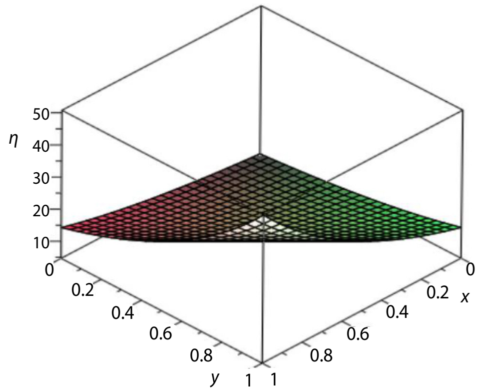

Figure 8. The shape presents predator $\eta(x, y, t)$ population density with appropriate parameter for *Case 2* when $\lambda = 0.8$, $\sigma = 0.04$, $\mu = 0.3$, $\hbar = -1$, and $t = 0.65$, with $\rho = 0.95$ and $\beta = 0.60$

Case 3. Lastly, we take the predator-prey equation of fractional order having the initial conditions:

$$\xi(x, y, 0) = \sqrt{(xy)}, \quad \eta(x, y, 0) = \mathrm{e}^{x+y} \tag{30}$$

By using the HATT to solve the eqs. (21) and (22) with initial condition (30), we get the following results:

$$\xi_0 = \sqrt{(xy)}, \quad \eta_0 = \mathrm{e}^{x+y}$$

$$\xi_1 = \frac{\hbar(x^2 + y^2 - 4\lambda x^2 y^2 + 4\sigma x^2 y^2 \mathrm{e}^{x+y})t^\rho}{4(xy)^{3/2}\Gamma(\rho+1)}, \quad \eta_1 = -\frac{\hbar \mathrm{e}^{x+y}(2 - \mu + \sigma\sqrt{xy})t^\beta}{\Gamma(\beta+1)} \tag{31}$$

$$\vdots$$

and so on, in this manner the remaining iterations can be achieved.

The HATT solutions of the eq. (1) having the initial condition (30) obtained in series form is presented:

$$\xi = \xi_0 + \xi_1 + \xi_2 + \cdots, \qquad \eta = \eta_0 + \eta_1 + \eta_2 + \cdots \tag{32}$$

In figs. 9-12, the effect of order of fractional derivative is shown on prey and predator populations with respect to time and space variables.

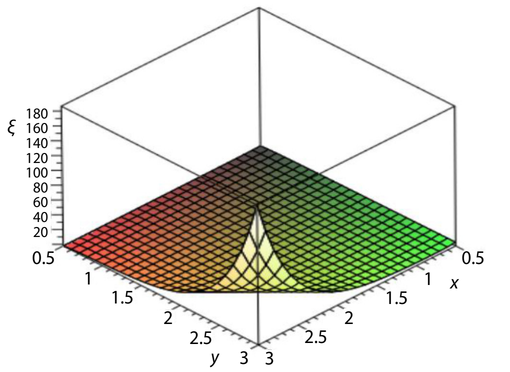

Figure 9. The shape presents the prey $\xi(x, y, t)$ population density with appropriate parameter for *Case 3* when $\lambda = 0.02$, $\sigma = 0.06$, $\mu = 0.3$, $\hbar = -1$, and $t = 0.5$, with $\rho = \beta = 1$

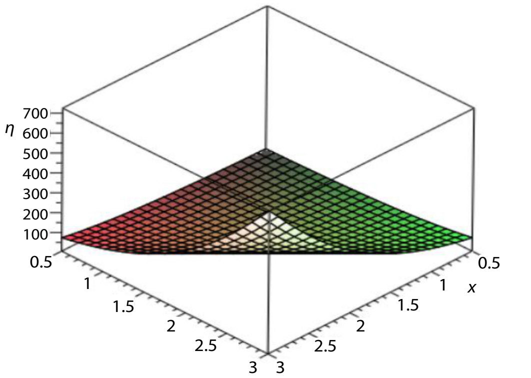

Figure 10. The shape presents predator $\eta(x, y, t)$ population density with appropriate parameter for *Case 3* when $\lambda = 0.02$, $\sigma = 0.06$, $\mu = 0.3$, $\hbar = -1$, and $t = 0.5$, with $\rho = \beta = 1$

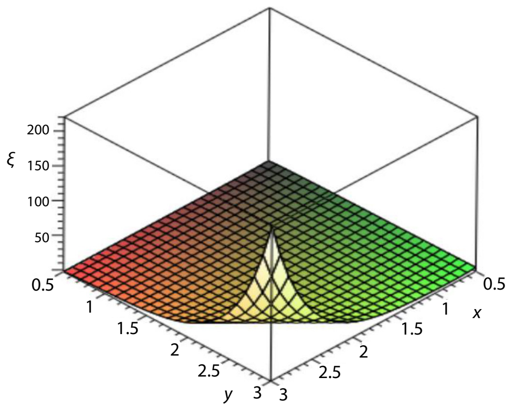

Figure 11. The shape presents the prey $\xi(x, y, t)$ population density with appropriate parameter for *Case 3* when $\lambda = 0.02$, $\sigma = 0.06$, $\mu = 0.3$, $\hbar = -1$, and $t = 0.5$, with $\rho = 0.95$ and $\beta = 0.75$

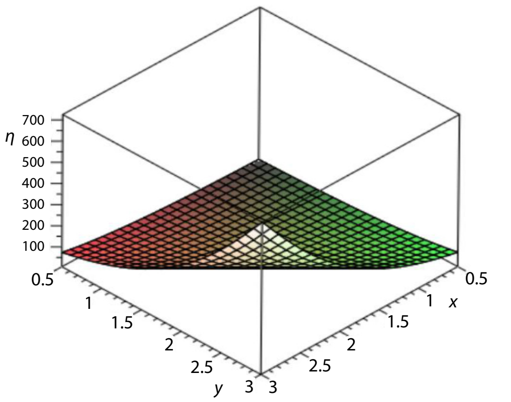

Figure 12. The shape presents predator $\eta(x, y, t)$ population density with appropriate parameter for *Case 3* when $\lambda = 0.02$, $\sigma = 0.06$, $\mu = 0.3$, $\hbar = -1$, and $t = 0.5$, with $\rho = 0.95$ and $\beta = 0.75$

Conclusion

In this study, the HATT has been efficiently put up to find the approximate solution which converges to the exact solution of time-fractional non-linear predator-prey population model subject to initial conditions. It is a significant outcome of these fractional ordered differential systems is that they gives higher degree of freedom and flexibility in the mathematical modelling and graphs show that the solution of these fractional systems is not only depends on time, but also depends on the order of fractional derivatives. The outcomes indicate that HATT a very strong and easy computational approach in obtaining analytical solution for various kinds of non-linear fractional differential equations.

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