PACKING CHROMATIC NUMBER OF TRANSFORMATION GRAPHS

by

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Graph coloring is an assignment of labels called colors to elements of a graph. The packing coloring was introduced by Goddard et al. [1] in 2008 which is a kind of coloring of a graph. This problem is NP-complete for general graphs. In this paper, we consider some transformation graphs and generalized their packing chromatic numbers.

Key words: *packing chromatic number, coloring of graphs, paths and cycles*

Introduction

Let G be a connected graph and $k \geq 1 \in Z$. A packing k-coloring of a graph G is a mapping $\pi : V(G) \rightarrow \{1, 2,...,k\}$ such that any two vertices of color i are at least $i + 1$. The packing chromatic number $\chi_p(G)$ of G is the smallest integer k for which G has packing k-coloring [2].

The packing coloring was introduced by Goddard *et al.* [1] as broadcast coloring. The signals of two stations that are using the same broadcast frequency will interfere unless they are stated sufficiently far apart. The distance in which the signals will propagate is directly related to the power of those signals. Bresar *et al.* [3] mentioned that this concept could have several additional applications.

The distance $d(u, v)$ between two vertices u and v in G is the minimum length of a path joining them if any vertex; otherwise $d(u, v) = \infty$ [4]. Let G be a connected graph and v be a vertex of G. The eccentricity $e(v)$ of v is the distance to a vertex farthest from v [4]. The diameter, $diam(G)$ of a connected graph G is the maximum distance between any two vertices of G. We denote the set $\{u \in V(G)|d(u, v) \leq k\}$ by $N_k[x]$. A set U of vertices in a graph G is independent if no two vertices in U are adjacent. The maximum number of vertices in an independent set of vertices of G is called the indepence number of G and is denoted by $\alpha(G)$ [5]. A k-vertex coloring of G is an assignment of k colors, 1, 2,...,k to the vertices of G. The coloring is proper if no two distinct adjacent vertices have the same color. Thus a proper k-vertex coloring of a graph G is a partition $(V_1, V_2,...,V_k)$ of V into k (possibly empty) independent sets. The chromatic number $\chi(G)$, of G is the minimum k for which G is k-colorable. If $\chi(G) = k$, G is said to be k-chromatic [6].

Preliminary

In this paper, the packing chromatic number of transformation graphs of path, cycle, wheel, complete and star graphs are given. Throughout this paper, we consider finite, simple, undirected graphs only. We refers to [5] for explained terminology and notations.

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Definition 1. Let G be a graph. We look for a partition of the V of G into disjoint sets $X_1, X_2,...,X_k$ called color classes. Each color class X_i should be an i-packing, a set of vertices where any distinct pair $u, v \in X_i$ satisfies $d(u, v) > i$ and this partition is called a k-coloring. The smallest integer k for which there exists a packing k-coloring of G is called the packing $\chi(G)$ [7].

Let $G = V[V(G), E(G)]$ be a graph, and x, y, z be three variables values + or – . The transformation graph G^{xyz} is the graph having $V(G) \cup E(G)$ as the vertex set, and for $\alpha, \beta \in V(G)$, $E(G)$ [8].

Proposition 2. If $diamG = 2$

$$\chi_p(G) \le \beta(G) + 1 \text{ [2]} \tag{1}$$

Theorem 3. Let G be a graph of order n. Then

$$\alpha(G) + \beta(G) = n \text{ [1]} \tag{2}$$

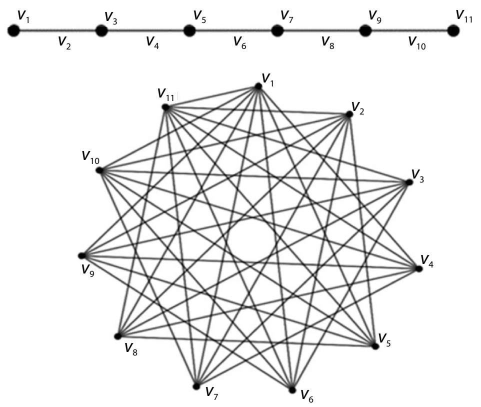

Figure 1. Labelings of path with 6 vertices P_6 and P_6^{---}

Main results

In this section we mention about the packing chromatic number of transformation graphs G^{---}, G^{+--}, G^{-+-}, and G^{++-} where G is either path, cycle, wheel, complete or star graph.

Theorem 4. Let P_n^{---} be the graph of order $2n - 1$. The packing coloring number of the P_n^{---}:

$$\chi_p\left(P_{\boldsymbol{n}}^{---}\right) = 2n - 3 \tag{3}$$

Proof. Let $v_1, v_2, v_3,...v_{2n-1} \in V(P_n^{---})$, $v_1, v_3, v_5,...v_{2n-1} \in V(P_n)$, and $v_2, v_4,...v_{2n-2} \in E(P_n)$. The $\alpha(P_n^{---}) = 3$ then $|X_1| = 3$. One can see an example of labeling in fig. 1.

In order to color whole graph we need two consider two cases.

Case 1. Let $v_1, v_{2n-1} \in V(P_n^{---})$ be end vertices of P_n and $v_2, v_{2n-2} \in V(P_n^{---})$ be pendant edges of P_n. Then

$$d(v_1, v_2) = d(v_1, v_3) = 2,\ d(v_1, v_4) = d(v_1, v_5) = \ldots = d(v_1, v_{2n-1}) = 1 \tag{4}$$

Similiarly:

$$d(v_2, v_3) = d(v_2, v_4) = 2,\ d(v_2, v_5) = d(v_2, v_6) = \ldots = d(v_2, v_{2n-1}) = 1 \tag{5}$$

$$\begin{gathered} d(v_{2n-2}, v_1) = d(v_{2n-2}, v_2) = \ldots = d(v_{2n-2}, v_{2n-5}) = 1 \\ d(v_{2n-2}, v_{2n-4}) = d(v_{2n-2}, v_{2n-3}) = d(v_{2n-2}, v_{2n-1}) = 2 \end{gathered} \tag{6}$$

Case 2. Let $v_3, v_4,...v_{2n-4}, v_{2n-3} \in V(P_n^{---})$.

$$d(v_3, v_4) = d(v_3, v_5) = 2,\ d(v_3, v_6) = d(v_3, v_7) = \ldots = d(v_3, v_{2n-1}) = 1 \tag{7}$$

The distance $v_3, v_4,...v_{2n-3} \in V(P_n^{---})$ of vertices can see similiarly.

Now consider $v_4, v_6,...v_{2n-4} \in V(P_n^{---})$.

$$d(v_4, v_5) = d(v_4, v_6) = 2, d(v_4, v_7) = d(v_4, v_8) = \ldots = d(v_4, v_{2n-1}) = 1 \tag{8}$$

One can see distances of $v_6, v_7, ... v_{2n-4}$.

Therefore, $diamP_n^{---} = 2$. Then other color classes contain one element:

$$\chi_p\left(P_n^{---}\right) \leq 2n-4+1 \tag{9}$$

$$\chi_p\left(P_n^{---}\right) \leq 2n-3 \tag{10}$$

As a result of *Proposition 2* and *Theorem 3* $\chi_p(P_n^{---}) \leq 2n - 3$:

Let 1 $\chi_p(P_n^{---}) = 2n - 4$. Since $\alpha(P_n^{---}) = 3$, then three vertices of P_n^{---} have color 1.

Also $diam(P_n^{---}) = 2$ thus rest of the vertices should have different colors. So, we need at least vertices to packing color the $P_n^{---}2n - 4 + 1$.

Thus:

$$\chi_p\left(P_n^{---}\right) = 2n-3$$

Figure 2 shows an example of packing coloring of P_6^{---}.

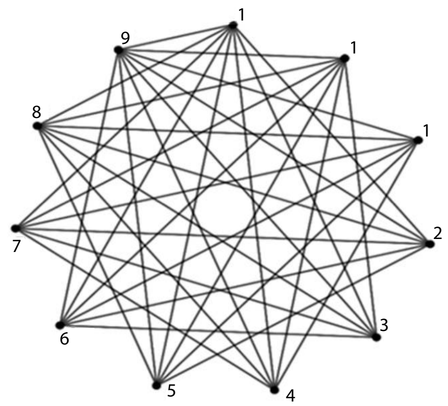

Figure 2. An example of packing coloring of P_6^{---}

Theorem 5. Let C_n^{---} be the graph of order $2n$, where $n > 3$. The packing coloring number of the C_n^{---}:

$$\chi_p\left(C_n^{---}\right) = 2n-2 \tag{11}$$

Proof. Let $v_1, v_2, ..., v_{2n}$ be the vertices of C_n^{---}, where $v_1, v_2, ... v_n$ are vertices, $v_{n+1}, v_{n+2}, ..., v_{2n}$ are edges of C_n, $n > 3$. There must be a vertex to which the color 1 is assigned. Let v_1 be a vertex of X_1. The other vertices v_i, which have $d(v_1, v_i)$. Since $\alpha(C_n^{---}) = 3$, then the three vertices (for instance v_1, v_{n+1}, v_2) will also be colored with color 1. Since $diam(C_n^{---}) = 2$, $d(v_3, v_i) > 2$ is not possible, the other vertices are colored with different colors:

$$\chi_p\left(C_n^{---}\right) \leq 2n-2 \tag{12}$$

Assume that $\chi_p(C_n^{---}) = 2n - 3$. By using same process as the proof of *Theorem 4* we obtain $\chi_p(C_n^{---}) \geq 2n - 2$:

$$\chi_p\left(C_n^{---}\right) = 2n-2$$

Figure 3 shows an example of packing coloring of C_5^{---}.

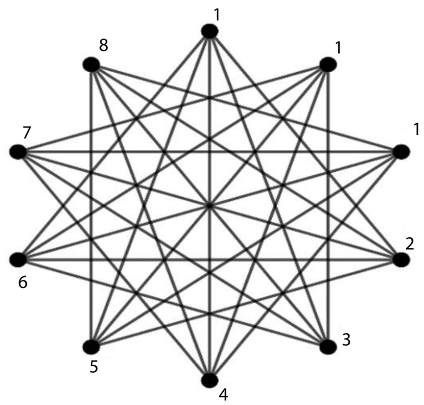

Figure 3. The $\chi_p(C_5^{---}) = 8$

*Theorem 6***.** Let W_n^{---} be the graph of order $3n-2$, where $n \geq 5$:

$$\chi_p\left(W_n^{---}\right) = 3n-4 \tag{13}$$

Proof. Let $v_1, v_2, ... v_{3n-2}$ be the vertices of W_n^{---}, where $v_1, v_2, ... v_n$ are vertices, $v_{n+1}, v_{n+2}, ... v_{3n-2}$ are edges of W_n, $n \geq 5$. The rest of the proof will use the same process as the previous ones.

An example of packing coloring of W_5^{---} is shown in fig. 4.

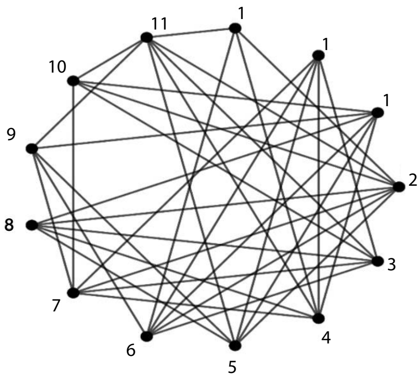

Figure 4 .The $\chi_p(W_5^{---}) = 11$

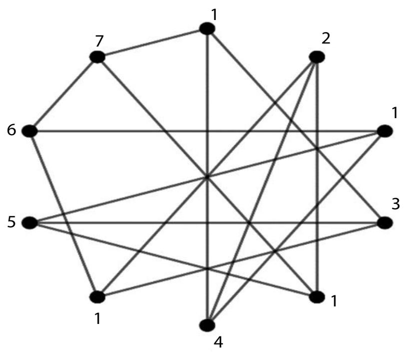

Figure 5. The $\chi_p(K_4^{---}) = 7$

Theorem 7. Let K_n^{---} be the graph of order $(n^2 + n)/2$, where $n \geq 4$:

$$\chi_p\left(K_n^{---}\right) = \frac{n^2 - n + 2}{2} \tag{14}$$

Proof. Let $v_1, v_2,...,v_{(n^2+n)/2} \in V(K_n^{---})$, $v_1, v_3,... v_{2n-1} \in V(K)$, $v_2, v_4,...,v_{2n}, v_{2n+1}, v_{(n^2+n)/2} \in E(K_n)$. Since $\alpha(K_n^{---}) = n$, then $|X_1| = n$. For the remaining vertices we will proceed same way as the previous ones:

$$\left(\frac{n^2+n}{2} - \frac{2n}{2} = \frac{n^2-n}{2}\right),\ \frac{n^2-n}{2} + 1 = \frac{n^2-n+2}{2}$$

So

$$\chi_p(K_n^{---}) = \frac{n^2 - n + 2}{2}$$

An example pf packing coloring of K_4^{---} is shown in fig. 5.

Theorem 8. Let W_n^{+--} be a graph of order $3n - 2$, where $n \geq 5$:

$$\chi_p\left(W_n^{+--}\right) = 2n - 1 \tag{15}$$

Proof. Let $v_1, v_2,...,v_{3n-2} \in V(W_n^{+--})$, $v_1, v_3,...v_{2n-1} \in V(W_n)$, $v_2, v_4,...,v_{2n}, v_{2n+1},...,v_{3n-2} \in E(W_n)$. Since $\alpha(W_n^{+--}) = n$, then $|X_1| = n$. The rest of the vertices can be colored in the same way as the previous ones:

Theorem 9. Let $K_{1,n-1}^{+--}$ be a graph of order $2n - 1$, where $n \geq 4$:

$$\chi_p\left(K_{1,n-1}^{+--}\right) = n \tag{16}$$

Proof. Let $v_1, v_2,...,v_{2n-1} \in V(K_{1,n-1}^{+--})$, $v_1, v_2,...v_n \in V(K_{1,n-1})$, $v_{n+1}, v_{n+2},...,v_{2n-1} \in E(K_{1,n-1})$. Since $\alpha(K_{1,n-1}^{+--}) = n$, then $|X_1| = n$. Thus

$$\chi_p\left(K_{1,n-1}^{+--}\right) = n$$

Theorem 10. Let P_n^{++-} be a graph of order $2n - 1$, where $n \geq 3$:

$$\chi_p(P_n^{++-}) = 2n - \left\lceil \frac{n}{2} \right\rceil \tag{17}$$

Proof. Let $v_1, v_2, v_3,...,v_{2n-1} \in V(P_n^{++-})$, $v_1, v_3, v_5,...v_{2n-1} \in V(P_n)$, and $v_2, v_4,...,v_{2n-2}, \in E(P_n)$. We should not consider n is either odd or even to find independence number of P_n^{++-}. Therefore, $\alpha(P_n^{++-}) = \lceil n/2 \rceil$. Then $|X_1| = \lceil n/2 \rceil$ and

$$\chi_p(P_n^{++-}) = 2n - \left\lceil \frac{n}{2} \right\rceil$$

Theorem 11. Let K_n^{++-} be a graph of order $(n^2 + n)/2$ and $n \geq 3$:

$$\chi_p\left(K_n^{++-}\right) = \frac{n^2 + n - 2}{2} \tag{18}$$

Theorem 12. Let $K_{1,n-1}^{++-}$ be a graph of order $2n - 1$, where $n \geq 4$:

$$\chi_p\left(K_{1,n-1}^{++-}\right) = n+1 \tag{19}$$

Proofs of *Theorem 11* and *Theorem 12* can be easily seen by using same process of previous ones.

Results about packing coloring numbers of some transformation graphs are summarized in tab. 1.

Table 1. Coloring number of some graphs

Graphs	$\chi_p(G)$
$\chi_p(P_n^{+--})$, where $n \geq \mathbf{3}$	$2n-3$
$\chi_p(C_n^{+--})$, where $n \geq \mathbf{3}$	$2n-2$
$\chi_p(K_n^{+--})$, where $n \geq \mathbf{3}$	$(n^2-n+2)/2$
$\chi_p(P_n^{-+-})$, where $n \geq 4$	$2n-3$
$\chi_p(C_n^{-+-})$, where $n \geq 3$	$2n-2$
$\chi_p(W_n^{-+-})$, where $n \geq 5$	$3n-4$
$\chi_p(K_n^{-+-})$, where $n \geq 4$	$(n^2-n+2)/2$
$\chi_p(C_n^{++-})$, where $n \geq 3$	$2n-1$
$\chi_p(W_n^{++-})$, where $n \geq 4$	$3n-3$

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