APPLICATION OF MATHEMATICAL METHODS FOR THE NON-LINEAR SEVENTH ORDER SAWADA-KOTERA-ITO DYNAMICAL WAVE EQUATION

by

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This article deal with finding travelling wave solutions for the 7th order Sawada-Kotera-Ito dynamical wave equation which describes the evolution of steeper waves of shorter wavelength than KdV equations using modified extended direct algebraic method. The new solutions derived have various physical structure, we also give graphic representation of the exact and stable solutions.

Key words: *higher KdV equations, mathematical method, soliton solutions*

Introduction

Finding exact solution of non-linear partial differential equations (NLPDE) perform an important part in the study of non-linear physical phenomena. In the last several years a lot of effective method used to find exact solution such as tanh method Malfliet [1], sech method Wazwaz [2], rational hyperbolic method Wazwaz [3] and exp-function method Abdou *et al.* [4] and so on. Understanding a lot of application in optical fibers, electromagnetic radiation, solid state physics to view some not all need to formulate the problem in mathematical model usually express as NLPDE. The solutions give a good explication of what happened and will happen later, hence we need analytical and numerical method to solve the model [5-14]. Higher order KdV equations in fluid dynamics was treated by a lot of researchers, Seadawy [15] get new exact solutions by using the variational method while Kabir *et al.* [16] used modified Kudryshov method to find exact solitary wave solution of higher order non-linear equations. Seadawy [17] apply the auxiliary equation of the direct algebraic method to obtain traveling wave solution for some different kinds of the generalized non-linear fifth-order of KdV equations. Ganji [18] obtain new exact solutions for seventh-order Sawada-Kotera-Ito, Lax and Kaup-Kupershmidt equations using Exp-function method. Feng [19] get more general solution of the 7th order Sawada-Kotera (sSK) using (G'/G) expansion method and show that it is effective and powerful tool for solving non-linear equations in mathematical physics. Shen *et al.* [20] have studied the description of shallow-water, hydromagnetic and ion-acoustic waves in fluids and plasmas,

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and its extensions, with the cubic non-linearity, higher-order linear dispersion and non-dynamics for their integrable properties through the Bell-polynomial procedure and Pomeau *et al.* [21] gave structural stability of the KdV solitons under a singular perturbation. Wazwaz [22] used the The Hirotas direct method and the tanhcoth method for multiple-soliton solutions of the Sawada-Kotera-Ito 7th order equation also Salas [23] use the Cole-Hopf transformation compute the exact solution sSKI. Jafari *et al.* [24] applied hes variational iteration method for solving sSK while El-Sayed and Kaya [25] implemented the Adomian decomposition method (ADM) for approximating the solution of the 7th order Sawada-Kotera (sSK) and a Lax's 7th order KdV (LsKdV) equations and obtained exact solitary-wave solutions and numerical solutions. Ganji *et al.* [26] used the HPM and VIM methods to find exact solution 7th order Sawada-Kotera dynamical wave equation (sSK). Zuhra *et al.* [27] applied optimal homotopy asymptotic method (OHAM) on generalize KdV (gKdV) equation with different coefficients to form 7th order Sawada Kotera-(sSK) equations. In this work we consider 7th order Sawada-Kotera-Ito dynamical wave equation (sSKI) [28] and use the modified extended direct algebraic method [29] to obtain new exact travelling wave solutions. The rest of the paper is organized in section, *Travelling wave solutions* we apply the method to sSK and listing the two families of the solutions according to auxiliary equations [30, 31] and plot the results of stable solutions. In section, *Results and discussion* are considered and finaly some conclusions are given.

Travelling wave solutions

Consider the 7th order Sawada-Kotera-Ito equation:

$$u_t + 252u^3u_x + 63u_x^3 + 378uu_xu_{xx} + 126u^2u_{xxx} + 63u_{xx}u_{xxx} + 42u_xu_{xxxx} + 21uu_{xxxxx} + u_{xxxxxxx} = 0 \tag{1}$$

Let the travelling wave solution:

$$u(x,t) = u(\xi) = \sum_{i=0}^{m} p_i\phi^i(\xi) \tag{2}$$

be the travelling wave solution of eq. (1) with $\phi(\xi)$ satisfies the following auxiliary equation:

$$\phi' = \sqrt{h_2\phi^2 + h_3\phi^3 + h_4\phi^4} \tag{3}$$

$\xi = kx - \omega t$, where, h_i, p_i $(i = 0, ...m)$ are arbitrary constants and k and ω are wave length and frequency, respectively. The positive constant m is determined later. Using eqs. (2) and (3), eq. (1) can be written:

$$\begin{aligned} &-\omega u' + 252ku^3u' + 63k^3u''' + 378uu'u'' + 126k^3u^2u''' + \\ &+63k^5u''u''' + 42k^5u'u^{(4)} + 21k^5uu^{(5)} + k^7u^{(7)} = 0 \end{aligned} \tag{4}$$

Families I

Balancing the non-linear term u^2u''' and the highest order derivative $u^{(7)}$ in eq. (4) using homogeneous balance method [32] gives $m = 2$, so the solution will be:

$$u(x,t) = u(\xi) = p_0 + p_1\phi(\xi) + p_2\phi^2(\xi) \tag{5}$$

Substituting from eq. (5) into eq. (4) and collecting coefficients then setting them equal to zero, we obtain a system of algebraic equations. Solving this system we can get the value of the parameters k, ω, p_0, p_1, p_2. We have three possibilities for the coefficients of the solution of the seventh-order Sawada-Kotera-Ito equation.

– Case I

$$p_1 = -k^2 h_3, \quad p_2 = -2k^2 h_4, \quad h_2 = \frac{h_3^2}{4h_4}$$
$$\omega = \frac{k[k^6 h_3^6 + 84k^4 h_3^4 h_4 p_0 + 2016k^2 h_3^2 h_4^2 p_0^2 + 16128 h_4^3 p_0^3]}{64h_4^3} \tag{6}$$

– Case II

$$p_0 = -\frac{1}{3}k^2 h_2, p_1 = -k^2 h_3, p_2 = -2k^2 h_4, \omega = -\frac{4}{3}k^7 h_2^3 \tag{7}$$

– Case III

$$p_0 = -\frac{k^2 h_3^2}{12h_4}, \; p_1 = -2k^2 h_3, \; p_2 = -4k^2 h_4, \; \omega = -\frac{k^7 h_3^6}{48h_4^3}, \; h_2 = \frac{h_3^2}{4h_4} \tag{8}$$

Solving eq. (3) we can find that ϕ takes one of the following forms:

$$\phi_1 = -\frac{h_2}{h_3}\left[1 + s\tanh\left(\frac{\sqrt{h_2}}{2}\xi + \xi_0\right)\right], \; h_2 > 0, \; s = \pm 1, \; h_2 = \frac{h_3^2}{4h_4} \tag{9}$$

$$\phi_2 = -\sqrt{\frac{h_2}{4h_4}}\left(1 + s\frac{\sinh(\sqrt{h_2}\xi + \xi_0)}{\cosh(\sqrt{h_2}\xi + \xi_0) + \rho}\right), \; h_2 > 0, \; h_2 = \frac{h_3^2}{4h_4}, \; s = \pm 1, \rho = \pm 1 \tag{10}$$

$$\phi_3 = -\frac{h_2}{h_3}\left[1 + s\frac{\cosh\left(\sqrt{h_2}\xi + \xi_0\right) + \rho\sqrt{\sigma^2 + 1}}{\sinh(\sqrt{h_2}\xi + \xi_0) + \sigma}\right], \; h_2 > 0, \; h_2 = \frac{h_3^2}{4h_4},$$
$$s = \pm 1, \; \rho = \pm 1, \; \sigma, \xi_0 \text{ are arbitrary} \tag{11}$$

$$\phi_4 = \frac{2h_2 \operatorname{sech}\left(\sqrt{h_2}\xi\right)}{\sqrt{h_3^2 - 4h_2 h_4} - h_4 \operatorname{sech}\left(\sqrt{h_2}\xi\right)}, \; h_2 > 0, \; h_3^2 - 4h_2 h_4 > 0, \; h_3 = h_4 \tag{12}$$

$$\phi_5 = \frac{2h_2 \operatorname{csch}\left(\sqrt{h_2}\xi\right)}{\sqrt{-h_3^2 + 4h_2 h_4} - h_4 \operatorname{csch}\left(\sqrt{h_2}\xi\right)}, h_2 > 0, \; h_3^2 - 4h_2 h_4 < 0, \; h_3 = h_4 \tag{13}$$

Solutions

Now substituting from eqs. (6)-(8) into eq. (5) considering the different form of ϕ_i, $i = 1, 2, 3, 4, 5$ from eqs. (9)-(13) we get the solution of the 7th order Sawada-Kotera-Ito equation:

$$u(x,t) = p_0 + p_1\phi(\xi) + p_2\phi^2(\xi) \tag{14}$$

– Case I

$$u_1 = \frac{h_3^2 k^2}{8h_4}\left(1 - s^2\tanh^2\left(\frac{1}{4}\sqrt{\frac{h_3^2}{h_4}}\left(kx - \frac{kt\left(h_3^6 k^6 + 84h_4 h_3^4 k^4 p_0 + 2016h_4^2 h_3^2 k^2 p_0^2 + 16128 h_4^3 p_0^3\right)}{64h_4^3}\right) + \xi_0\right)\right) + p_0$$

$$u_2=\frac{1}{4}\sqrt{\frac{h_3^2}{h_4^2}}h_3k^2\left(\frac{s\sinh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(kx-\frac{kt\left(h_3^6k^6+84h_4h_3^4k^4p_0+2016h_4^2h_3^2k^2p_0^2+16128h_4^3p_0^3\right)}{64h_4^3}\right)+\xi_0\right)}{\cosh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(kx-\frac{kt\left(h_3^6k^6+84h_4h_3^4k^4p_0+2016h_4^2h_3^2k^2p_0^2+16128h_4^3p_0^3\right)}{64h_4^3}\right)+\xi_0\right)+\rho}+1\right)-$$

$$-\frac{h_3^2k^2}{8h_4}\left(\frac{s\sinh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(kx-\frac{kt\left(h_3^6k^6+84h_4h_3^4k^4p_0+2016h_4^2h_3^2k^2p_0^2+16128h_4^3p_0^3\right)}{64h_4^3}\right)+\xi_0\right)}{\cosh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(kx-\frac{kt\left(h_3^6k^6+84h_4h_3^4k^4p_0+2016h_4^2h_3^2k^2p_0^2+16128h_4^3p_0^3\right)}{64h_4^3}\right)+\xi_0\right)+\rho}+1\right)^2+p_0$$

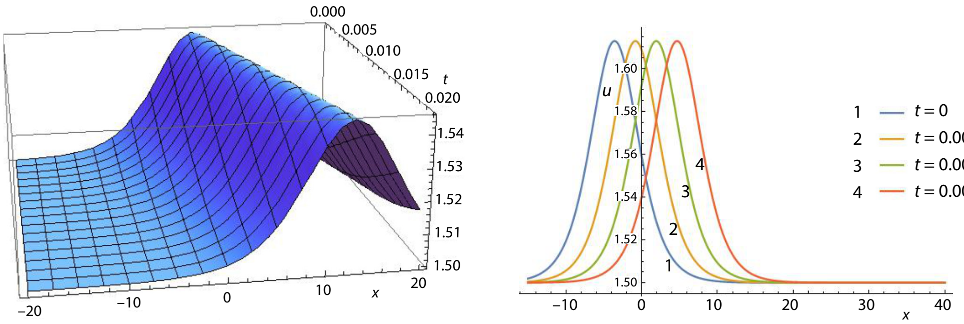

Figure 1. Bright soliton solutions of u_1 and u_2

$$u_3=\frac{h_3^2k^2}{4h_4}\left(\frac{s\left(\cosh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(kx-\frac{kt\left(h_3^6k^6+84h_3^4k^4p_0+2016h_4^2h_3^2k^2p_0^2+1612h_4^3p_0^3\right)}{64h_4^3}\right)+\xi_0\right)+\rho\sqrt{\sigma^2+1}\right)}{\sinh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(kx-\frac{kt\left(h_3^6k^6+84h_3^4k^4p_0+2016h_4^2h_3^2k^2p_0^2+1612h_4^3p_0^3\right)}{64h_4^3}\right)+\xi_0\right)+\sigma}+1\right)-$$

$$-\frac{h_3^2k^2}{8h_4}\left(\frac{s\left(\cosh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(kx-\frac{kt\left(h_3^6k^6+84h_3^4k^4p_0+2016h_4^2h_3^2k^2p_0^2+1612h_4^3p_0^3\right)}{64h_4^3}\right)+\xi_0\right)+\rho\sqrt{\sigma^2+1}\right)}{\sinh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(kx-\frac{kt\left(h_3^6k^6+84h_3^4k^4p_0+2016h_4^2h_3^2k^2p_0^2+1612h_4^3p_0^3\right)}{64h_4^3}\right)+\xi_0\right)+\sigma}+1\right)^2+p_0$$

– Case II

$$u_4=-\frac{h_3^2k^2}{24h_4}\left(3s^2\tanh^2\left(\frac{1}{4}\sqrt{\frac{h_3^2}{h_4}}\left(kx-\frac{h_3^6k^7t}{48h_4^3}\right)+\xi_0\right)-1\right)$$

$$u_5=\frac{h_3k^2}{24h_4}\left(3h_3\left(\frac{s\sinh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(kx-\frac{h_3^6k^7t}{48h_4^3}\right)+\xi_0\right)}{\cosh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(kx-\frac{h_3^6k^7t}{48h_4^3}\right)+\xi_0\right)+\rho}+1\right)^2+\right.$$

$$\left.+6h_3\left(\frac{s\sinh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(kx-\frac{h_3^6k^7t}{48h_4^3}\right)+\xi_0\right)}{\cosh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(kx-\frac{h_3^6k^7t}{48h_4^3}\right)+\xi_0\right)+\rho}+1\right)-2h_3\right)$$

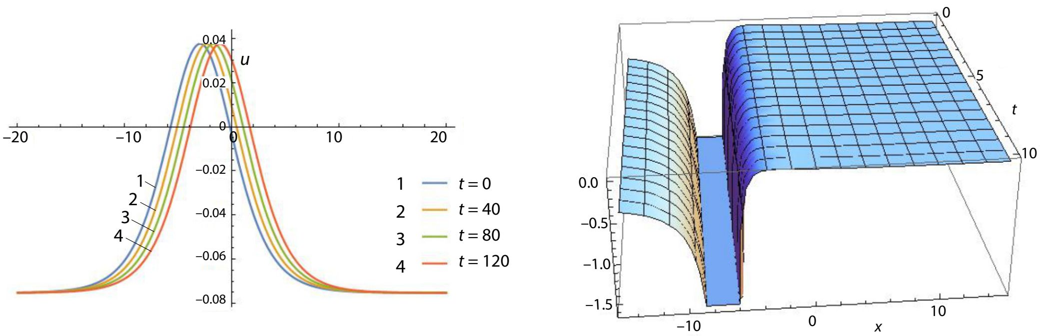

Figure 2. Bright soliton solution of u_4 and u_5

$$u_6=\frac{h_3^2k^2}{24h_4}\left(-3\left(\frac{s\left(\cosh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(kx-\frac{h_3^6k^7t}{48h_4^3}\right)+\xi_0\right)+\rho\sqrt{\sigma^2+1}\right)}{\sinh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(kx-\frac{h_3^6k^7t}{48h_4^3}\right)+\xi_0\right)+\sigma}+1\right)^2+\right.$$

$$\left.+6\left(\frac{s\left(\cosh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(kx-\frac{h_3^6k^7t}{48h_4^3}\right)+\xi_0\right)+\rho\sqrt{\sigma^2+1}\right)}{\sinh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(kx-\frac{h_3^6k^7t}{48h_4^3}\right)+\xi_0\right)+\sigma}+1\right)-2\right)$$

$$u_7=-\frac{h_2h_4k^2}{3\left(\sqrt{h_4\left(h_4-4h_2\right)}-h_4\operatorname{sech}\left(\sqrt{h_2}\left(kx-\frac{4}{3}h_2^3k^7t\right)\right)\right)^2}.$$

$$\cdot\left(4h_2\left(6\,\mathrm{sech}^2\left(\sqrt{h_2}\left(kx-\frac{4}{3}h_2^3k^7t\right)\right)-1\right)+\right.$$

$$\left.+h_4\left(1-5\,\mathrm{sech}^2\left(\sqrt{h_2}\left(kx-\frac{4}{3}h_2^3k^7t\right)\right)\right)+4\sqrt{h_4\left(h_4-4h_2\right)}\,\mathrm{sech}\left(\sqrt{h_2}\left(kx-\frac{4}{3}h_2^3k^7t\right)\right)\right)$$

$$u_8=\frac{h_2k^2}{3\left(\sqrt{\left(4h_2-h_4\right)h_4}-h_4\mathrm{csch}\left(\sqrt{h_2}\left(kx-\frac{4}{3}h_2^3k^7t\right)\right)\right)^2}\left(-6h_4\mathrm{sech}\left(\sqrt{h_2}\left(kx-\frac{4}{3}h_2^3k^7t\right)\right)\cdot\right.$$

$$\left.\cdot\left(-h_4\mathrm{csch}\left(\sqrt{h_2}\left(kx-\frac{4}{3}h_2^3k^7t\right)\right)+4h_2\mathrm{sech}\left(\sqrt{h_2}\left(kx-\frac{4}{3}h_2^3k^7t\right)\right)+\sqrt{\left(4h_2-h_4\right)h_4}\right)\right)-\frac{h_2k^2}{3}$$

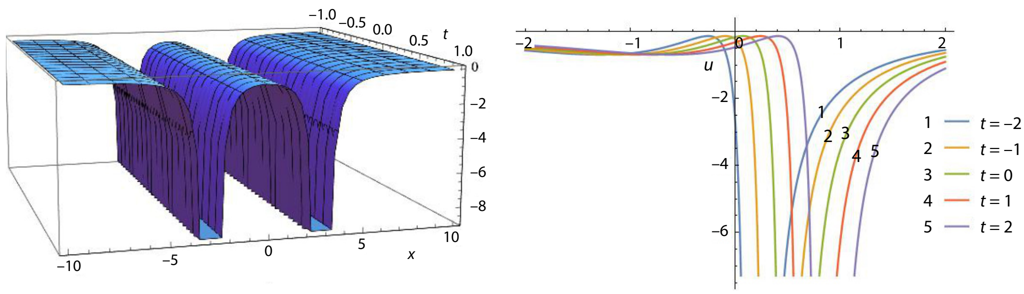

Figure 3. Double dark solitons wave solution of u_7 and dark soliton solution of u_8

– Case III

$$u_9=-\frac{h_3^2k^2}{12h_4}\left(3s^2\tanh^2\left(\frac{1}{4}\sqrt{\frac{h_3^2}{h_4}}\left(\frac{h_3^6k^7t}{48h_4^3}+kx\right)+\xi_0\right)-2\right)$$

$$u_{10}=\frac{h_3k^2}{12h_4}\left(-3h_3\left(\frac{s\sinh\left(\frac{1}{2}\sqrt{h_3^2}\left(\frac{h_3^6k^7t}{48h_4^3}+kx\right)+\xi_0\right)}{\cosh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(\frac{h_3^6k^7t}{48h_4^3}+kx\right)+\xi_0\right)+\rho}+1\right)^2+\right.$$

$$\left.+\frac{6h_3}{h_4}\left(\frac{s\sinh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(\frac{h_3^6k^7t}{48h_4^3}+kx\right)+\xi_0\right)}{\cosh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(\frac{h_3^6k^7t}{48h_4^3}+kx\right)+\xi_0\right)+\rho}+1\right)-\frac{h_3}{h_4}\right)$$

$$u_{11}=\frac{h_3^2k^2}{12h_4}\left(-3\left(\frac{s\left(\cosh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(\frac{h_3^6k^7t}{48h_4^3}+kx\right)+\xi_0\right)+\rho\sqrt{\sigma^2+1}\right)}{\sinh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(\frac{h_3^6k^7t}{48h_4^3}+kx\right)+\xi_0\right)+\sigma}+1\right)^2+\right.$$

$$\left.+6\left(\frac{s\left(\cosh\cosh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(\frac{h_3^6k^7t}{48h_4^3}+kx\right)+\xi_0\right)+\rho\sqrt{\sigma^2+1}\right)}{\sinh\left(\frac{1}{2}\sqrt{\frac{h_3^2}{h_4}}\left(\frac{h_3^6k^7t}{48h_4^3}+kx\right)+\xi_0\right)+\sigma}+1\right)-1\right)$$

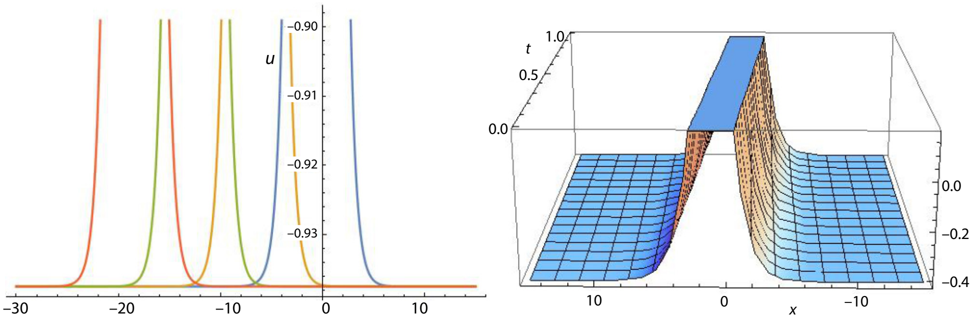

Figure 4. Periodic wave solution of u_9 and bright solitary wave solution of u_{11}

Families 2

Here $\phi(\xi)$ satisfies the auxiliary equation $\phi' = \sqrt{h_2\phi^2 + h_4\phi^4 + h_6\phi^6}$ and the solution can be expressed:

$$u(x,t)=u(\xi)=\sum_{i=0}^{n}q_i\phi^i(\xi) \tag{15}$$

Balancing the non-linear term $u^2 u'''$ and the highest order derivative $u^{(7)}$ in eq. (4) using homogeneous balance method gives $n = 4$, so the solution will be:

$$u(x,t)=u(\xi)=q_0+q_1\phi(\xi)+q_2\phi^2(\xi)+q_3\phi^3(\xi)+q_4\phi^4(\xi) \tag{16}$$

substituting from eq. (16) into eq. (4) and collecting coefficients then setting them equal to zero, we obtain a system of algebraic equations. Solving this system we can get the value of the parameters k, ω, q_0, q_1, q_2, q_3, and q_4 here we have only one case for the coefficients.

– Case I

$$q_0=-\frac{4}{3}k^2h_2,\quad q_1=q_3=0,\quad q_2=-4k^2h_4,\quad q_4=-8k^2h_6,\quad w=-\frac{256}{3}k^7h_2^3 \tag{17}$$

Now substituting from eq. (17) into eq. (16), we obtain the solutions of 7th order Sawada-Kotera-Ito equation:

$$u_{12}=\frac{4}{3}h_2k^2\left(-\frac{6h_2h_4^2h_6\operatorname{sech}^4\left(\sqrt{h_2}\left(\frac{256}{3}h_2^3k^7t+kx\right)\right)}{\left(h_4^2-h_2h_6\left(s\tanh\left(\sqrt{h_2}\left(\frac{256}{3}h_2^3k^7t+kx\right)\right)+1\right)^2\right)^2}+\right.$$

$$\left.+\frac{3h_4^2\operatorname{sech}^2\left(\sqrt{h_2}\left(\frac{256}{3}h_2^3k^7t+kx\right)\right)}{h_4^2-h_2h_6\left(s\tanh\left(\sqrt{h_2}\left(\frac{256}{3}h_2^3k^7t+kx\right)\right)+1\right)^2}-1\right)$$

$$u_{13}=\frac{4}{3}h_2k^2\left(\frac{6h_4}{h_4-\sqrt{h_4^2-4h_2h_6}\,s\cosh\left(2\sqrt{h_2}\left(\frac{256}{3}h_2^3k^7t+kx\right)\right)}-\right.$$

$$\left.-\frac{24h_2h_6}{\left(h_4-\sqrt{h_4^2-4h_2h_6}\,s\cosh\left(2\sqrt{h_2}\left(\frac{256}{3}h_2^3k^7t+kx\right)\right)\right)^2}-1\right)$$

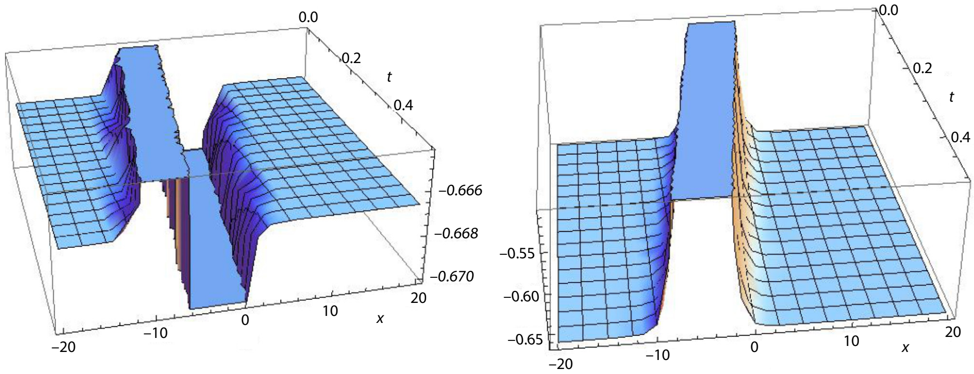

Figure 5. Single bright and single dark solitary wave solution of u_{12} and bright solitary wave solution of u_{13} with $h_2 = 2$, $h_4 = 5$, $h_6 = 1$, $s = -1$, $p_0 = 1$, $k = 0.5$

$$u_{14}=\frac{4}{3}h_2k^2\left(\frac{6h_4}{h_4-\sqrt{4h_2h_6-h_4^2}\,s\sinh\left(2\sqrt{h_2}\left(\frac{256}{3}h_2^3k^7t+kx\right)\right)}-\right.$$

$$\left.-\frac{24h_2h_6}{\left(h_4-\sqrt{4h_2h_6-h_4^2}\,s\sinh\left(2\sqrt{h_2}\left(\frac{256}{3}h_2^3k^7t+kx\right)\right)\right)^2}-1\right)$$

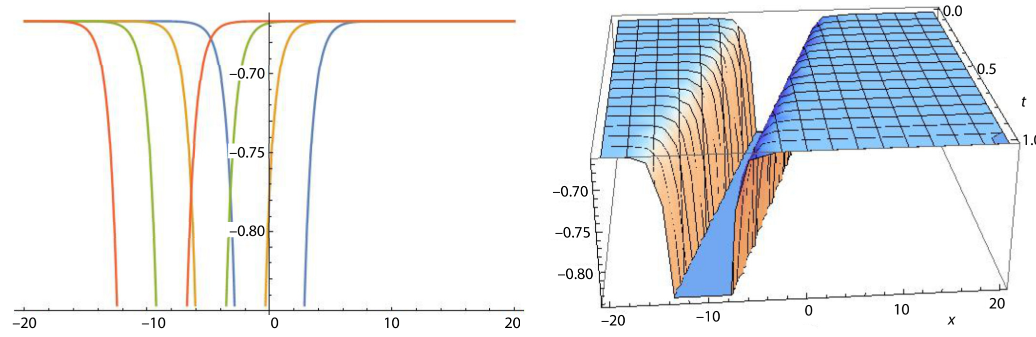

Figure 6. Dark soliton solution of u_{13} with $h_2 = 2$, $h_4 = 5$, $h_6 = 1$, $s = 1$, $p_0 = 1$, $k = 0.5$ and solitary wave solution of u_{13} with $h_2 = 2$, $h_4 = 5$, $h_6 = 1$, $s = 1$, $p_0 = 1$, $k = 0.5$

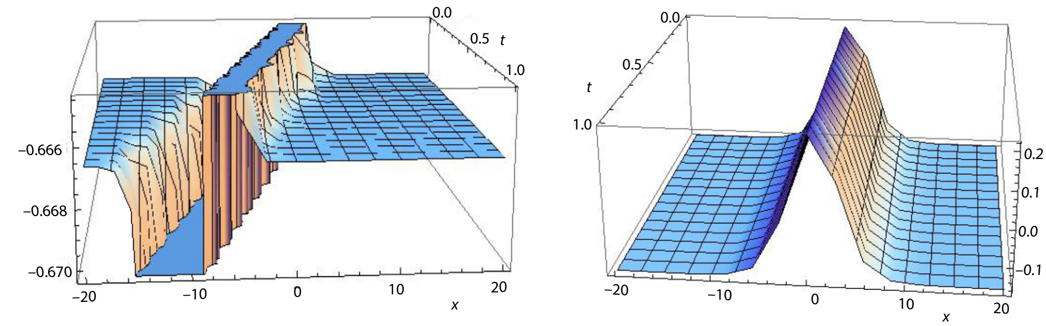

Figure 7. Singular soliton wave solution of u_{14} and solitary wave solution of u_{15} with $h_4 = 2$, $h_6 = 3$, $p_0 = 1$, $k = 0.5$

$$u_{15} = \frac{k^2 h_4^2}{3h_6}\left(-1 - \frac{3h_4^2 \operatorname{sech}^4\left(\frac{1}{2}\sqrt{\frac{h_4^2}{h_6}}\left(\frac{4h_4^6 k^7 t}{3h_6^3} + kx\right)\right)}{\left(\sqrt{h_4^2}\, s \tanh\left(\frac{1}{2}\sqrt{\frac{h_4^2}{h_6}}\left(\frac{4h_4^6 k^7 t}{3h_6^3} + kx\right)\right) + h_4^2\right)^2} + \right.$$
$$\left. + \frac{6h_4 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{h_4^2}{h_6}}\left(\frac{4h_4^6 k^7 t}{3h_6^3} + kx\right)\right)}{\sqrt{h_4^2}\, s \tanh\left(\frac{1}{2}\sqrt{\frac{h_4^2}{h_6}}\left(\frac{4h_4^6 k^7 t}{3h_6^3} + kx\right)\right) + h_4^2}\right)$$

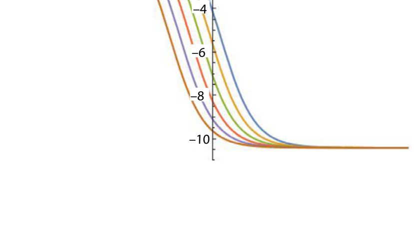

Figure 8. Solitary wave solution of u_{15} with $h_4 = -1$, $h_6 = 0.2$, $s = -1$, $p_0 = 1$, $k = 0.5$

$$u_{16}=-\frac{h_4^2k^2}{3h_6}-\frac{h_4^4k^2\operatorname{csch}^4\left(\frac{1}{2}\sqrt{\frac{h_4^2}{h_6}}\left(\frac{4h_4^6k^7t}{3h_6^3}+kx\right)\right)}{h_6\left(\sqrt{h_4^2}\,s\coth\left(\frac{1}{2}\sqrt{\frac{h_4^2}{h_6}}\left(\frac{4h_4^6k^7t}{3h_6^3}+kx\right)\right)+h_4^2\right)^2}-$$

$$-\frac{2h_4^3k^2\operatorname{csch}^2\left(\frac{1}{2}\sqrt{\frac{h_4^2}{h_6}}\left(\frac{4h_4^6k^7t}{3h_6^3}+kx\right)\right)}{h_6\left(\sqrt{h_4^2}\,s\coth\left(\frac{1}{2}\sqrt{\frac{h_4^2}{h_6}}\left(\frac{4h_4^6k^7t}{3h_6^3}+kx\right)\right)+h_4^2\right)}$$

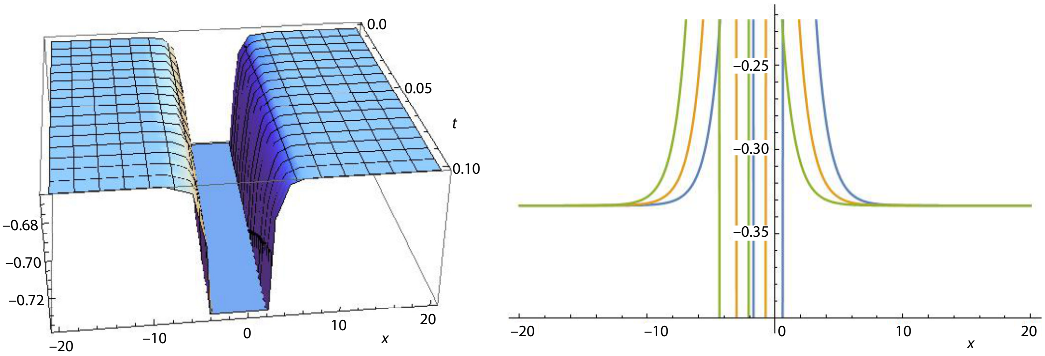

Figure 9. Solitary wave solution of u_{15} with $h_2 = 2$, $h_4 = -4$, $s = -1$, $p_0 = 1$, $k = 0.5$ and bright soliton solution of u_{16} with $h_2 = -2$, $h_6 = 1$, $s = 1$, $p_0 = 1$, $k = 0.5$

$$u_{17}=-\frac{4}{3}h_2k^2+4h_2k^2\left(s\tanh\left(\frac{1}{2}\sqrt{h_2}\left(\frac{256}{3}h_2^3k^7t+kx\right)\right)+1\right)-$$

$$-\frac{8h_2^2h_6k^2\left(s\tanh\left(\frac{1}{2}\sqrt{h_2}\left(\frac{256}{3}h_2^3k^7t+kx\right)\right)+1\right)^2}{h_4^2}$$

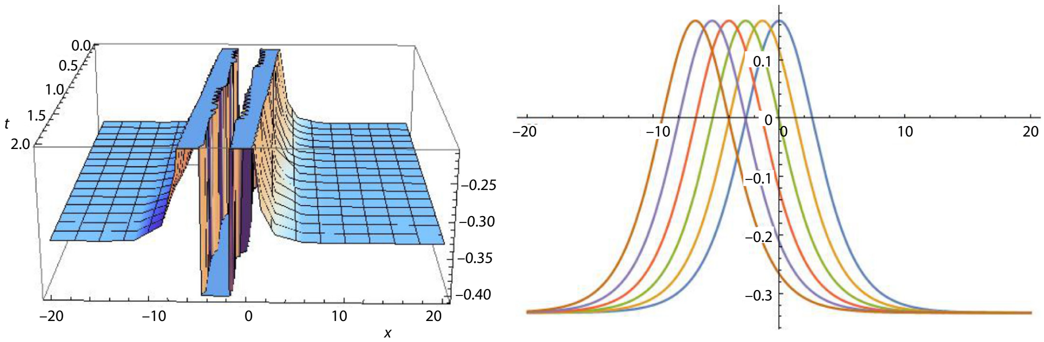

Figure 10. Solitary wave solution of u_{16} and bright solitary wave solution of u_{17}

Results and discussion

In the literature we noticed that the numerical approximations by ADM applied to sSK showed a high degree of accuracy the order of the errors was $O(h_8)$, while the homotopy analysis method (HAM) applied by Arora and Sharma [33] got the approximate solutions of Sawada-Kotera-Ito equation, Lax equation, and Kaup-Kuperschmidt equation, respectively, in very few iteration by choosing a suitable value of auxiliary parameter, h, which controls the convergence of the method and by comparing with the known exact solution, it is shown that the obtained approximate solutions have a very high accuracy the absolute errors were approximately of order $O(h_{12})$, Jafari *et al.* [24] applied VIM to sSK and LsKdV equations also, comparisons were made between Hes variational iteration method and Adomian decomposition method (ADM) for sSK and LsKdV equations the VIM reduces the volume of calculations without requiring to compute the Adomian polynomials. In [22] the single-soliton solution and other solutions as well were derived by using the tanhcoth method. The tanhcoth method has the advantage of determining more than one solution. A combination of Hirotas method and Heremans method were used to formally derive multiple-soliton solutions of the completely integrable SKIto seventh-order equation, the study confirmed the belief that these two methods are powerful techniques to handle non-linear dispersive equations of any order. However, the tanhcoth method may give more than one single-soliton solution, whereas the direct method gives one-soliton solution. The Hirotas direct method gives N-soliton solutions for completely integrable equations, whereas the tanh method does not have this capability. We noticed also that the traveling wave solutions for Sawada-Kotera-Ito equation established by using the exp-function method is very powerful and efficient technique in finding exact solutions. In our work here we established different forms of the exact solutions according to three cases in *Family 1* and one case from *Family 2* with solitary wave, bright soliton, singular solitary, double dark, periodic wave, single bright and single dark solitary solutions.

Conclusion

Modified extended direct algebraic method was used to find exact solution the non-linear seventh order Sawada-Kotera Ito equation and get new form of soliton solutions which can be considered as a good result to understand some physical properties of dynamical wave equations.The graphical analysis discuss the behavior of the soliton solutions.

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References

[1] Malfliet, W., Hereman, W., The Tanh Method: Exact Solutions of Non-Linear Evolution and Wave Equations, *Physica Scripta*, *54* (1996), 6, 563
[2] Wazwaz, A. M., The Tanh-Coth and the Sech Methods for Exact Solutions of the Jaulent-Miodek Equation, *Physics Letters A, 366* (2007), 1-2, pp. 85-90
[3] Wazwaz, A. M., New Travelling Wave Solutions to the Boussinesq and the Klein-Gordon Equations, *Communications in Non-Linear Science and Numerical Simulation*, *13* (2008), 5, pp. 889-901
[4] Abdou, M., *et al.*, New Application of Exp-Function Method for Improved Boussinesq Equation, *Physics Letters A, 369* (2007), 5-6, pp. 469-475
[5] Seadawy, A. R., The 2-D Interaction of a Shear Flow with a Free Surface in a Stratified Fluid and Its a Solitary Wave Solutions Via Mathematical Methods, *European Physical Journal Plus*, *132* (2017), 518

[6] Tariq K. U., Seadawy, A. R., Bistable Bright-Dark Solitary Wave Solutions of the (3+1)-Dimensional Breaking Soliton, Boussinesq Equation with Dual Dispersion and Modified Korteweg-de Vries-Kadomtsev-Petviashvili Equations and Their Applications, *Results in Physics*, *7* (2017), Mar., pp. 1143-1149
[7] Seadawy, A. R., El-Rashidy, K., Rayleigh-Taylor Instability of the Cylindrical Flow with Mass and Heat Transfer, *The Pramana - Journal of Physics, 87* (2016), 20
[8] Seadawy, A., R., et al., Stability Analysis of New Exact Traveling Wave Solutions of New Coupled KdV and New Coupled Zakharov-Kuznetsov Systems, *The European Physical Journal Plus, 132* (2017), 162
[9] Seadawy, A. R., The 3-D Weakly Non-Linear Shallow Water Waves Regime and Its Travelling Wave Solutions, *International Journal of Computational Methods, 15*, (2018), 3, 1850017
[10] Khater, A. H., *et al*., General Soliton Solutions for Non-Linear Dispersive Waves in Convective Type Instabilities, *Physica Scripta, 74* (2006), 3, pp. 384-393
[11] Helal, M. A. Seadawy, A. R., Benjamin-Feir-Instability in Non-Linear Dispersive Waves, *Computers and Mathematics with Applications, 64* (2012), 11, pp. 3557-3568
[12] Khater, A. H., *et al*., Variational Method for the Non-Linear Dynamics of an Elliptic Magnetic Stagnation-line, *The European Physical Journal D, 39*, (2006), 2, pp. 237-245
[13] Seadawy A. R., Approximation Solutions of Derivative Non-Linear Schrodinger Equation with Computational Applications by Variational Method, *The European Physical Journal Plus, 130* (2015), 182
[14] Seadawy A. R., Ion Acoustic Solitary Wave Solutions of 2-D Non-Linear KadomtsevPetviashvili-Burgers Equation in Quantum Plasma, *Mathematical methods and applied Sciences, 40* (2017), 5, pp. 1598-1607
[15] Seadawy, A., New Exact Solutions for the KdV Equation with Higher Order Non-Linearity by Using the Variational Method, *Computers and Mathematics with Applications*, *62* (2011), 10, pp. 3741-3755
[16] Kabir, M., Modified Kudryashov Method for Finding Exact Solitary Wave Solutions of Higher-Order Non-Linear Equations, *Mathematical methods in the Applied Sciences, 34* (2011), 2, pp. 213-219
[17] Seadawy, A., The Generalized Non-Linear Higher Order of KdV Equations from the Higher Order Non-Linear Schrdinger Equation, *Optik, 139* (2017), June, pp. 31-43
[18] Ganji, D., *et al.*, New Exact Solutions for Seventh-Order Sawada-Kotera-Ito, Lax and Kaup-Kupershmidt Equations Using Expfunction Method, *Mathematical Methods in the Applied Sciences, 33* (2010), 2, pp. 167-176
[19] Feng, J., New Traveling Wave Solutions to the Seventh-Order Sawada-Kotera Equation, *Journal of applied mathematics and informatics, 28* (2010), 5-6, pp. 1431-1437
[20] Shen, Y., *et al.*, Improved Bell-Polynomial Procedure for the Higher-Order Korteweg-de Vries Equations in Fluid Dynamics, *Applied Mathematics and Computation, 274* (2016), Feb, pp. 403-413
[21] Pomeau, Y., *et al.*, Structural Stability of the Korteweg-de Vries Solitons under a Singular Perturbation, *Physica D: Non-linear Phenomena, 31* (1988), 1, pp. 127-134
[22] Wazwaz, A. M., The Hirotas Bilinear Method and the Tanh-Coth Method for Multiple-Soliton Solutions of the Sawada-Kotera-Kadomtsev-Petviashvili Equation, *Applied Mathematics and Computation, 200* (2008), 1, pp. 160-166
[23] Salas, A. H., *et al.*, Computing Exact Solutions to a Generalized Lax-Sawada-Kotera-Ito Seventh-Order KdV Equation, *Mathematical Problems in Engineering, 2010* (2010), ID 524567
[24] Jafari, H., *et al.*, Application of Hes Variational Iteration Method for Solving Seventh Order Sawada-Kotera Equations, *Applied Mathematical Sciences, 2* (2008), 9-12, pp. 471-477
[25] El-Sayed, S. M., Kaya, D., An Application of the ADM to Seven-Order Sawada-Kotara equations, *Applied mathematics and computation, 157* (2004), 1, pp. 93-101
[26] Ganji, D., *et al.*, The HPM and VIM Methods for Finding the Exact Solutions of the Non-Linear Dispersive Equations and Seventh-Order Sawada-Kotera Equation, *International Journal of Modern Physics B, 23* (2009), 01, pp. 39-52
[27] Zuhra, S., *et al.*, Generalized Seventh Order Korteweg-de Vries Equations by Optimal Homotopy Aysmptotic Method, *Sci. Int, 27* (2015), 4, pp. 3023-3032
[28] Kaya, D., Inan, I., On the Sawada-Kotare equation, *Journal Inst. Math. Comp. Sei, 14* (2001), pp. 117-119
[29] Hubert, M., *et al.*, Optical Solitons in Parabolic Law Medium with Weak Non-Local Non-Linearity Using Modified Extended Direct Algebraic Method, *Optik, 161* (2018), Feb., pp. 180-186
[30] Yomba, E., A Generalized Auxiliary Equation Method and Its Application Non-Linear Klein-Gordon and Generalized Non-Linear Camassa-Holm Equations, *Physics Letters A, 372* (2008), 7, pp. 1048-1060
[31] Wang, M., *et al.*, Application of a Homogeneous Balance Method to Exact Solutions of Non-Linear Equations in Mathematical Physics, *Physics Letters A, 216* (1996), 1-5, pp. 67-75

[32] Wang, M., Exact Solutions for a Compound KdV-Burgers Equation, *Physics Letters A, 213* (1996), 5-6, pp. 279-287
[33] Arora, R., Sharma, H., Application of HAM to Seventh Order KdV Equations, *International Journal of System Assurance Engineering and Management*, *9* (2018), 1, pp. 131-138

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