MULTIDIMENSIONAL GENERAL CONVEXITY FOR STOCHASTIC PROCESSES AND ASSOCIATED WITH HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES

by

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In this study, we idetified multidimensional general convex stochastic processes. Concordantly, we obtained some important results related stochastic processes. Moreover, we derived some Hermite-Hadamard type integral inequalities for these stochastic processes.

Key words: *multidimensional general convex stochastic processes, mean-square integral, Hermite-Hadamard type integral inequality*

Introduction

In the literature, the following inequality is well-known as Hermite-Hadamard type integral inequality (HHII) for convex functions (CF) [1]:

$$F\left(\frac{p+q}{2}\right) \le \frac{1}{q-p}\int_p^q F(x)\mathrm{d}x \le \frac{F(p)+F(q)}{2}$$

In this context, many researchers studied on HHII for φ-CF, general convex functions (GCF) in literature, see [2-8]. For example, in 1999, Youness [2] defined E-convexity such that a function $F:\mathbb{R}^n \to \mathbb{R}$ is said to be E-convex on a set $M \subset \mathbb{R}^n$ if there is a map $E:\mathbb{R}^n \to \mathbb{R}^n$ such that M is an E-convex set:

$$F\left[\lambda E(t)+(1-\lambda)E(s)\right] \le \lambda F\left[E(t)\right]+(1-\lambda)F\left[E(s)\right]$$

for each $t,s \in M$ and $\lambda \in [0,1]$. Using Youness's definition, Sarikaya *et al.* [3] showed such that a function $F:[p,q] \subset \mathbb{R} \to \mathbb{R}$ is called GCF on the real number interval $[p,q]$, if the following inequality holds*:*

$$F\left[\lambda\varphi(t)+(1-\lambda)\varphi(s)\right] \le \lambda F\left[\varphi(t)\right]+(1-\lambda)F\left[\varphi(s)\right]$$

for all $t,s \in [p,q]$, $\lambda \in [0,1]$ and $\varphi:[p,q] \to [p,q]$, $\varphi(p) < \varphi(q)$ is a function. Also, Cristescu [8] obtained HHII for GCF:

$$F\left[\frac{\varphi(p)+\varphi(q)}{2}\right] \le \frac{1}{\varphi(p)-\varphi(q)}\int_{\varphi(p)}^{\varphi(q)} F(t)\mathrm{d}t \le \frac{F\left[\varphi(p)\right]+F\left[\varphi(q)\right]}{2}$$

Besides, Set *et al.* [9] established the following GCF on the co-ordinates:

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Let $\Delta: = [u_1, v_1] \times [u_2, v_2] \subseteq [0,\infty)^2$; $u_1 < v_1$; $u_2 < v_2$; $\varphi_i:[u_i, v_i] \longrightarrow [u_i, v_i]$, $i = 1,2$ be a continuous function. A function $F:\Delta \longrightarrow \mathbb{R}$ is called GCF on Δ, if the following inequality holds:

$$F\left[\lambda\varphi_1(t_1)+(1-\lambda)\varphi_1(t_2),\lambda\varphi_2(s_1)+(1-\lambda)\varphi_2(s_2)\right]\leq \lambda F\left[\varphi_1(t_1),\varphi_2(s_1)\right]+(1-\lambda)F\left[\varphi_1(t_2),\varphi_2(s_2)\right]$$

for all (t_1, s_1), $(t_2, s_2) \in \Delta$ and $\lambda \in [0,1]$. If the above inequality is reversed then F is said to *be* φ-concave on Δ.

Other definition of GCF is defined by Set *et al.* [9]

A function $F:\Delta \to \mathbb{R}$ is called co-ordinated φ-convex on Δ, if the following partial mappings $F_{\varphi_2}:[u_1, v_1] \to \mathbb{R}$, $F_{\varphi_2}(u):= F(u_2, \varphi_2)$ and $F_{\varphi_1}:[u_2, v_2] \to \mathbb{R}$, $F_{\varphi_1}(v):= F(\varphi_1,v)$ are defined φ-convex for all $\varphi_1 \in [u_1, v_1]$ and $\varphi_2 \in [u_2, \varphi_2]$. Then:

$$F\left(\frac{\varphi_1(u_1)+\varphi_1(v_1)}{2},\frac{\varphi_2(u_2)+\varphi_2(v_2)}{2}\right)\leq\frac{1}{2\left[\varphi_1(v_1)-\varphi_1(u_1)\right]}\int\limits_{\varphi_1(u_1)}^{\varphi_1(v_1)}F\left(t,\frac{\varphi_2(u_2)+\varphi_2(v_2)}{2}\right)\mathrm{d}t+$$

$$+\frac{1}{2\left[\varphi_2(v_2)-\varphi_2(u_2)\right]}\int\limits_{\varphi_2(u_2)}^{\varphi_2(v_2)}F\left(\frac{\varphi_1(u_1)+\varphi_1(v_1)}{2},s\right)\mathrm{d}s\leq$$

$$\leq\frac{1}{\left[\varphi_1(v_1)-\varphi_1(u_1)\right]\left[\varphi_2(v_2)-\varphi_2(u_2)\right]}\int\limits_{\varphi_1(u_1)}^{\varphi_1(v_1)}\int\limits_{\varphi_2(u_2)}^{\varphi_2(v_2)}F(t,s)\,\mathrm{d}t\mathrm{d}s\leq$$

$$\leq\frac{1}{4\left[\varphi_1(v_1)-\varphi_1(u_1)\right]}\int\limits_{\varphi(u_1)}^{\varphi(u_2)}\left\{F\left[t,\varphi_2(u_2)\right]+F\left[t,\varphi_2(v_2)\right]\right\}\mathrm{d}t+$$

$$+\frac{1}{4\left[\varphi_2(v_2)-\varphi_2(u_2)\right]}\int\limits_{\varphi(v_1)}^{\varphi(v_2)}\left\{F\left[\varphi_1(u_1),s\right]+F\left[\varphi_1(v_1),s\right]\right\}\mathrm{d}s\leq$$

$$\leq\frac{1}{4}\left\{F\left[\varphi_1(u_1),\varphi_2(u_2)\right]+F\left[\varphi_1(u_1),\varphi_2(v_2)\right]+F\left[\varphi_1(v_1),\varphi_2(u_2)\right]+F\left[\varphi_1(v_1),\varphi_2(v_2)\right]\right\}$$

Besides, De la Cal *et al.* [10] obtained HHII for MCF in 2006. Elahi *et al.* [11] derived HHII for s-MCF. Viloria *et al.* [12] verified HHII for harmonically MCF.

Correspondingly, $\xi:I \times \Omega \longrightarrow \mathbb{R}$ is called a stochastic process (SP), if $t \in I \subset \mathbb{R}$ is considered of a time parameter, such that $(\Omega, \Im, P)$ is an arbitrary probability space. There are satisfactory evidence on similar results for SP in the literature, belong to Nikodem [13], Shaked and Shanthikumar [14], Skowronski [15], Kotrys [16].

Nowadays, Set *et al.* [17] studied on co-ordinated CSP. Moreover, Sarikaya *et al.* [18] investigated φ_h-*CSP*. Let us show this definition for $h(\lambda) = \lambda$.

Let us consider a function $\varphi:[u,v]\longrightarrow[u,v]$, $u < v$. *Then* $\xi:[u,v] \times \Omega\longrightarrow\mathbb{R}$ *is called* GCSP *on* $[u,v] \subseteq [0,\infty)$:

$$\xi\left(\left(\lambda\varphi(t)+(1-\lambda)\varphi(s)\right),\cdot\right)\leq\lambda\xi(\varphi(t),\cdot)+(1-\lambda)\xi\left(\varphi(s),\cdot\right),\ (\text{a.e.})$$

for all $t,s \in [u,v]$ and $\lambda \in [0,1]$. If the function φ is continuous increasing, then

$$\xi\left(\frac{\varphi(u)+\varphi(v)}{2},\cdot\right)\leq\frac{1}{\varphi(v)-\varphi(u)}\int\limits_{\varphi(u)}^{\varphi(v)}\xi(t,\cdot)\,\mathrm{d}t\leq\frac{\xi\left(\varphi(u),\cdot\right)+\xi\left(\varphi(v),\cdot\right)}{2},\ (\text{a.e.})$$

In 2018, Karahan *et al.* [19] investigated MCSP. There are well-known definitions and some fundamentals about SP in the literature, see [13-19]. Note that, throughout paper the symbol (*a. e.*) means *almost everywhere* and the stochastic process ξ is mean-square integrable.

Our claim is to define multidimensional general convex stochastic processes (MGC-SP) and to obtain some HHII for these processes.

Main results

In this section, we identified MGCSP and proved HHII for these processes. Let be a continuous increasing function φ_i: $[u_i,v_i]\longrightarrow[u_i,v_i]$ and for $i = 1,2,...,n$, $n \geq 2$.

$$\mathfrak{D}^n := \prod_{i=1}^{n}[u_i,v_i] \subseteq [0,\infty)^n$$

$$\Delta_i^+ := \varphi_i(u_i)+\varphi_i(v_i); \quad \Delta_i^- := \varphi_i(v_i)-\varphi_i(u_i) \text{ such that } \varphi_i(v_i)<\varphi_i(u_i)$$

$$\boldsymbol{\varphi(t)} := \left(\wedge_{k=1}^{n}\varphi_k(t_k)\right) \equiv \left(\varphi_1(t_1),\ldots,\varphi_n(t_n)\right) \equiv \varphi(t_1,\ldots,t_n) \in \mathfrak{D}^n$$

Let us give definition of multidimensional general convexity for SP:

Definition 1: Assume that the function φ:$\mathfrak{D}^n\longrightarrow\mathfrak{D}^n$ is a continuous increasing. Then ξ:$\mathfrak{D}^n\times\Omega\longrightarrow\mathbb{R}$ is called MGCSP if:

$$\xi\left(\left(\lambda\boldsymbol{\varphi(t)}+(1-\lambda)\boldsymbol{\varphi(s)}\right),\cdot\right) \leq \lambda\xi\left(\boldsymbol{\varphi(t)},\cdot\right)+(1-\lambda)\xi\left(\boldsymbol{\varphi(s)},\cdot\right),(\text{a.e.})$$

for all $\lambda \in [0,1]$.

Definition 2: ξ:$\mathfrak{D}^n\times\Omega\longrightarrow\mathbb{R}$ is called MGCSP on $\mathfrak{D}^n$ if the following partial SP $\xi^i_{\varphi_i(t_i)}$:$[u_i,v_i]\longrightarrow\mathbb{R}$ are general convex on $[u_i,v_i]$.

$$\xi^i_{\varphi_i(t_i)}(t,\cdot) := \xi\left(\left(\wedge_{k=1}^{i-1}\varphi_k(t_k),t,\wedge_{k=i+1}^{n}\varphi_k(t_k)\right),\cdot\right),(\text{a.e.})$$

$$\text{for all } \xi^i_{\varphi_i(t_i)} \in [u_i,v_i],\ i=1,2,\ldots,n$$

Lemma 3: Every general CSP ξ:$\mathfrak{D}^n\times\Omega\longrightarrow\mathbb{R}$ is general convex on n-co-ordinates almost everywhere, not the other way round.

Proof. Let ξ:$\mathfrak{D}^n\times\Omega\longrightarrow\mathbb{R}$ be a MGCSP. Using the definition of $\xi^i_{\varphi_n(t_n)}$, we get:

$$\xi^i_{\varphi_n(t_n)}\left(\left(\lambda\varphi(t)+(1-\lambda)\varphi(s)\right),\cdot\right) = \xi\left(\left(\wedge_{k=1}^{i-1}\varphi_k(t_k),\lambda\varphi(t)+(1-\lambda)\varphi(s),\wedge_{k=i+1}^{n}\varphi_k(t_k)\right),\cdot\right) \leq$$

$$\leq \lambda\xi\left(\left(\wedge_{k=1}^{i-1}\varphi_k(t_k),\varphi(t),\wedge_{k=i+1}^{n}\varphi_k(t_k)\right),\cdot\right)+(1-\lambda)\xi\left(\left(\wedge_{k=1}^{i-1}\varphi_k(t_k),\varphi(s),\wedge_{k=i+1}^{n}\varphi_k(t_k)\right),\cdot\right) =$$

$$= \lambda\xi^i_{\varphi_n(t_n)}\left(\varphi(t),\cdot\right)+(1-\lambda)\xi^i_{\varphi_n(t_n)}\left(\varphi(s),\cdot\right),(\text{a.e.})$$

On the other hand, assume that ξ:$[0,1]^n\times\Omega\longrightarrow\mathbb{R}$:

$$\xi\left(\boldsymbol{\varphi(t)},\cdot\right) := \varphi_1(t_1)\varphi_2(t_2)\ldots\varphi_n(t_n)$$

This process is clearly a MGCSP. But for:

$$\boldsymbol{\varphi(t)} = (1,1,\ldots,0),\ \boldsymbol{\varphi(s)} = (0,1,\ldots,1) \in [0,1]^n$$

we have

$$\xi\left(\left(\lambda\boldsymbol{\varphi(t)}+(1-\lambda)\boldsymbol{\varphi(s)}\right),\cdot\right) = \xi\left(\left(\lambda,1,1,\ldots,(1-\lambda)\right),\cdot\right) = \lambda(1-\lambda)$$

$$\lambda\xi\left(\boldsymbol{\varphi(t)},\cdot\right)+(1-\lambda)\xi\left(\boldsymbol{\varphi(s)},\cdot\right) = \lambda.0+(1-\lambda).0 = 0$$

This gives:

$$\xi\left(\left(\lambda\boldsymbol{\varphi}(\boldsymbol{t})+(1-\lambda)\boldsymbol{\varphi}(\boldsymbol{s})\right),\cdot\right)>\lambda\xi\left(\boldsymbol{\varphi}(\boldsymbol{t}),\cdot\right)+(1-\lambda)\xi\left(\boldsymbol{\varphi}(\boldsymbol{s}),\cdot\right)$$

for all $\lambda \in [0,1]$, that is, X is not general convex $[0,1]^n$.

Remark 4: If $\xi:\mathfrak{D}^n\times\Omega\longrightarrow\mathbb{R}$ is MGCSP, then $\xi^i_{\varphi_n(t_n)}:[u_i,v_i]\times\Omega\longrightarrow\mathbb{R}$ is GCSP such that:

$$\xi^i_{\varphi_n(t_n)}\left(\frac{\Delta_i^+}{2},\cdot\right)\le\frac{1}{\Delta_i^-}\int_{\varphi_i(u_i)}^{\varphi_i(v_i)}\xi^i_{\varphi_n(t_n)}\left(t_i,\cdot\right)\mathrm{d}t_i\le\frac{\xi^i_{\varphi_n(t_n)}\left(\varphi_i(u_i),\cdot\right)+\xi^i_{\varphi_n(t_n)}\left(\varphi_i(v_i),\cdot\right)}{2},(\text{a.e.})\tag{1}$$

Theorem 5: Let $\xi:\mathfrak{D}^n\times\Omega\longrightarrow\mathbb{R}$ be a MGCSP. Then:

$$\sum_{i=1}^{n}\frac{1}{\Delta_i^-}\int_{\varphi_i(u_i)}^{\varphi_i(v_i)}\left(\xi^i_{\varphi_n(u_n)}(t_i,\cdot)+\xi^i_{\varphi_n(v_n)}(t_i,\cdot)\right)\mathrm{d}t_i\le$$

$$\le\frac{n}{2}\left[\xi\left(\boldsymbol{\varphi}(\boldsymbol{u}),\cdot\right)+\xi\left(\boldsymbol{\varphi}(\boldsymbol{v}),\cdot\right)\right]+\frac{1}{2}\sum_{i=1}^{n}\left[\xi^i_{\varphi_n(u_n)}\left(\varphi_i(u_i),\cdot\right)+\xi^i_{\varphi_n(v_n)}\left(\varphi_i(v_i),\cdot\right)\right],(\text{a.e.})\tag{2}$$

Proof. Using the left hand of eq. (1) by $\xi^i_{\varphi_n(u_n)}(\varphi_i(u_i),\cdot)=\xi(\boldsymbol{\varphi}(\boldsymbol{u}),\cdot)$ and $\xi^i_{\varphi_n(v_n)}(\varphi_i(v_i),\cdot)=\xi(\boldsymbol{\varphi}(\boldsymbol{v}),\cdot)$ for each $i=1,...,n$, then almost everywhere:

$$\frac{1}{\Delta_i^-}\int_{\varphi_i(u_i)}^{\varphi_i(v_i)}\xi^i_{\varphi_n(u_n)}(t_i,\cdot)\mathrm{d}t_i\le\frac{\xi^i_{\varphi_n(u_n)}\left(\varphi_i(u_i),\cdot\right)+\xi^i_{\varphi_n(u_n)}\left(\varphi_i(v_i),\cdot\right)}{2}\le\frac{\xi\left(\boldsymbol{\varphi}(\boldsymbol{u}),\cdot\right)+\xi^i_{\varphi_n(u_n)}\left(\varphi_i(v_i),\cdot\right)}{2}$$

$$\frac{1}{\Delta_i^-}\int_{\varphi_i(u_i)}^{\varphi_i(v_i)}\xi^i_{\varphi_n(v_n)}(t_i,\cdot)\mathrm{d}t_i\le\frac{\xi^i_{\varphi_n(v_n)}\left(\varphi_i(u_i),\cdot\right)+\xi^i_{\varphi_n(v_n)}\left(\varphi_i(v_i),\cdot\right)}{2}\le\frac{\xi^i_{\varphi_n(v_n)}\left(\varphi_i(u_i),\cdot\right)+\xi\left(\boldsymbol{\varphi}(\boldsymbol{v}),\cdot\right)}{2}$$

All of sides of the above inequalities by integrating on $[\varphi_i(u_i),\ \varphi_i(v_i)]$:

$$\frac{1}{\Delta_i^-}\int_{\varphi_i(u_i)}^{\varphi_i(v_i)}\left[\xi^i_{\varphi_n(u_n)}\left(t_i,\cdot\right)+\xi^i_{\varphi_n(v_n)}\left(t_i,\cdot\right)\right]\mathrm{d}t_i\le$$

$$\le\frac{\xi\left(\boldsymbol{\varphi}(\boldsymbol{u}),\cdot\right)+\xi^i_{\varphi_n(u_n)}\left(\varphi_i(v_i),\cdot\right)+\xi\left(\boldsymbol{\varphi}(\boldsymbol{v}),\cdot\right)+\xi^i_{\varphi_n(v_n)}\left(\varphi_i(u_i),\cdot\right)}{2}$$

Taking summation from 1 to n, this completes the proof.

Theorem 6: Let $\xi:\mathfrak{D}^n\times\Omega\longrightarrow\mathbb{R}$ be a MGCSP. Then:

$$\sum_{i=1}^{n-1}\xi\left(\left(\wedge_{k=1}^{i-1}\varphi_k(t_k),\frac{\Delta_i^+}{2},\frac{\Delta_{i+1}^+}{2},\wedge_{k=i+2}^{n}\varphi_k(t_k),\right),\cdot\right)\le\sum_{i=1}^{n-1}\frac{1}{\Delta_i^-}\int_{\varphi_i(u_i)}^{\varphi_i(v_i)}\xi^{i+1}_{\varphi_{i+1}(t_n)}\left(\frac{\Delta_{i+1}^+}{2},\cdot\right)\mathrm{d}t_i\le$$

$$\le\sum_{i=1}^{n-1}\frac{1}{\Delta_i^-\Delta_{i+1}^-}\int_{\varphi_i(u_i)}^{\varphi_i(v_i)}\int_{\varphi_{i+1}(u_{i+1})}^{\varphi_{i+1}(v_{i+1})}\xi^{i+1}_{\varphi_{i+1}(t_n)}\left(t_{i+1},\cdot\right)\mathrm{d}t_{i+1}\mathrm{d}t_i\le$$

$$\le\sum_{i=1}^{n-1}\frac{1}{2\Delta_i^-}\int_{\varphi_i(u_i)}^{\varphi_i(v_i)}\left[\xi^{i+1}_{\varphi_{i+1}(t_n)}\left(\varphi\left(u_{i+1}\right),\cdot\right)+\xi^{i+1}_{\varphi_{i+1}(t_n)}\left(\varphi\left(v_{i+1}\right),\cdot\right)\right]\mathrm{d}t_i\tag{3}$$

$$\leq \frac{1}{4}\sum_{i=1}^{n-1}\begin{bmatrix} \xi((\wedge_{k=1}^{i-1}\varphi_k(t_k),\ \varphi_i(u_i),\ \varphi_{i+1}(u_{i+1}),\ \wedge_{k=i+2}^{n}\varphi_k(t_k)),\cdot) \\ +\xi((\wedge_{k=1}^{i-1}\varphi_k(t_k),\ \varphi_i(v_i),\ \varphi_{i+1}(u_{i+1}),\ \wedge_{k=i+2}^{n}\varphi_k(t_k)),\cdot) \\ +\xi((\wedge_{k=1}^{i-1}\varphi_k(t_k),\ \varphi_i(u_i),\ \varphi_{i+1}(v_{i+1}),\ \wedge_{k=i+2}^{n}\varphi_k(t_k)),\cdot) \\ +\xi((\wedge_{k=1}^{i-1}\varphi_k(t_k),\ \varphi_i(v_i),\ \varphi_{i+1}(v_{i+1}),\ \wedge_{k=i+2}^{n}\varphi_k(t_k)),\cdot) \end{bmatrix},\ (\text{a.e.}) \tag{3}$$

Proof. Using eq. (1) by $\xi^{i+1}_{\varphi_{i+1}(t_n)}$, then almost everywhere:

$$\xi^{i+1}_{\varphi_{i+1}(t_n)}\left(\frac{\Delta^+_{i+1}}{2},\cdot\right) \leq \frac{1}{\Delta^-_{i+1}}\int_{\varphi_{i+1}(u_{i+1})}^{\varphi_{i+1}(v_{i+1})}\xi^{i+1}_{\varphi_{i+1}(t_n)}(t_{i+1},\cdot)\,\mathrm{d}t_{i+1} \leq \frac{\xi^{i+1}_{\varphi_{i+1}(t_n)}\left(\varphi_{i+1}(u_{i+1}),\cdot\right)+\xi^{i+1}_{\varphi_{i+1}(t_n)}\left(\varphi_{i+1}(v_{i+1}),\cdot\right)}{2}$$

All of sides of the above inequality by integrating on $[(\varphi_i(u_i), (\varphi_i(v_i)]$:

$$\frac{1}{\Delta^-_i}\int_{\varphi_i(u_i)}^{\varphi_i(v_i)}\xi^{i+1}_{\varphi_{i+1}(t_n)}\left(\frac{\Delta^+_{i+1}}{2},\cdot\right)\mathrm{d}t_i \leq \frac{1}{\Delta^-_i\Delta^-_{i+1}}\int_{\varphi_i(u_i)}^{\varphi_i(v_i)}\int_{\varphi_{i+1}(u_{i+1})}^{\varphi_{i+1}(v_{i+1})}\xi^{i+1}_{\varphi_{i+1}(t_n)}(t_{i+1},\cdot)\,\mathrm{d}t_{i+1}\mathrm{d}t_i \leq$$

$$\leq \frac{1}{2\Delta^-_i}\int_{\varphi_i(u_i)}^{\varphi_i(v_i)}\left(\xi^{i+1}_{\varphi_{i+1}(t_n)}\left(\varphi_{i+1}(u_{i+1}),\cdot\right)+\xi^{i+1}_{\varphi_{i+1}(t_n)}\left(\varphi_{i+1}(v_{i+1}),\cdot\right)\right)\mathrm{d}t_i \tag{4}$$

Applying HHII to the left hand of eq. (4) for each $i \in \{1,..., n-1\}$:

$$\xi\left(\left(\wedge_{k=1}^{i-1}\varphi_k(t_k),\frac{\Delta^+_i}{2},\frac{\Delta^+_{i+1}}{2},\wedge_{k=i+2}^{n}\varphi_k(t_k)\right),\cdot\right) \leq \frac{1}{\Delta^-_i}\int_{\varphi_i(u_i)}^{\varphi_i(v_i)}\xi^{i+1}_{\varphi_{i+1}(t_n)}\left(\frac{\Delta^+_{i+1}}{2},\cdot\right)\mathrm{d}t_i \tag{5}$$

and also applying HHII to the right hand of eq. (4):

$$\frac{1}{2}\left[\frac{1}{\Delta^-_i}\int_{\varphi_i(u_i)}^{\varphi_i(v_i)}\xi^{i+1}_{\varphi_{i+1}(t_n)}\left(\varphi_{i+1}(u_{i+1}),\cdot\right)\mathrm{d}t_i + \frac{1}{\Delta^-_i}\int_{\varphi_i(u_i)}^{\varphi_i(v_i)}\xi^{i+1}_{\varphi_{i+1}(t_n)}\left(\varphi_{i+1}(v_{i+1}),\cdot\right)\mathrm{d}t_i\right]$$

$$\leq \frac{1}{4}\begin{bmatrix} \xi((\wedge_{k=1}^{i-1}\varphi_k(t_k),\ \varphi_i(u_i),\ \varphi_{i+1}(u_{i+1}),\ \wedge_{k=i+2}^{n}\varphi_k(t_k)),\cdot) \\ +\xi((\wedge_{k=1}^{i-1}\varphi_k(t_k),\ \varphi_i(v_i),\ \varphi_{i+1}(u_{i+1}),\ \wedge_{k=i+2}^{n}\varphi_k(t_k)),\cdot) \\ +\xi((\wedge_{k=1}^{i-1}\varphi_k(t_k),\ \varphi_i(u_i),\ \varphi_{i+1}(v_{i+1}),\ \wedge_{k=i+2}^{n}\varphi_k(t_k)),\cdot) \\ +\xi((\wedge_{k=1}^{i-1}\varphi_k(t_k),\ \varphi_i(v_i),\ \varphi_{i+1}(v_{i+1}),\ \wedge_{k=i+2}^{n}\varphi_k(t_k)),\cdot) \end{bmatrix} \tag{6}$$

for each $i \in \{1,..., n-1\}$. After using the inequalities (5) and (6) in eq. (4) and then taking summation from 1 to $n = 1$, we have eq. (3).

Remark 7: Using *Theorem 6* for $n = 1$, then the classical HHII for 2-D GCSP.

Theorem 8: Let $\xi:\mathfrak{D}^n\times\Omega\longrightarrow\mathbb{R}$ be a MGCSP. Then:

$$\xi\left(\left(\frac{\Delta^+_1}{2},\ldots,\frac{\Delta^+_{n-1}}{2},\frac{\Delta^+_n}{2}\right),\cdot\right) \leq \frac{1}{\prod_{i=1}^{n}\Delta^-_i}\int_{\varphi_1(u_1)}^{\varphi_1(v_1)}\cdots\int_{\varphi_n(u_n)}^{\varphi_n(v_n)}\xi(t_1,\ldots,t_n,\cdot)\,\mathrm{d}t_n\ldots\mathrm{d}t_1 \leq$$

$$\leq \frac{1}{2^n}\sum_{\boldsymbol{\delta}\in l_i(n)}\xi\left(\boldsymbol{\delta\varphi}(\boldsymbol{u})+(1-\boldsymbol{\delta})\boldsymbol{\varphi}(\boldsymbol{v}),\cdot\right),(\text{a.e.}) \tag{7}$$

where

$$l_i(n) := \left\{ \boldsymbol{\delta} \in \mathbb{N}_0^n : \delta_i \le 1, |\boldsymbol{\delta}| = n+1-i, i = 1,\ldots,n+1 \right\}$$

$$|\boldsymbol{\delta}| := \delta_1 + \ldots + \delta_n \in \mathbb{N}; \boldsymbol{\delta\varphi}(\boldsymbol{u}) := \left(\delta_1\varphi_1(u_1),\ldots,\delta_n\varphi_n(u_n)\right) \in \mathbb{N}_0^n$$

Proof. Using eq. (1), we get the following inequality for $\xi^n_{\varphi_n(t_n)}$ almost everywhere:

$$\xi^n_{\varphi_n(t_n)}\left(\frac{\Delta_n^+}{2},\cdot\right) \le \frac{1}{\Delta_n^-}\int\limits_{\varphi_n(u_n)}^{\varphi_n(v_n)} \xi^n_{\varphi_n(t_n)}(t_n,\cdot)\mathrm{d}t_n \le \frac{\xi^n_{\varphi_n(t_n)}(\varphi_n(u_n),\cdot)+\xi^n_{\varphi_n(t_n)}(\varphi_n(v_n),\cdot)}{2} \tag{8}$$

Also, using the same method in the proof of *Theorem 6* by the inequality eq. (8), then almost everywhere

$$\xi\left(\left(\wedge_{k=1}^{n-2}\varphi_k(t_k),\frac{\Delta_{n-1}^+}{2},\frac{\Delta_n^+}{2}\right),\cdot\right) \le \frac{1}{\Delta_{n-1}^-\Delta_n^-}\int\limits_{\varphi_{n-1}(u_n)}^{\varphi_{n-1}(v_n)}\int\limits_{\varphi_n(u_n)}^{\varphi_n(v_n)} \xi^n_{\varphi_n(t_n)}(t_n,\cdot)\mathrm{d}t_n\mathrm{d}t_{n-1}$$

$$\le \frac{1}{2^2}\left[\begin{array}{l}\xi\left(\left(\wedge_{k=1}^{n-2}\varphi_k(t_k),\varphi_{n-1}(u_{n-1}),\varphi_n(u_n)\right),\cdot\right)+\xi\left(\left(\wedge_{k=1}^{n-2}\varphi_k(t_k),\varphi_{n-1}(v_{n-1}),\varphi_n(u_n)\right),\cdot\right)\\ +\xi\left(\left(\wedge_{k=1}^{n-2}\varphi_k(t_k),\varphi_{n-1}(u_{n-1}),\varphi_n(v_n)\right),\cdot\right)+\xi\left(\left(\wedge_{k=1}^{n-2}\varphi_k(t_k),\varphi_{n-1}(v_{n-1}),\varphi_n(v_n)\right),\cdot\right)\end{array}\right] \tag{9}$$

Integrating eq. (9) on $[\varphi_{n-2}(u_{n-2}),\ \varphi_{n-2}(v_{n-2})]$:

$$\frac{1}{\Delta_{n-2}^-}\int\limits_{\varphi_{n-2}(u_{n-2})}^{\varphi_{n-2}(v_{n-2})}\xi\left(\left(\wedge_{k=1}^{n-2}\varphi_k(t_k),\frac{\Delta_{n-1}^+}{2},\frac{\Delta_n^+}{2}\right),\cdot\right)\mathrm{d}t_{n-2} \le$$

$$\le \frac{1}{\prod_{i=n-2}^{n}\Delta_i^-}\int\limits_{\varphi_{n-2}(u_{n-2})}^{\varphi_{n-2}(v_{n-2})}\int\limits_{\varphi_{n-1}(u_{n-1})}^{\varphi_{n-1}(v_{n-1})}\int\limits_{\varphi_n(u_n)}^{\varphi_n(v_n)}\xi^n_{\varphi_n(t_n)}(t_n,\cdot)\mathrm{d}t_n\mathrm{d}t_{n-1}\mathrm{d}t_{n-2}$$

$$\le \frac{1}{\Delta_{n-2}^-}\int\limits_{u_{n-2(u_{n-2})}}^{\varphi_{n-2(v_{n-2})}}\frac{1}{2^2}\left[\begin{array}{l}\xi((\wedge_{k=1}^{n-2}\varphi_k(t_k),\ \varphi_{n-1}(u_{n-1}),\ \varphi_n(u_n)),\cdot)\\ +\xi((\wedge_{k=1}^{n-2}\varphi_k(t_k),\ \varphi_{n-1}(v_{n-1}),\ \varphi_n(u_n)),\cdot)\\ +\xi((\wedge_{k=1}^{n-2}\varphi_k(t_k),\ \varphi_{n-1}(u_{n-1}),\ \varphi_n(v_n)),\cdot)\\ +\xi((\wedge_{k=1}^{n-2}\varphi_k(t_k),\ \varphi_{n-1}(v_{n-1}),\ \varphi_n(u_n)),\cdot)\end{array}\right]\mathrm{d}t_{n-2} \tag{10}$$

Again using the same method in the proof of *Theorem 6* by the inequality eq. (10), then almost everywhere:

$$\xi\left(\left(\wedge_{k=1}^{n-3}\varphi_k(t_k),\frac{\Delta_{n-2}^+}{2},\frac{\Delta_{n-1}^+}{2},\frac{\Delta_n^+}{2}\right),\cdot\right) \le \frac{1}{\prod_{i=n-2}^{n}\Delta_i^-}\int\limits_{\varphi_{n-2}(u_{n-2})}^{\varphi_{n-2}(v_{n-2})}\int\limits_{\varphi_{n-1}(u_{n-1})}^{\varphi_{n-1}(v_{n-1})}\int\limits_{\varphi_n(u_n)}^{\varphi_n(v_n)}\xi^n_{\varphi_n(t_n)}(t_n,\cdot)\mathrm{d}t_n\mathrm{d}t_{n-1}\mathrm{d}t_{n-2} \le$$

$$\leq \frac{1}{2^3}\begin{bmatrix} \xi((\wedge_{k=1}^{n-3}\varphi_k(t_k),\ \varphi_{n-2}(u_{n-2}),\ \varphi_{n-1}(u_{n-1}),\ \varphi_n(u_n)),\cdot) \\ +\xi((\wedge_{k=1}^{n-3}\varphi_k(t_k),\ \varphi_{n-2}(v_{n-2}),\ \varphi_{n-1}(u_{n-1}),\ \varphi_n(u_n)),\cdot) \\ +\xi((\wedge_{k=1}^{n-3}\varphi_k(t_k),\ \varphi_{n-2}(u_{n-2}),\ \varphi_{n-1}(v_{n-1}),\ \varphi_n(u_n)),\cdot) \\ +\xi((\wedge_{k=1}^{n-3}\varphi_k(t_k),\ \varphi_{n-2}(v_{n-2}),\ \varphi_{n-1}(v_{n-1}),\ \varphi_n(u_n)),\cdot) \\ +\xi((\wedge_{k=1}^{n-3}\varphi_k(t_k),\ \varphi_{n-2}(u_{n-2}),\ \varphi_{n-1}(u_{n-1}),\ \varphi_n(v_n)),\cdot) \\ +\xi((\wedge_{k=1}^{n-3}\varphi_k(t_k),\ \varphi_{n-2}(v_{n-2}),\ \varphi_{n-1}(u_{n-1}),\ \varphi_n(v_n)),\cdot) \\ +\xi((\wedge_{k=1}^{n-3}\varphi_k(t_k),\ \varphi_{n-2}(u_{n-2}),\ \varphi_{n-1}(v_{n-1}),\ \varphi_n(v_n)),\cdot) \\ +\xi((\wedge_{k=1}^{n-3}\varphi_k(t_k),\ \varphi_{n-2}(v_{n-2}),\ \varphi_{n-1}(v_{n-1}),\ \varphi_n(v_n)),\cdot) \end{bmatrix}$$

Thus, using inductive method and taking into account $\xi^n_{\varphi_n(t_n)}(t_n,\cdot) := \xi(t_1,...,t_n,\cdot)$, we get eq. (7).

Example 9: Let $\xi:\mathfrak{D}^n\times\Omega\longrightarrow\mathbb{R}$ be a MGCSP. Then:

$$\xi\left(\left(\frac{\varphi_1(u_1)+\varphi_1(v_1)}{2},\frac{\varphi_2(u_2)+\varphi_2(v_2)}{2},\frac{\varphi_3(u_3)+\varphi_3(v_3)}{2}\right),\cdot\right)$$

$$\leq \frac{1}{\prod_{i=n-2}^{3}\left(\varphi_i\left(v_i\right)-\varphi_i(u_i)\right)}\int_{\varphi_1(u_1)}^{\varphi_1(v_1)}\int_{\varphi_2(u_2)}^{\varphi_2(v_2)}\int_{\varphi_3(u_3)}^{\varphi_3(v_3)}\xi\left((t_1,t_2,t_3),\cdot\right)\mathrm{d}t_3\mathrm{d}t_2\mathrm{d}t_1$$

$$\leq \frac{1}{2^3}\begin{bmatrix} \xi((\varphi_1(u_1),\ \varphi_2(u_2),\ \varphi_3(u_3)),\cdot)+\xi((\varphi_1(v_1),\ \varphi_2(u_2),\ \varphi_3(u_3)),\cdot) \\ +\xi((\varphi_1(u_1),\ \varphi_2(v_2),\ \varphi_3(u_3)),\cdot)+\xi((\varphi_1(v_1),\ \varphi_2(v_2),\ \varphi_3(u_3)),\cdot) \\ +\xi((\varphi_1(u_1),\ \varphi_2(u_2),\ \varphi_3(v_3)),\cdot)+\xi((\varphi_1(v_1),\ \varphi_2(u_2),\ \varphi_3(v_3)),\cdot) \\ +\xi((\varphi_1(u_1),\ \varphi_2(v_2),\ \varphi_3(v_3)),\cdot)+\xi((\varphi_1(v_1),\ \varphi_2(v_2),\ \varphi_3(v_3)),\cdot) \end{bmatrix},\text{(a.e.)}$$

Indeed, according to *Theorem 8* for $n = 3$, we get:

$$\xi^3_{\varphi_3(t_3)}\left(\varphi_3(t_3),\cdot\right) := \xi\left(\left(\varphi_1(t_1),\varphi_2(t_2),\varphi_3(t_3)\right),\cdot\right) := \xi\left(\left(t_1,t_2,t_3\right),\cdot\right)$$

$$l_i\left(3\right) := \left\{\boldsymbol{\delta}\in\mathbb{N}_0^3 : \delta_i \leq 1, \left|\boldsymbol{\delta}\right| = 4-i\right\}, i = 1,2,3,4$$

then

$$l_1(3) = \{(1,1,1)\};\ l_2(3) = \{(0,1,1),(1,0,1),\ (1,1,0)\},$$
$$l_3(3) = \{(0,0,1),(0,1,0),\ (1,0,0),\};\ l_4(3) = \{(0,0,0)\}$$

therefore

$$\xi\left(\left(\varphi_1(u_1),\varphi_2(u_2),\varphi_3(u_3)\right),\cdot\right)$$
$$= \xi\left(\begin{pmatrix} (1,1,1)\left(\varphi_1(u_1),\varphi_2(u_2),\varphi_3(u_3)\right) \\ +\left[(1,1,1)-(1,1,1)\right]\left(\varphi_1(v_1),\varphi_2(v_2),\varphi_3(v_3)\right) \end{pmatrix},\cdot\right)$$

for

$$\boldsymbol{\varphi}(\boldsymbol{u}) = \left(\varphi_1(u_1),\varphi_2(u_2),\varphi_3(u_3)\right),\ \boldsymbol{\varphi}(\boldsymbol{v}) = \left(\varphi_1(v_1),\varphi_2(v_2),\varphi_3(v_3)\right)$$

thus

$$\sum_{\boldsymbol{\delta}\in l_1(3)}\xi\left(\boldsymbol{\delta\varphi}(\boldsymbol{u})+(1-\boldsymbol{\delta})\boldsymbol{\varphi}(\boldsymbol{v}),\cdot\right)=\xi\left(\left(\varphi_1(u_1),\varphi_2(u_2),\varphi_3(u_3)\right),\cdot\right)$$

Similarly by l_2 (3), l_3 (3), and l_4 (3), respectively, we get almost everywhere:

$$\sum_{\boldsymbol{\delta}\in l_2(3)}\xi\left(\boldsymbol{\delta\varphi}(\boldsymbol{u})+(1-\boldsymbol{\delta})\boldsymbol{\varphi}(\boldsymbol{v}),\cdot\right)=\xi\left(\left(\varphi_1(u_1),\varphi_2(v_2),\varphi_3(v_3)\right),\cdot\right)+$$

$$+\xi\left(\left(\varphi_1(u_1),\varphi_2(v_2),\varphi_3(u_3)\right),\cdot\right)+\xi\left(\left(\varphi_1(u_1),\varphi_2(u_2),\varphi_3(v_3)\right),\cdot\right);$$

$$\sum_{\boldsymbol{\delta}\in l_3(3)}\xi\left(\boldsymbol{\delta\varphi}(\boldsymbol{u})+(1-\boldsymbol{\delta})\boldsymbol{\varphi}(\boldsymbol{v}),\cdot\right)=\xi\left(\left(\varphi_1(v_1),\varphi_2(v_2),\varphi_3(u_3)\right),\cdot\right)+$$

$$+\xi\left(\left(\varphi_1(v_1),\varphi_2(u_2),\varphi_3(v_3)\right),\cdot\right)+\xi\left(\left(\varphi_1(u_1),\varphi_2(v_2),\varphi_3(v_3)\right),\cdot\right);$$

$$\sum_{\boldsymbol{\delta}\in l_4(3)}\xi\left(\boldsymbol{\delta\varphi}(\boldsymbol{u})+(1-\boldsymbol{\delta})\boldsymbol{\varphi}(\boldsymbol{v}),\cdot\right)=\xi\left(\left(\varphi_1(v_1),\varphi_2(v_2),\varphi_3(v_3)\right),\cdot\right)$$

Finally, using all of the above equalities in eq. (7), we obtain the desired result in this example.

Conclusion

The fundamental contribution of this study to literature has been the introduction of general convexity for multidimensional stochastic processes principally. Besides, some characteristics of these processes was indicated specifically. Finally, some Hermite-Hadamard type integral inequalities for these processes were obtained mathematically. We hope that original results can be obtained for different processes using the methods in this study.

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References

[1] Hadamard, J., Étude sur les propriétés des fonctions entières et en particulier d.une function considerée par Riemann (in French), *Journal Math Pures Appl*., *58* (1893), pp. 171-215
[2] Youness, E. A., E-Convex Sets, E-Convex Functions and E-Convex Programming, *Journal Optimiz. Theory App*., *102* (1999), 2, pp. 439-450
[3] Sarikaya, M. Z., *et al*., On Some Generalized Integral Inequalities For Φ-Convex Functions, *Studia Univ. Babeş-Bolyai Mathematica, 60* (2015), 3, pp. 367–377
[4] Syau, Y. R., Lee, E. S., Some Properties of E-Convex Functions, *Appl. Math. Lett*., *18* (2005), 9, pp. 1074-1080
[5] Martinez-Legaz, J. E., Singer, I., On *φ*-Convexity of Convex Functions, *Linear Algebra and its Applications*, *278* (1998), 1-3, pp. 163-181
[6] Shaikh A. A., *et al*., Some Results on *φ*-Convex Functions and Geodesic Φ-Convex Functions, *Differential Geometry-Dynamical Systems*, *20* (2018), June, pp. 159-169
[7] Dragomir, S. S., Inequalıtıes of Jensen Type for *φ*-Convex Functions, *Fasciculi Mathematici, 55* (2015), 1, pp.35-52
[8] Cristescu, G., Hadamard-Type Inequalities for *φ*-Convex Functions, *Annals of the University of Oradea*, *Fascicle of Man. and Tech. Eng., CD-Rom Edition*, *12* (2004), 3

[9] Set, E., *et al*., Hadamard Type Inequalities for φ-Convex Functions on the Co-ordinates, *Tbilisi Mathematical Journal*, *7* (2014), 2, pp. 51-60
[10] De la Cal, J., Carcamo J., Multidimensional Hermite-Hadamard Inequalities and the Convex Order, *Journal of Mathematical Analysis and Applications*, *324* (2006), 1, pp. 248-261
[11] Ellahi, H., *et al*., Hadamard's Inequality for s-Convex Function on *n*-Co-ordinates, *Proceedings*, 1st ICAM, Attock, Pakistan, 2015
[12] Viloria, J. M., Cortez, M. V., Hermite-Hadamard Type Inequalities for Harmonically Convex Functions on *n*-Co-ordinates, *Appl. Math. Inf. Sci. Lett*., *6* (2018), 2, pp. 1-6
[13] Nikodem, K., On Convex Stochastic Processes, *Aequat. Math*., 20 (1980), 1, pp. 184-197
[14] Shaked, M., Shanthikumar, J. G., Stochastic Convexity and Its Applications, *Advances in Applied Probability*, *20* (1988), 2, pp. 427-446
[15] Skowronski, A., On Some Properties of J-Convex Stochastic Processes, *Aequat. Math*., *44* (1992), 2-3, pp. 249-258
[16] Kotrys, D., Hermite-Hadamard Inequality for Convex Stochastic Processes, *Aequat. Math*., *83* (2012), 1-2, pp. 143-151
[17] Set, E., *et al*., Hermite-Hadamard Inequalities for Co-ordinates Convex Stochastic Processes, *Mathematica Aeterna*, *5* (2015), 2, pp. 363-382
[18] Sarikaya, M. Z., *et al*., Hermite-Hadamard Type Inequality for φ_h-Convex Stochastic Processes, *Proceedings*, AIP Conference, *1726* (2016), 020076, 1
[19] Karahan, V., Okur, N., Hermite-Hadamard Type Inequalities for Convex Stochastic Processes On *n*-Co-ordinates, *Turk. J. Math. Comput. Sci., 10* (2018), May, pp. 256-262

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