GENERAL SOLUTIONS FOR THE MIXED BOUNDARY VALUE PROBLEM ASSOCIATED TO HYDROMAGNETIC FLOWS OF A VISCOUS FLUID BETWEEN SYMMETRICALLY HEATED PARALLEL PLATES

by

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Exact general solutions for hydromagnetic flows of an incompressible viscous fluid between two horizontal infinite parallel plates are established when the upper plate is fixed and the inferior one applies a time-dependent shear stress to the fluid. Porous effects are taken into consideration and the problem in discussion is completely solved for moderate values of the Hartman number. It is found that the fluid velocity and the non-trivial shear stress satisfy PDE of the same form and the motion characteristics do not depend of magnetic and porous parameters independently but only by a combination of them that is called the effective permeability. For illustration, as well as to bring to light some physical insight of results that have been obtained, three special cases are considered and the influence of Reynolds number as well as combined porous and magnetic effects on the fluid motion are graphically underlined and discussed for motions due to constant or ramped-type shear stresses on the boundary. The starting solutions corresponding to motions induced by the lower plate that applies constant or oscillatory shear stresses to the fluid are presented as sum of steady-state and transient solutions and the required time to reach the steady-state is graphically determined. This time is greater for motions due to sine as compared to cosine oscillating shear stresses on the boundary. The steady-state is rather obtained in the presence of a magnetic field or porous medium.

Key words: *viscous fluid, parallel plates, general solutions, mixed boundary value problem*

Introduction

The fluid-flow between parallel plates is a classical fluid mechanics problem having many applications like polymer processing, power transmission equipment, transient loading of mechanical components and many others. It can be generated, for instance, by one of plates that is moving in its plane or applies a shear stress to the fluid. The first exact solutions corresponding to the motions between two horizontal parallel plates induced by the upper plate that is moving in its plane, more exactly for the plane Couette flow between parallel plates, seem to be those of Schlichting [1] and Sinha and Choudhary [2] without, respectively with suction. Other interesting solutions for the same motions of incompressible viscous fluids have been

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established by Rajagopal [3] and Erdogan [4]. An extension of the previous solutions has been provided by Mendiburu *et al.* [5] when a constant or time-dependent pressure gradient is taken into consideration.

Unsteady hydromagnetic Couette flows between parallel plates have been extensively studied due to their multiple applications such as geothermal systems, MHD generators, polymer technology, petroleum industry, nuclear reactors*, etc.* The interaction between the electrical conducting fluid and the magnetic field induces effects with important applications in physics, chemistry, engineering, horticulture and hydrology. Exact solutions for the MHD Couette flow of viscous fluids between two infinite parallel plates have been obtained by Singh and Okwoyo [6]. The influence of a periodic pressure gradient on the unsteady MHD flow has been early obtained by Singh and Ram [7] and recently, Kiema *et al.* [8] found exact solutions for the steady MHD Couette flow between parallel plates using Sumudu transform. A numerical study of the unsteady MHD flow with magnetic field lines fixed relative to the moving upper plate has been presented by Onyango *et al.* [9].

In the same time, flows of incompressible viscous fluids through porous media have received an increasing attention due to their practical applications in geophysical and astrophysical studies, agricultural engineering, petroleum industries and oil reservoir technology. A numerical study of the fluid motion through horizontal channels of porous media has been presented by Al-Hadhrami [10]. The influence of porous medium on the unsteady MHD Couette flow of incompressible viscous fluids between two infinite parallel plates could be also brought to light, for instance, from the results of Kesavaiah *et al.* [11], Venkateswarlu *et al.* [12], Das and Ojha [13] and a part of their references. Unfortunately, a two parameter approach that is used in their graphical representations regarding the effects of magnetic and porous parameters is superfluous or even misleading [14, 15].

Furthermore, in previousmentioned papers as well as in many others from the literature are solved motion problems with velocity on the boundary although in many practical problems the shear stress is given on the boundary [16]. Actually, in Newtonian mechanics force is the cause and kinematics is the effect [17] and the no slip condition may not be applicable for flows of polymeric fluids which can slide on the boundary. Consequently, boundary conditions on stresses are significant and Renardy [18] showed how well-posed boundary value problems can be formulated. Unsteady MHD natural-convection boundary-layer flow of a viscous, incompressible, and electrically conducting dusty fluid past an impulsively moving vertical plate with ramped temperature in the presence of thermal radiation and transverse magnetic field was studied by Nandkeolyar *et al.* [19]. Prakash *et al.* [20] studied the combined effects of thermal radiation, buoyancy force and magnetic field on oscillatory flow of a conducting optically thin dusty fluid through a vertical channel filled with a saturated porous medium. Das *et al.* [21, 22] investigated the fully developed mixed convective flow in a vertical channel filled with nanofluids ian the presence of thermal radiation and transverse magnetic field. The transient natural-convection in a vertical channel filled with nanofluids was studied by Das *et al.* [23] and thermal radiation was taken into consideration. To the best of our knowledge, exact solutions for mixed boundary value problems corresponding to unsteady MHD flows between horizontal parallel plates do no exist or are rare in the existing literature.

The main purpose of this note is to provide general solutions for the unsteady MHD flow of an incompressible viscous fluid between two infinite horizontal parallel plates when the upper plate is fixed and the lower one applies a time-dependent shear stress to the fluid. For completion, the porous effects are taken into consideration and it is found that the fluid velocity and the corresponding non-trivial shear stress satisfy partial differential equations of the same form and the motion characteristics do not depend on magnetic and porous effects independently but only by a combination of them that is called the effective permeability. Consequently, a two parameter approach as usually used in the literature is superfluous or even misleading. In order to bring to light some physical insight of results which have been obtained, three special cases with engineering applications are considered and combined magnetic and porous effects on the motions due to constant or ramp-type shear stresses on the boundary are graphically depicted and discussed. The starting solutions are presented as sum of steady-state (permanent) and transient components, and the required time to reach the steady-state is graphically determined.

Statement of the problem

Let us consider an electrically conducting incompressible viscous fluid at rest between two infinite horizontal parallel plates in the presence of a porous medium and of a uniform magnetic field of strength, B, acting normal to the plates as shown in fig. 1. At the moment $t = 0^+$ the inferior plate applies a time-dependent shear stress $-Sf(t)$ to the fluid. Here, S is a constant shear stress, the dimensionless function $f(\cdot)$ is piecewise continuous and $f(0) = 0$. Due to the shear the fluid is gradually moved and its velocity has the form [3, 4]:

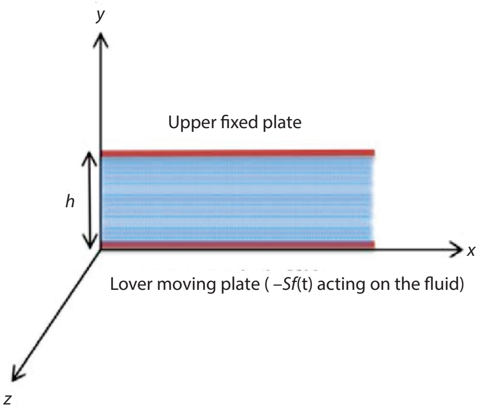

Figure 1. Flow geometry

$$v = v(y,t) = [v(y,t), 0, 0] \tag{1}$$

in a suitable Cartesian co-ordinate system x, y, and z. For such motions, the continuity equation is identically satisfied.

We also assume there is no surplus electric charge distribution present in the fluid and Hall effects are not significant due to the moderate values of Hartman number. In these conditions, the constitutive equation of incompressible viscous fluids and the motion equations reduce to the relevant partial differential equations:

$$\tau(y,t) = \mu \frac{\partial v(y,t)}{\partial y}, \quad \rho \frac{\partial v(y,t)}{\partial t} = \frac{\partial \tau(y,t)}{\partial y} - \sigma B^2 v(y,t) - \frac{\mu}{k} v(y,t) \tag{2}$$

where $\tau(y, t)$ is the non-trivial shear stress, ρ and μ are the density, respectively the viscosity of the fluid, σ – the electrically conductivity, and k – the permeability of porous medium. The corresponding initial and boundary conditions:

$$v(y,0) = 0, \quad \tau(y,0) = 0, \quad 0 \le y \le h \tag{3}$$

$$\tau(0,t) = \mu \left. \frac{\partial v(y,t)}{\partial y} \right|_{y=0} = -Sf(t), \quad v(h,t) = 0, \quad t \ge 0 \tag{4}$$

where h is the distance between plates. In order to determine solutions which are independent of the flow geometry, we use the next non-dimensional variables and functions:

$$y^* = \frac{y}{h}, \; t^* = \frac{S}{\mu} t, \; v^* = \frac{\mu}{hS} v, \; \tau^* = \frac{\tau}{S}, \; f^*(t^*) = f\left(\frac{\mu}{S} t^*\right) \tag{5}$$

Introducing eq. (5) in eqs. (2)-(4) and dropping out the star notation we obtain the following dimensionless initial and boundary value problem:

$$\tau(y,t) = \frac{\partial v(y,t)}{\partial y}, \quad \mathrm{Re}\frac{\partial v(y,t)}{\partial t} = \frac{\partial \tau(y,t)}{\partial y} - K_{\mathrm{eff}} v(y,t) \tag{6}$$

$$v(y,0) = 0, \quad \tau(y,0) = 0, \quad 0 \le y \le 1 \tag{7}$$

$$\left.\frac{\partial v(y,t)}{\partial y}\right|_{y=0} = -f(t), \quad v(1,t) = 0, \quad t \ge 0 \tag{8}$$

where $\mathrm{Re} = Vh/\nu$ is the Reynolds number, $V = \mathrm{S}h/\mu$ – the characteristic velocity, $K_{\mathrm{eff}} = M + 1/K$ – the called the effective permeability [24] while:

$$M = \frac{\sigma B^2}{\mu} h^2 \quad \text{and} \quad K = \frac{k}{h^2} \tag{9}$$

are the non-dimensional magnetic and porous parameters. Eliminating $\tau(y,t)$ or $v(y,t)$ between eqs. (6), we get the governing equations:

$$\mathrm{Re}\frac{\partial v(y,t)}{\partial t} = \frac{\partial^2 v(y,t)}{\partial y^2} - K_{\mathrm{eff}} v(y,t) \quad \text{or} \quad \mathrm{Re}\frac{\partial \tau(y,t)}{\partial t} = \frac{\partial^2 \tau(y,t)}{\partial y^2} - K_{\mathrm{eff}} \tau(y,t) \tag{10}$$

for the fluid velocity, respectively the corresponding shear stress $\tau(y, t)$. Equations (10) clearly show that the fluid velocity $v(y, t)$ and the adequate shear stress $\tau(y, t)$, as well as in the case of unidirectional motions on an infinite plate [25, 26], satisfy PDE which are identical as form. This result, which is also true in the dimensional case, is of a fundamental importance. It allow us, for instance, to get exact solutions for motions generated by the two parallel plates that applies shear stresses to the fluid if the solutions corresponding to motions due to the plates which are moving in their planes are known. As an example, we consider the flow of incompressible viscous fluids between two infinite horizontal parallel plates that applies the same constant shear stress, S, to the fluid in the absence of magnetic and porous effects. Assuming again that the whole system is at rest at the moment $t = 0$ and bearing in mind the dimensional form of eq. (10-2) and the result of Erdogan [4] from eq. (12), it results that the distribution of the dimensional shear stress in the flow domain of this motion is given by the equality:

$$\tau(y,t) = S\left\{1 - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cos\left[\frac{(2n+1)\pi}{2h}y\right]\exp\left[-\frac{(2n+1)^2\pi^2}{4h^2}\nu t\right]\right\}, \quad -h \le y \le h, t \ge 0$$

if $2h$ is the distance between plates and the origin of the co-ordinate system is at the middle of the distance between them. The corresponding velocity field:

$$v(y,t) = \frac{8Sh}{\mu\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\left[\frac{(2n+1)\pi}{2h}y\right]\left\{1 - \exp\left[-\frac{(2n+1)^2\pi^2}{4h^2}\nu t\right]\right\}, \quad -h \le y \le h, t \ge 0$$

is immediately obtained substituting the expression of the shear stress in eq. (2) in which porous and magnetic effects are neglected. Direct computations clearly show that, with the view to the identity (A1) from *Appendix*, the governing equations and all imposed initial and boundary conditions are satisfied. Furthermore, coming back to our initial problem, eq. (10) clearly shows that the fluid velocity as well as the shear stress do not depend on the parameters M and K independently, but only by a combination of them that is called the effective permeability. Consequently, the study of MHD flows of incompressible viscous fluids in such motions with/

without porous effects is exactly the same problem and a two parameter approach is superfluous or even misleading. More exactly, this problem admits the same solution for an infinite set of values of parameters M and K which correspond to the same value of the effective permeability K_{eff}. In the following, the partial differential eq. (10-1) together with the conditions (7-1) and (8) will be solved using the finite Fourier cosine transform.

Solution of the problem

Multiplying eq. (10-1) by $\cos(\mu_n y)$, where $\mu_n = (2n + 1)\pi/2$, integrating the result with respect to y from zero to one and using the identity:

$$\int_0^1 \frac{\partial^2 v(y,t)}{\partial y^2}\cos(\mu_n y)\mathrm{d}y = -\mu_n^2\, v_{Fn}(t) - \left.\frac{\partial v(y,t)}{\partial y}\right|_{y=0} + (-1)^n \mu_n v(1,t) \tag{11}$$

and the initial and boundary conditions (7-1) and (8), we find:

$$\frac{\mathrm{d}v_{Fn}(t)}{\mathrm{d}t} + \frac{\mu_n^2 + K_{\text{eff}}}{\text{Re}} v_{Fn}(t) = \frac{f(t)}{\text{Re}}, \quad t > 0 \tag{12}$$

where the finite Fourier cosine transform:

$$v_{Fn}(t) = \int_0^1 v(y,t)\cos(\mu_n y)\mathrm{d}y \tag{13}$$

of $v(y, t)$ has to satisfy the condition:

$$v_{Fn}(0) = 0, \quad n = 0,1,2,3... \tag{14}$$

Solving the ODE eq. (12) with the initial condition (14) and applying the inverse Fourier cosine transform, see eq. (A2) from *Appendix*, we find for the dimensionless velocity field the expression:

$$v(y,t) = \frac{2}{\text{Re}}\sum_{n=0}^{\infty}\cos(\mu_n y)\int_0^t f(t-s)\exp\left(-\frac{\mu_n^2 + K_{\text{eff}}}{\text{Re}}s\right)\mathrm{d}s \tag{15}$$

or equivalently:

$$v(y,t) = 2f(t)\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{\mu_n^2 + K_{\text{eff}}} - 2\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{\mu_n^2 + K_{\text{eff}}}\int_0^t f'(t-s)\exp\left(-\frac{\mu_n^2 + K_{\text{eff}}}{\text{Re}}s\right)\mathrm{d}s \tag{16}$$

The solutions (15) and (16) clearly satisfy the initial and boundary conditions (7-1) and (8-2). However, under these forms, the boundary condition (8-1) seems to be unsatisfied. In order to avoid this drawback, we use eq. (A3) from *Appendix* and write $v(y, t)$ in the equivalent but suitable form:

$$\begin{aligned} v(y,t) = (1-y)f(t) - 2K_{\text{eff}} f(t)\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{\mu_n^2(\mu_n^2 + K_{\text{eff}})} - \\ -2\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{\mu_n^2 + K_{\text{eff}}}\int_0^t f'(t-s)\exp\left(-\frac{\mu_n^2 + K_{\text{eff}}}{\text{Re}}s\right)\mathrm{d}s \end{aligned} \tag{17}$$

which clearly satisfies all imposed initial and boundary conditions. The corresponding non-trivial shear stress $\tau(y, t)$ can be immediately obtained introducing anyone of eqs. (15)-(17) in (6-1). Making $M = 0$ or $K \to \infty$ into aforementioned relations, the solution corresponding to the motion through a porous medium, respectively the solution corresponding to the MHD flow is obtained. In the absence of magnetic and porous effects eq. (17), for instance, reduces:

$$v(y,t)=(1-y)f(t)-2\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{\mu_n^2}\int_0^t f'(t-s)\exp\left(-\frac{\mu_n^2}{\mathrm{Re}}s\right)\mathrm{d}s \tag{18}$$

The dimensionless frictional force $\tau(t)$ per unit area exerted by the fluid on the stationary plate, as it results from eqs. (6-1) and (17), is given:

$$\tau_s(t)=\left.\frac{\partial v(y,t)}{\partial y}\right|_{y=1}=-f(t)+2K_{\mathrm{eff}}f(t)\sum_{n=0}^{\infty}\frac{(-1)^n}{\mu_n(\mu_n^2+K_{\mathrm{eff}})}+$$
$$+2\sum_{n=0}^{\infty}\frac{(-1)^n\mu_n}{\mu_n^2+K_{\mathrm{eff}}}\int_0^t f'(t-s)\exp\left(-\frac{\mu_n^2+K_{\mathrm{eff}}}{\mathrm{Re}}s\right)\mathrm{d}s \tag{19}$$

while the volume flux $Q(t)$ per unit width of a plane normal to the flow:

$$Q(t)=\int_0^1 v(y,t)\mathrm{d}y=\frac{f(t)}{2}-2K_{\mathrm{eff}}f(t)\sum_{n=0}^{\infty}\frac{(-1)^n}{\mu_n^3(\mu_n^2+K_{\mathrm{eff}})}-$$
$$-2\sum_{n=0}^{\infty}\frac{(-1)^n}{\mu_n(\mu_n^2+K_{\mathrm{eff}})}\int_0^t f'(t-s)\exp\left(-\frac{\mu_n^2+K_{\mathrm{eff}}}{\mathrm{Re}}s\right)\mathrm{d}s \tag{20}$$

Finally, it is worth pointing out the fact that the obtained expressions for the velocity field $v(y, t)$ can generate exact solutions for any motion with technical relevance of this type. Consequently, the problem in discussion is completely solved. For illustration, as well as to bring to light some physical insight of results that have been obtained, three special cases with engineering applications will be here considered.

f(t) = H(t) (Constant shear stress on the boundary)

By substituting $f(t)$ by $H(t)$ (the Heaviside unit step function) in anyone of eqs. (15)-(18), we get the velocity field corresponding to the motion induced by the lower plate that applies a constant shear stress to the fluid. eqs. (17) and (18), for instance, take the simplified forms, see also the property (A4), from *Appendix*:

$$v(y,t)=1-y-2K_{\mathrm{eff}}\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{\mu_n^2(\mu_n^2+K_{\mathrm{eff}})}-2\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{\mu_n^2+K_{\mathrm{eff}}}\exp\left(-\frac{\mu_n^2+K_{\mathrm{eff}}}{\mathrm{Re}}t\right) \tag{21}$$

$$v(y,t)=1-y-2\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{\mu_n^2}\exp\left(-\frac{\mu_n^2}{\mathrm{Re}}t\right) \tag{22}$$

whose steady (permanent) components:

$$v_p(y)=1-y-2K_{\mathrm{eff}}\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{\mu_n^2(\mu_n^2+K_{\mathrm{eff}})}\quad\text{and}\quad v_p(y)=1-y \tag{23}$$

The shear stress distribution in the flow domain is immediately obtained introducing anyone of previous relations in eq. (6-1). Equations (19) and (20) also take the simpler forms:

$$\tau_s(t)=-1+2K_{\mathrm{eff}}\sum_{n=0}^{\infty}\frac{(-1)^n}{\mu_n(\mu_n^2+K_{\mathrm{eff}})}+2\sum_{n=0}^{\infty}\frac{(-1)^n\mu_n}{\mu_n^2+K_{\mathrm{eff}}}\exp\left(-\frac{\mu_n^2+K_{\mathrm{eff}}}{\mathrm{Re}}t\right) \tag{24}$$

$$Q(t)=\frac{1}{2}-2K_{\mathrm{eff}}\sum_{n=0}^{\infty}\frac{(-1)^n}{\mu_n^3(\mu_n^2+K_{\mathrm{eff}})}-2\sum_{n=0}^{\infty}\frac{(-1)^n}{\mu_n(\mu_n^2+K_{\mathrm{eff}})}\exp\left(-\frac{\mu_n^2+K_{\mathrm{eff}}}{\mathrm{Re}}t\right) \tag{25}$$

which tend to the asymptotic solutions:

$$\tau_{sp}=\tau_s(\infty)=-1+2K_{\mathrm{eff}}\sum_{n=0}^{\infty}\frac{(-1)^n}{\mu_n(\mu_n^2+K_{\mathrm{eff}})},\quad Q_p=Q(\infty)=\frac{1}{2}-2K_{\mathrm{eff}}\sum_{n=0}^{\infty}\frac{(-1)^n}{\mu_n^3(\mu_n^2+K_{\mathrm{eff}})} \tag{26}$$

when t tends to infinity. The fluid velocity $v(y, t)$, as well as the frictional force $\tau_s(t)$ and the volume flux $Q(t)$, given by eqs. (21), (24), and (25), as it was to be expected, are zero at the initial moment $t = 0$, see eqs. (A3) and (A5) from *Appendix*. Direct computations easy show that the steady component $v_p(y)$ and the asymptotic shear stress $\tau_s(\infty)$ given by eqs. (23-1) and (26-1) can be written in the equivalent forms:

$$v_p(y)=\frac{1}{\sqrt{K_{\mathrm{eff}}}}\frac{\sinh\left[(1-y)\sqrt{K_{\mathrm{eff}}}\right]}{\cosh\left(\sqrt{K_{\mathrm{eff}}}\right)},\quad \tau_{sp}=-\frac{1}{\cosh\left(\sqrt{K_{\mathrm{eff}}}\right)} \tag{27}$$

which in the absence of magnetic and porous effects (when $K_{\mathrm{eff}} \to 0$) reduce:

$$v_p(y)=1-y,\quad \tau_{sp}=-1 \tag{28}$$

For validation, the equivalence of eqs. (27-1) and (27-2) with eq. (23-1), respectively, (26-1) is proved by fig. 2.

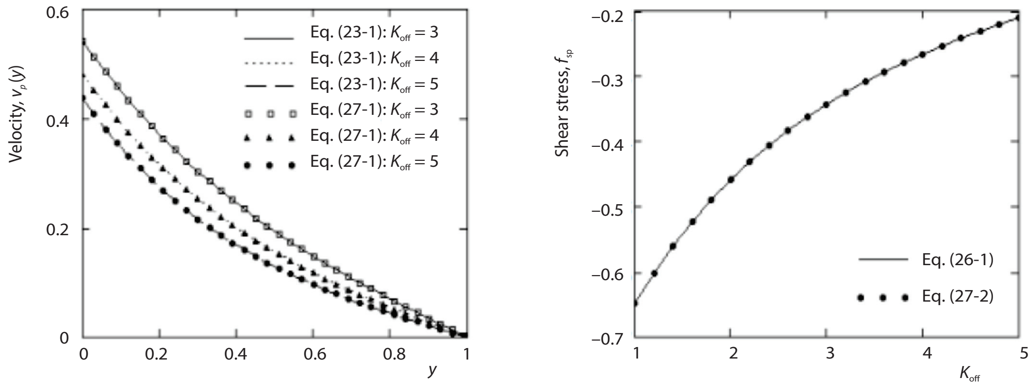

Figure 2. Profiles of the velocity $v_p(y)$ given by eqs. (23-1) and (27-1) against y and the shear stress τ_{sp} given by eqs. (26-1) and (27-2) as a function of K_{eff}

f(t) = tH(t) (Ramp-type shear stress on the boundary)

The velocity field corresponding to the motion induced by the inferior plate that applies a ramp-type shear stress [27] to the fluid, as it results from eqs. (17) and (18) are given:

$$v(y,t)=(1-y)t-2tK_{\mathrm{eff}}\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{\mu_n^2(\mu_n^2+K_{\mathrm{eff}})}-2\mathrm{Re}\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{(\mu_n^2+K_{\mathrm{eff}})^2}\left[1-\exp\left(-\frac{\mu_n^2+K_{\mathrm{eff}}}{\mathrm{Re}}t\right)\right] \tag{29}$$

respectively

$$v(y,t)=(1-y)t-2Re\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{\mu_n^4}\left[1-\exp\left(-\frac{\mu_n^2}{Re}t\right)\right] \tag{30}$$

This motion is unsteady and remains unsteady and similar solutions with ramp-type shear stress on the boundary are obtained in [21], eqs. (28) and (29) . The frictional force on the stationary plate and the volume flux, as it results from eqs. (19) and (20):

$$\tau_s(t) = -t + 2tK_{\text{eef}}\sum_{n=0}^{\infty}\frac{(-1)^n}{\mu_n(\mu_n^2+K_{\text{eff}})} + 2\text{Re}\sum_{n=0}^{\infty}\frac{(-1)^n\mu_n}{(\mu_n^2+K_{\text{eff}})^2}\left[1-\exp\left(-\frac{\mu_n^2+K_{\text{eff}}}{\text{Re}}t\right)\right] \tag{31}$$

$$Q(t) = \frac{t}{2} - 2tK_{\text{eff}}\sum_{n=0}^{\infty}\frac{(-1)^n}{\mu_n^3(\mu_n^2+K_{\text{eff}})} - 2\text{Re}\sum_{n=0}^{\infty}\frac{(-1)^n}{\mu_n(\mu_n^2+K_{\text{eff}})^2}\left[1-\exp\left(-\frac{\mu_n^2+K_{\text{eff}}}{\text{Re}}t\right)\right] \tag{32}$$

Denoting by $v_0(y, t)$, $\tau_0(y, t)$, and $v_1(y, t)$, $\tau_1(y, t)$, the dimensionless velocity and shear stress fields corresponding to the two problems (with constant, respectively ramp-type shear stress on the boundary), it is easy to observe:

$$v_1(y,t) = \int_0^t v_0(y,s)\text{d}s,\ \ \tau_1(y,t) = \int_0^t \tau_0(y,s)\text{d}s \tag{33}$$

Furthermore, it is not difficult to show:

$$\begin{aligned} v_n(y,t) &= (n!)\int_0^t\int_0^{s_1}\int_0^{s_2}\dots\int_0^{s_{n-1}} u_0(y,s_n)\text{d}s_1\text{d}s_2\dots\text{d}s_n \\ \tau_n(y,t) &= (n!)\int_0^t\int_0^{s_1}\int_0^{s_2}\dots\int_0^{s_{n-1}} \tau_0(y,s_n)\text{d}s_1\text{d}s_2\dots\text{d}s_n \end{aligned} \tag{34}$$

where $v_n(y, t)$, $\tau_n(y, t)$ are the dimensionless velocity and shear stress fields corresponding to motions due to the lower plate that applies shear stresses of the form $t^nH(t)$ to the fluid.

f(t) = H(t) cos(ωt) or H(t) sin(ωt)
(Oscillatory shear stresses on the boundary)

Replacing $f(t)$ by $H(t)\cos(\omega t)$ or $H(t)\sin(\omega t)$ in eq. (17) and bearing in mind the property (A4) regarding the Dirac delta function $\delta(\cdot)$, we find for the velocity field the expressions:

$$\begin{aligned} v_c(y,t) &= (1-y)\cos(\omega t) - 2K_{\text{eff}}\cos(\omega t)\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{\mu_n^2(\mu_n^2+K_{\text{eff}})} - 2\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{\mu_n^2+K_{\text{eff}}}\cdot \\ &\cdot\exp\left(-\frac{\mu_n^2+K_{\text{eff}}}{\text{Re}}t\right) + 2\omega\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{\mu_n^2+K_{\text{eff}}}\int_0^t\sin(\omega s)\exp\left[-\frac{\mu_n^2+K_{\text{eff}}}{\text{Re}}(t-s)\right]\text{d}s \end{aligned} \tag{35}$$

respectively

$$\begin{aligned} v_s(y,t) &= (1-y)\sin(\omega t) - 2K_{\text{eff}}\sin(\omega t)\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{\mu_n^2(\mu_n^2+K_{\text{eff}})} - \\ &- 2\omega\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{\mu_n^2+K_{\text{eff}}}\int_0^t\cos(\omega s)\exp\left[-\frac{\mu_n^2+K_{\text{eff}}}{\text{Re}}(t-s)\right]\text{d}s \end{aligned} \tag{36}$$

Evaluating the integrals from eqs. (35) and (36), the starting solutions $v_c(y, t)$ and $v_s(y, t)$ can be written as sums of steady-state (permanent) and transient solutions:

$$v_c(y,t) = v_{cp}(y,t) + v_{ct}(y,t),\quad v_s(y,t) = v_{sp}(y,t) + v_{st}(y,t) \tag{37}$$

where

$$v_{cp}(y,t) = (1-y)\cos(\omega t) - 2\cos(\omega t)\sum_{n=0}^{\infty}\frac{K_{\text{eff}}[(\mu_n^2+K_{\text{eff}})^2+(\omega\text{Re})^2]+(\omega\text{Re})^2\mu_n^2}{(\mu_n^2+K_{\text{eff}})^2+(\omega\text{Re})^2}\cdot$$
$$\cdot\frac{\cos(\mu_n y)}{\mu_n^2(\mu_n^2+K_{\text{eff}})} + 2\omega\text{Re}\sin(\omega t)\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{(\mu_n^2+K_{\text{eff}})^2+(\omega\text{Re})^2} \tag{38}$$

$$v_{ct}(y,t) = -2\sum_{n=0}^{\infty}\frac{(\mu_n^2+K_{\text{eff}})\cos(\mu_n y)}{(\mu_n^2+K_{\text{eff}})^2+(\omega\text{Re})^2}\exp\left(-\frac{\mu_n^2+K_{\text{eff}}}{\text{Re}}t\right) \tag{39}$$

$$v_{sp}(y,t) = (1-y)\sin(\omega t) - 2\omega\text{Re}\cos(\omega t)\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{(\mu_n^2+K_{\text{eff}})^2+(\omega\text{Re})^2} -$$
$$-2\sin(\omega t)\sum_{n=0}^{\infty}\frac{K_{\text{eff}}[(\mu_n^2+K_{\text{eff}})^2+(\omega\text{Re})^2]+(\omega\text{Re})^2\mu_n^2}{(\mu_n^2+K_{\text{eff}})^2+(\omega\text{Re})^2}\frac{\cos(\mu_n y)}{\mu_n^2(\mu_n^2+K_{\text{eff}})} \tag{40}$$

$$v_{st}(y,t) = 2\omega\text{Re}\sum_{n=0}^{\infty}\frac{\cos(\mu_n y)}{(\mu_n^2+K_{\text{eff}})^2+(\omega\text{Re})^2}\exp\left(-\frac{\mu_n^2+K_{\text{eff}}}{\text{Re}}t\right) \tag{41}$$

Direct computations show that the steady-state components $v_{cp}(y,t)$ and $v_{sp}(y,t)$ of $v_c(y,t)$ and $v_s(y,t)$ can be presented in the simple but equivalents forms:

$$v_{cp}(y,t) = \text{Re}\left\{\frac{sh[\beta(1-y)]}{\beta\text{ch}(\beta)}\text{e}^{i\omega t}\right\},\quad v_{sp}(y,t) = Im\left\{\frac{sh[\beta(1-y)]}{\beta ch(\beta)}\text{e}^{i\omega t}\right\} \tag{42}$$

where

$$\beta = \sqrt{i\omega\text{Re}+K_{\text{eff}}}$$

Re and *Im* represent the real, respectively, the imaginary part of that which follows and *i* is the imaginary unit. Indeed, the equivalence of the corresponding steady-state solutions given by eqs. (38) and (42-1), respectively, (40) and (42-2) is graphically proved by figs. 2 and 3.

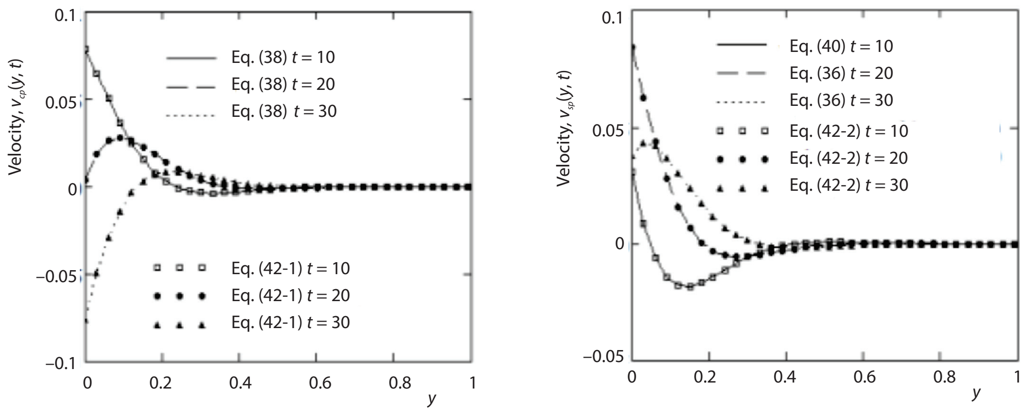

Figure 3. Profiles of the velocities $v_{cp}(y, t)$ and $v_{sp}(y, t)$ given by eqs. (38) and (42-1), respectively, (40) and (42-2) for for Re = 70, K_{eff} = 3, ω = 2, and different values of the time

Thermal transport

For the problem formulated in section *Statement of the problem* , the energy equation can be written [28, 29]:

$$\rho c_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{43}$$

where c_p is the specific heat, $T(y, t)$ – the fluid temperature, k – the thermal conductivity of fluid. The second term on the right side is the viscous dissipation term; it is the cause of a temperature rise in the flow. Along with eq. (43), we consider the following initial and boundary conditions:

$$T(y,0) = T_1,\ \ T(0,t) = T_1,\ \ T(h,t) = T_1,\ \ T_1 \neq 0 \tag{44}$$

Using the non-dimensional variables (5) and transformation:

$$\theta = \frac{T - T_1}{T_1} \tag{45}$$

Equations (43) and (44) become (neglecting the star notations):

$$\text{Pe}\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial y^2} + \text{Br}\left(\frac{\partial v}{\partial y}\right)^2 \tag{46}$$

$$\theta(y,0) = 0,\ \ \theta(0,t) = 0,\ \ \theta(h,t) = 1 \tag{47}$$

where Pe = Re, Pe is the Peclet number, Pr = ν/α – the Prandtl number, $\alpha = k/\rho c_p$ – the thermal diffusivity, and Br = $\mu V^2/kT_1$ – the Brinkman number we denote by $w(y, t)$ the function:

$$w(y,t) = \text{Br}\left(\frac{\partial v}{\partial y}\right)^2 \tag{48}$$

where

$$\frac{\partial v}{\partial y} = -f(t) + 2K_{\text{eff}} f(t)\sum_{n=0}^{\infty}\frac{\sin(\mu_n y)}{\mu_n\left(\mu_n^2 + K_{\text{eff}}\right)} + 2\sum_{n=0}^{\infty}\frac{\mu_n \sin(\mu_n y)}{\left(\mu_n^2 + K_{\text{eff}}\right)}\int_0^t f'(t-s)\exp\left(\frac{\mu_n^2 + K_{\text{eff}}}{\text{Re}}\right)\text{d}s \tag{49}$$

Applying the Laplace transformation eq. (46) and using the initial condition (47-1) we obtain:

$$\frac{\partial^2 \bar{\theta}(y,s)}{\partial y^2} = \text{Pe}s\bar{\theta}(y,s) - \bar{w}(y,s) \tag{50}$$

where

$$\bar{\chi}(y,s) = \int^{\infty} \chi(y,t)e^{\ st}\text{d}t$$

denotes the Laplace transform of the function $\chi(y, t)$. Function $\bar{\theta}(y, s)$ has to satisfy conditions:

$$\bar{\theta}(0,s) = 0,\ \ \bar{\theta}(1,s) = 0 \tag{51}$$

Using the finite sine Fourier transforms:

$$\tilde{\bar{\theta}}(k,s) = \int_0^1 \bar{\theta}(y,s)\sin(k\pi y)\text{d}y \tag{52}$$

into eq. (50) we obtain the transformed temperature:

$$\tilde{\bar{\theta}}(k,s) = \frac{1}{\text{Pe}s + k^2\pi^2}\tilde{\bar{w}}(k,s) \tag{53}$$

where

$$\tilde{w}(k,t) = \int_0^1 w(y,t)\sin(k\pi y)\mathrm{d}y = \int_0^1 \mathrm{Br}\left[\frac{\partial v(y,t)}{\partial y}\right]^2 \sin(k\pi y)\mathrm{d}y \tag{54}$$

Applying the inverse Laplace and Fourier transform we obtained the temperature field:

$$\theta(y,t) = 2\sum_{k=1}^{\infty}\tilde{\theta}(k,t)\sin(k\pi y) = 2\sum_{k=1}^{\infty}\sin(k\pi y)\frac{1}{\mathrm{Pe}}\int_0^t \mathrm{e}^{-\frac{k^2\pi^2}{\mathrm{Pe}}(t-\sigma)}\tilde{w}(k,\sigma)\mathrm{d}\sigma \tag{55}$$

Numerical results and conclusions

The hydromagnetic-flow problem of incompressible viscous fluids between two horizontal infinite parallel plates is analytically studied when the upper plate is stationary and the lower plate applies an arbitrary shear stress to the fluid. Exact expressions for the velocity field $v(y, t)$, the frictional force exerted by the fluid on the stationary plate $\tau_s(t)$ and the volume flux $Q(t)$ per unit width of a plane normal to the flow are obtained when porous effects are taken into consideration. It is worth pointing out the fact these characteristic entities do not depend of the magnetic and porous parameters M and K independently but only by a combination of them $K_{\mathrm{eff}} = M + 1/K$ which is called the effective permeability. As a result, the investigation of the hydromagnetic-flow of incompressible viscous fluids in such motions with/without porous effects is exactly the same problem and a two parameter approach is superfluous or even misleading.

For illustration, as well as to bring to light some physical insight of results that have been obtained, three special classes of motions with engineering applications are considered and the influence of Reynolds number, as well as the combined porous and magnetic effects on the fluid-flow, is graphically underlined in figs. 4-6 for motions due to constant or ramp-type shear stresses on the boundary. The starting solutions corresponding to motions induced by the lower plate that applies constant or oscillatory shear stresses to the fluid are written as sums of steady-state and transient solutions. In order to be sure of their correctness, in both cases, the steady-state solutions are presented in two different forms whose equivalence is graphically proved by figs. 2 and 3. The required time to reach the steady-state for motions induced by constant or oscillatory shear stresses on the boundary is graphically determined in figs. 7-10.

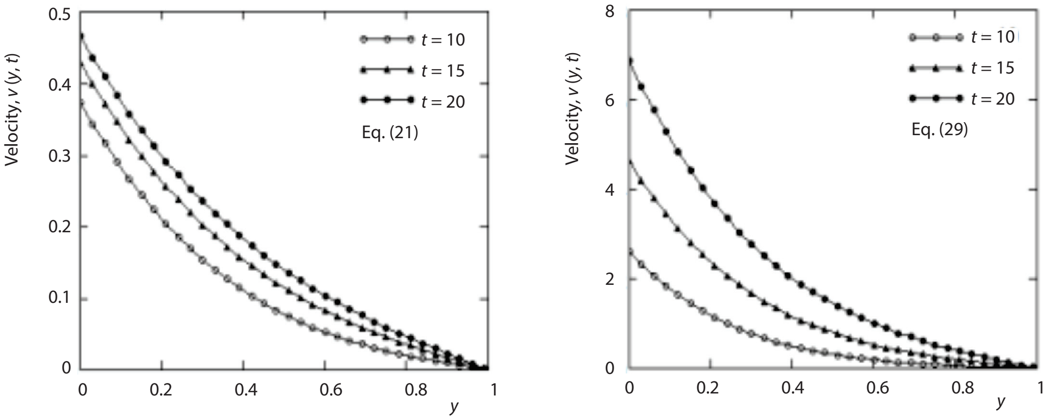

Figure 4. Profiles of the velocities $v(y, t)$ given by eqs. (21) and (29) for Re = 70, K_{eff} = 3, and different values of the time t

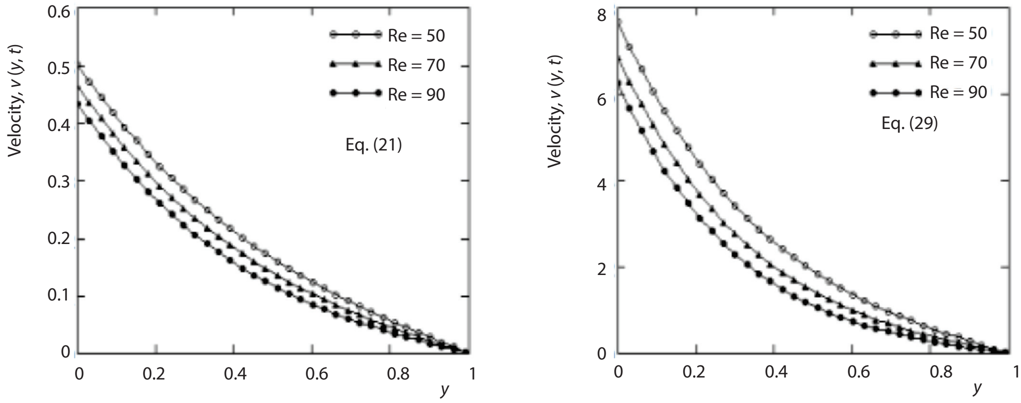

Figure 5. Profiles of the velocities $v(y, t)$ given by eqs. (21) and (29) for t = 2, K_{eff} = 3, and different values of the of the Reynolds number

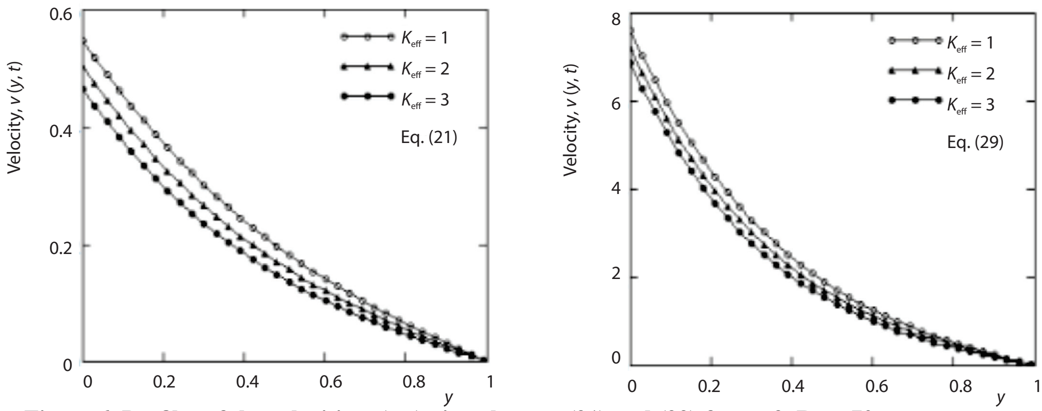

Figure 6. Profiles of the velocities $v(y, t)$ given by eqs. (21) and (29) for t = 2, Re = 70, and different values of the of the K_{eff}

This is the time after which the diagrams of starting solutions tend to superpose over those of the corresponding steady-state solutions. In practice, this time is important for those who want to eliminate the transients from their experiments.

In figs. 4-6, for comparison, the profiles of the dimensionless velocity $v(y, t)$ corresponding to motions induced by the lower plate that applies constant or ramp-type shear stresses to the fluid are presented for different values of the time, t, Reynolds number, and the effective permeability, K_{eff}. In all cases, the fluid velocity smoothly decreases from maximum values on the moving plate to the zero value on the stationary plate. It is an increasing function with respect to time and decreases for increasing values of Reynolds number or K_{eff}. As it was to be expected, the fluid velocity is appreciably greater for fluid motions due to the ramp-type shear stress on the boundary.

From fig. 6 it also results that the fluid velocity decreases in the presence of a magnetic field or of the porous medium because these graphical representations are also valid for M = 1, 2 or 3 in the absence of porous medium or for K = 1, 1/2 or 1/3 in the absence of the magnetic field. Consequently, the fluid velocity decreases for increasing values of M and increases for increasing values of K. However, such a conclusion is not quite opportune since the same graphical representations correspond to an infinite set of parameter values M and K that corresponds to the same effective permeability K_{eff}.

The flow due to the lower plate that applies a constant shear to the fluid becomes steady or permanent and the required time to reach the steady-state for such a motion is graphically determined in figs. 7 and 8 for two values of Reynolds number and three values of K_{eff}. This is the time after which the diagrams of the starting solution (21) are almost identical to those of steady-state solution (23). As it results from these figures, it is a decreasing function with respect to K_{eff} and increases for increasing values of Reynolds number. Consequently, the steady-state is rather obtained in the presence of porous medium or a magnetic field.

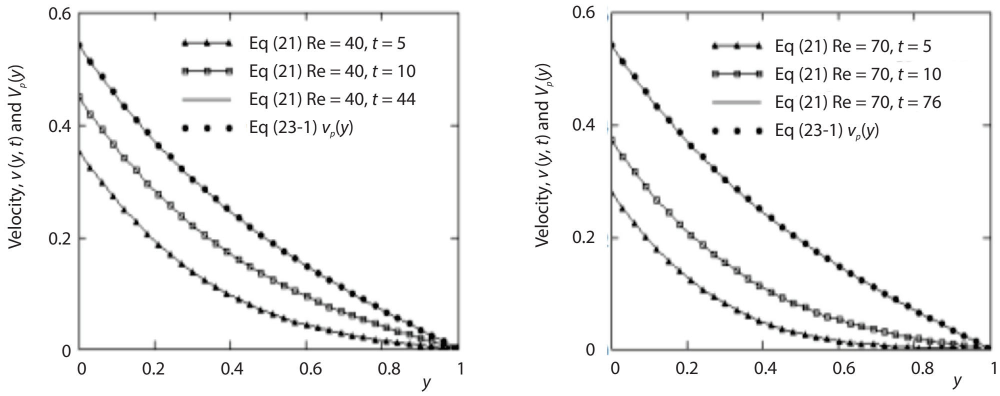

Figure 7. Required time to reach the steady-state for motion due to a constant shear on the boundary for K_{eff} = 3 and two values of the Reynolds number

The required time to reach the steady-state for motions induced by cosine or sine oscillating shear stresses on the boundary is determined in figs. 9 and 10 for three values of the Reynolds number and two of K_{eff}. As before, it is a decreasing function with respect to K_{eff} and increases with regard to Reynolds number. In addition, it is greater for motions produced by sine oscillations as compared to cosine oscillations of the shear stress on the boundary. This is obvious since for such motions the shear stress on the boundary is zero at time $t = 0$.

In fig. 10, unlike fig. 9, the diagrams of starting solutions are separately presented at different times because the profiles of steady-state solutions are almost identical for $K_{\text{eff}} = 1$ or 3.

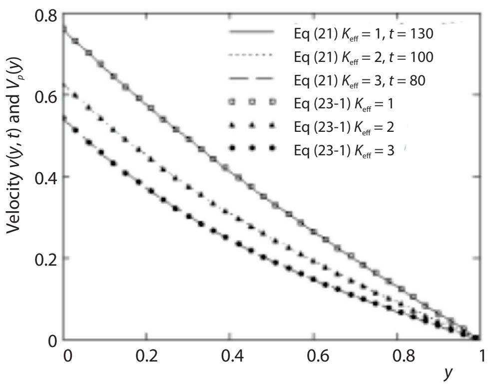

Figure 8. Required time to reach the steady-state for motion due to a constant shear on the boundary for Re = 70 and different values of the effective permeability K_{eff}

Figures 11 and 12 have been sketched to show the variation of the non-dimensional fluid temperature in the channel. To plot these graphs we used the following values of non-dimensional parameters. Profiles of the temperature in fig. 11 are plotted for the Brinkman number and for three different values of the time, t. It is observed from this figure that values of temperature increase from the value zero on the bottom wall to a maximum values attained near the moving wall and decreases to zero on the upper wall. This behavior is in accordance with the velocity field which has big variation near the moving wall of the channel therefore, the dissipation effects are stronger. It is known that the viscous dissipation produces heat due to the drag between fluid particles and it increases the fluid temperature. The bigger fluid velocity

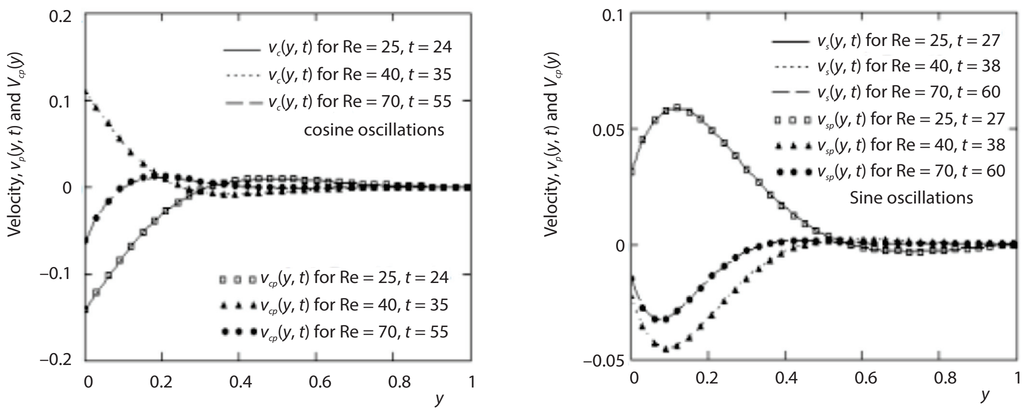

Figure 9. Required time to reach the steady-state for motions induced by cosine or sine oscillations of the shear on the boundary for $K_{\text{eff}} = 3$, $\omega = 2$, and different values of Reynolds number

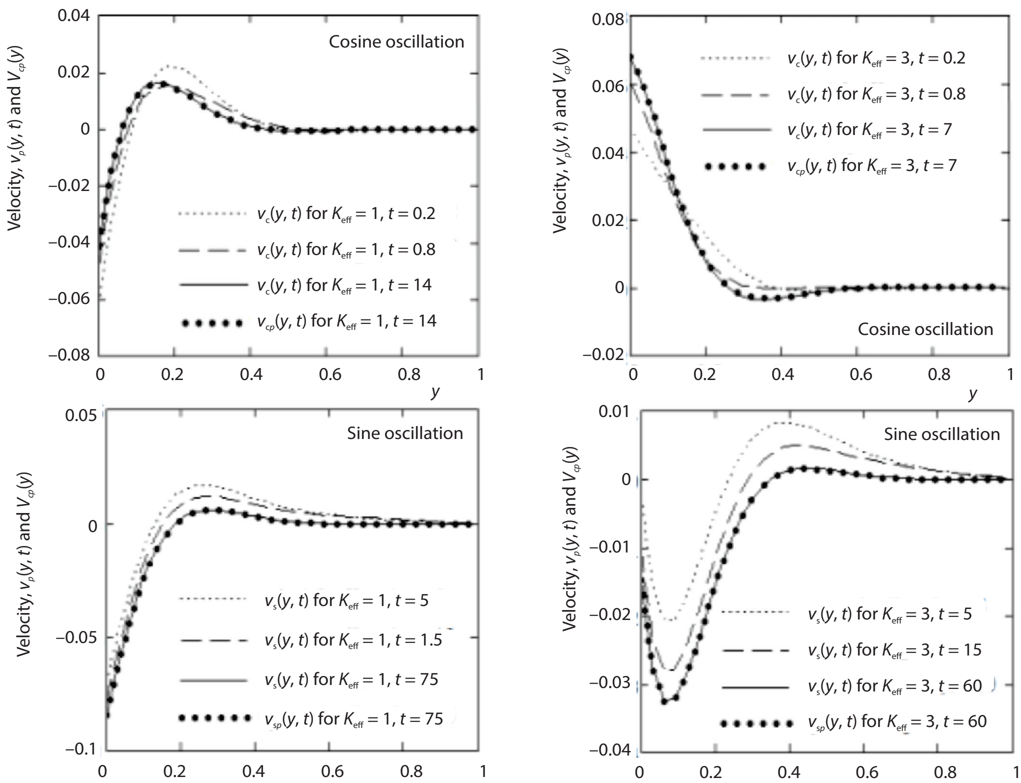

Figure 10. Required time to reach the steady-state for motions induced by cosine or sine oscillations of the shear on the boundary for Re = 70, $\omega = 2$, and two values of K_{eff}

will generate bigger drag between fluid particles and consequently larger viscous heating of the fluid. The influence of the Brinkman number on the fluid temperature is shown in fig. 12. As expected, the fluid temperature increases with the Brinkman number because, at larger values of Brinkman number, the heat produced by the viscous dissipation increases and hence the fluid temperature increases.

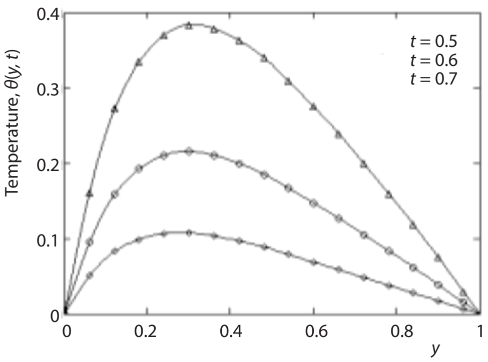

Figure 11. Profiles of the non-dimensional temperature $\theta(y, t)$ for different values of the time, t

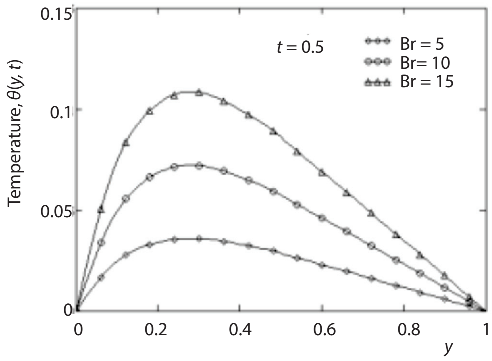

Figure 12. Profiles of the non-dimensional temperature $\theta(y, t)$ for different values of the Brinkman number

Conclusions

The main results that have been obtained by means of this study.

- Exact solutions are established for the motion of incompressible viscous fluids between two horizontal infinite parallel plates when the inferior plate applies a time-dependent shear stress to the fluid and porous and magnetic effects are taken into consideration.
- Governing equations for velocity and shear stress in such motions are identical as form.
- Motion characteristics do not depend of the magnetic and porosity parameters M and K independently but only by a combination of them $K_{\text{eff}} = M + 1/K$ which was called the effective permeability. A two parameter approach is redundant or even misleading.
- Velocity and shear stress fields $v_n(y, t)$ and $\tau_n(y, t)$ associated to motions induced by ramp-type shear stresses $t^n H(t)$on the boundary are presented as simple or multiple integrals of $v_0(y, t)$ and $\tau_n(y, t)$ corresponding to motions due to a constant shear on the boundary.
- The fluid-flows slower in the presence of a magnetic field or porous medium.
- Required time to reach the steady-state is an increasing function with respect to the Reynolds number and decreases for increasing values of K_{eff}. Consequently, the steady-state is rather obtained in the presence of a magnetic field or porous medium.
- The required to reach the steady-state is higher for motions induced by the lower plate that applies sine as compared to cosine oscillating shear stresses to the fluid. This is obvious since in the case of these motions the shear stress on the boundary is zero at time $t = 0$.
- The temperature field has a maximum value near the moving wall where the velocity variations are bigger, so the viscous dissipation effects are stronger.

Appendix

$$\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cos\left[\frac{(2n+1)\pi}{2h}y\right] = 1, \quad -h < y < h \tag{A1}$$

$$v(y,t) = 2\sum_{n=0}^{\infty} v_{Fn}(t)\cos(\mu_n y) \quad \text{if} \quad v_{Fn}(t) = \int_0^1 v(y,t)\cos(\mu_n y)\,\mathrm{d}y \tag{A2}$$

$$1 - y = 2\sum_{n=0}^{\infty}\frac{1}{\mu_n^2}\cos(\mu_n y) \quad \text{for} \quad 0 \le y \le 1, \quad \mu_n = \frac{(2n+1)\pi}{2} \tag{A3}$$

$$H'(t) = \delta(t) \text{ (the Dirac delta function) and } \int_0^t \delta(t) f(t-s)\mathrm{d}s = f(t) \tag{A4}$$

$$\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 1, \quad \frac{32}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} = 1 \tag{A5}$$

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