ABUNDANT LUMP SOLUTIONS AND INTERACTION SOLUTIONS OF A (3+1)-D KADOMTSEV-PETVIASHVILI EQUATION

by

Xiaoqing GAO[a]***, Sudao BILIGE***[a*]***, Jianqing LÜ***[a]***, Yuexing BAI***[b]***, Runfa ZHANG***[a]***, and Temuer CHAOLU***[b]

[a] Department of Mathematics, Inner Mongolia University of Technology, Hohhot, China
[b] College of Arts and Sciences, Shanghai Maritime University, Shanghai, China

Original scientific paper
https://doi.org/10.2298/TSCI1904437G

In this paper, abundant lump solutions and two types of interaction solutions of the (3+1)-D Kadomtsev-Petviashvili equation are obtained by the Hirota bilinear method. Some contour plots with different determinant values are sequentially given to show that the corresponding lump solution tends to zero when the determinant approaches to zero. The interaction solutions with special parameters are plotted to elucidate the solution properties.

Key words: *(3+1)-D Kadomtsev-Petviashvili equation, lump solution, interaction solutions, Hirota bilinear*

Introduction

In soliton theory, lump solutions have attracted more and more attention. As a kind of rational function solutions, lump solutions localize in all directions in the space. More importantly, collision among different solitons will occur. There are two kinds of collision, either elastic or in-elastic collisions. It was reported that lump solutions will keep their shapes, amplitudes, velocities after the collision with soliton solutions, which means the collision is completely elastic. While many other collisions are completely in-elastic. On the basis of different conditions, the collision will be changed essentially. Particular examples of lump solutions are found in many integrable equations [1-3]. Recently, Ma and Zhou [4] gave a way to get lump solutions of non-linear evolution equations (NLEE) by using the Hirota bilinear method and gave a theoretical proof of the existence. Based on this method, the researchers have studied the lump solutions and the interaction phenomenons of many NLEE [5-17].

In this paper, we would like to pay close attention to the (3+1)-D Kadomtsev-Petviashvili (KP) equation and acquire a common kind of lump solutions and two types of interaction solitons by using symbolic computation with MATHEMATICA.

The (3+1)-D KP equation [18] is:

$$u_{xt} + 6u_x^2 + 6uu_{xx} - u_{xxxx} - u_{yy} + u_{zz} = 0 \tag{1}$$

Firstly, we give the Hirota bilinear form of eq. (1) to search for the quadratic function solutions to the corresponding (3+1)-D bilinear KP equation, then derive a kind of lump solutions with the symbolic computation. Secondly, two types of interaction solutions will be obtained. We present the interaction solutions of lump solution with one stripe soliton by

* Corresponding author, e-mail: inmathematica@126.com

combining positive quadratic function with exponential function. We will obtain the solutions called kinky breather-soliton solutions. The dynamic properties of these solutions are shown by some figures under some special parameters. Finally, a sort of conclusion will be given at the end of the paper.

Lump solution of the (3+1)-D KP equation

Substitution of the dependent variable transformation:

$$u=-2(\ln f)_{xx} \tag{2}$$

with

$$f=f(x,y,t,z) \tag{3}$$

into eq. (1) yields the bilinear representation for eq. (1), which is:

$$(D_x D_t - D_x^4 - D_y^2 + D_z^2) f\cdot f = 0 \tag{4}$$

where f is a real function with respect to variables x, y, t, and z, and the derivatives $D_x D_t$, D_x^4, D_y^2 and D_z^2 are all bilinear derivative operators defined by:

$$D_x^\alpha D_y^\beta D_t^\gamma (f\cdot g)=$$

$$=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^\alpha \left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^\beta \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^\gamma f(x,y,t)g(x',y',t')|x=x',\quad y=y',\quad t=t' \tag{5}$$

To construct a bilinear KP equation by means of eq. (2), we consider Hirota bilinear form of eq. (1):

$$\text{KP}(f):=2(ff_{xt}-f_x f_t - ff_{xxxx}+4f_x f_{xxx}-3f_{xx}^2-ff_{yy}+f_y^2+ff_{zz}-f_z^2)=0 \tag{6}$$

To search for quadratic function solution to the bilinear KP eq. (6), we suppose:

$$f=g^2+h^2+a_{11}$$
$$g=a_1x+a_2y+a_3t+a_4z+a_5,\ h=a_6x+a_7y+a_8t+a_9z+a_{10} \tag{7}$$

where $a_i (1\le i\le 11)$ are real parameters to be determined later. For eq. (6), a direct symbolic computation with f in eq. (7) leads to the following set of constraining equations for the parameters.

Case 1

$$a_1=\frac{a_2^2a_3+2a_2a_7a_8-2a_4a_8a_9-a_3(a_4^2+a_7^2-a_9^2)}{a_3^2+a_8^2},\quad a_2=a_2,\quad a_3=a_3$$
$$a_4=a_4,\quad a_5=a_5,\quad a_6=\frac{2a_2a_3a_7-a_2^2a_8+a_4^2a_8+a_7^2a_8-2a_3a_4a_9-a_8a_9^2}{a_3^2+a_8^2} \tag{8}$$
$$a_7=a_7,\quad a_8=a_8,\quad a_9=a_9,\quad a_{10}=a_{10}$$

$$a_{11} = -\frac{1}{(a_3^2 + a_8^2)(a_2^2 - a_4^2)a_8^2 + a_3(-2a_2a_7a_8 + 2a_4a_8a_9) + a_3^2(a_7^2 - a_9^2)} \cdot$$

$$\cdot 3[a_2^4 + a_4^4 - 8a_2a_4a_7a_9 + (a_7^2 - a_9^2)^2 + 2a_4^2(a_7^2 + a_9^2) + 2a_2^2(-a_4^2 + a_7^2 + a_9^2)]^2$$

where $a_2, a_3, a_4, a_5, a_7, a_8, a_9, a_{10}$ are random parameters, which needs to satisfy the following determinant conditions:

$$a_3^2 + a_8^2 \neq 0, (a_2^2 - a_4^2)a_8^2 + a_3(-2a_2a_7a_8 + 2a_4a_8a_9) + a_3^2(a_7^2 - a_9^2) \neq 0 \tag{9}$$

By substituting eq. (8) into eq. (7), we obtain a kind of positive quadratic function solutions to the bilinear KP eq. (6). In turn, we gain a kind of lump solutions to the (3+1)-D KP eq. (1) through the transformation (2):

$$u = -\frac{4[-2(a_1g + a_6h)^2 + (a_1^2 + a_6^2)f]}{f^2} \tag{10}$$

where the functions *f, g*, and *h* are defined by eq. (7).

Case 2

$$\begin{gathered} a_1 = a_1, \quad a_2 = a_2, \quad a_3 = \frac{a_2^2a_6^2 - 2a_1a_4a_6a_9 + a_1^2(a_2^2 - a_4^2 + a_9^2)}{a_1(a_1^2 + a_6^2)}, \quad a_4 = a_4 \\ a_5 = a_5, \quad a_6 = a_6, \quad a_7 = \frac{a_2a_6}{a_1}, \quad a_8 = \frac{a_2^2a_6^3 - 2a_1^3a_4a_9 + a_1^2a_6(a_2^2 + a_4^2 - a_9^2)}{a_1^2(a_1^2 + a_6^2)} \\ a_9 = a_9, \quad a_{10} = a_{10}, \quad a_{11} = \frac{3(a_1^2 + a_6^2)^3}{(a_4a_6 - a_1a_9)^2} \end{gathered} \tag{11}$$

where $a_1, a_2, a_4, a_5, a_6, a_9, a_{10}$ are random parameters, which needs to meet the following condition:

$$a_1(a_1^2 + a_6^2) \neq 0, \quad a_4a_6 - a_1a_9 \neq 0 \tag{12}$$

By substituting eq. (11) into eq. (7), we gain a kind of positive quadratic function solutions to the bilinear KP eq. (6), yielding a kind of lump solutions to the (3+1)-D KP eq. (1) through the transformation (2):

$$u = -\frac{2[-(2a_1g + 2a_6h)^2 + 2(a_1^2 + a_6^2)f]}{f^2} \tag{13}$$

where the functions *f, g*, and *h* are defined by eq. (7).

Considering a special case of $z = y$, and assigning parameters to the following given values:

$$a_1 = -2, \quad a_2 = 3, \quad a_4 = 4, \quad a_5 = -2, \quad a_6 = 2, \quad a_9 = -1, \quad a_{10} = 2 \tag{14}$$

and substituting them into eqs. (11) and (7), we can acquire *f, g,* and *h*. We write down *f*:

$$f=\frac{128}{3}+\left(2-\frac{5t}{4}+2x-7y\right)^2+\left(2+\frac{25t}{4}+2x-4y\right)^2 \tag{15}$$

Through the transformation (2), we obtain the lump solution:

$$u=-\frac{768[375t^2-60t(8+8x-31y)-32(-26+6x^2+x(12-33y)-33y+42y^2)]}{[975t^2+60t(8+8x-13y)+8(152+24x^2+x(48-132y)-132y+195y^2)]^2} \tag{16}$$

The plots when $t = 0$ are depicted in figs. 1-3.

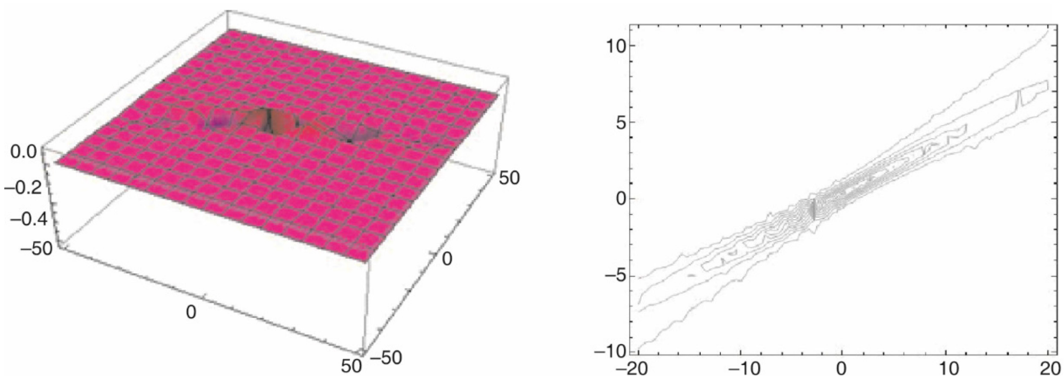

Figure 1. The 3-D plot and contour plot

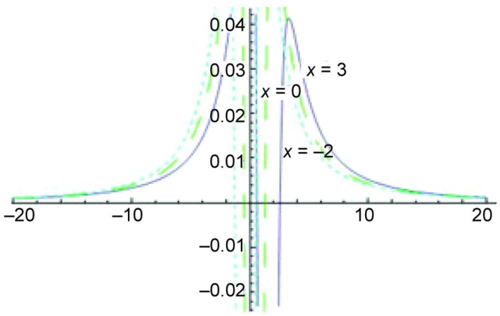

Figure 2. The profiles comparison of analytic approximation *y* with numerical results

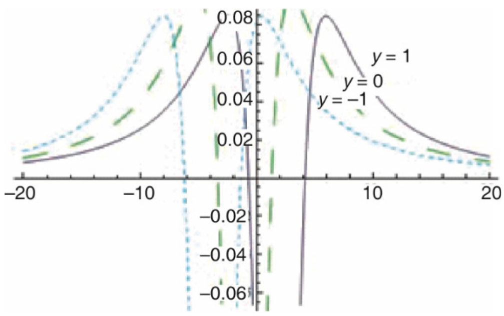

Figure 3. The profiles comparison of analytic approximation *x* with numerical results

Interaction solution of the (3+1)-D KP equation

Interaction solutions describe more diverse non-linear phenomena in nature. In this section, we handle the (3+1)-D KP equation for acquiring new interaction solutions by the Hirota bilinear method.

Lump-kink solution of the (3+1)-D KP equation

This section focuses on the interaction solution between the lump solution and the exponential function, called a lump-kink solution. Adding the exponential function to eq. (7), we have:

$$f = g^2 + h^2 + \exp(n) + a_{16} \tag{17}$$

where

$$\begin{cases} g = a_1 x + a_2 y + a_3 t + a_4 z + a_5 \\ h = a_6 x + a_7 y + a_8 t + a_9 z + a_{10} \\ n = a_{11} x + a_{12} y + a_{13} t + a_{14} z + a_{15} \end{cases} \tag{18}$$

where $a_i (1 \le i \le 16)$ are free real constants. By inserting eq. (17) into eq. (6), we obtain the following results.

Case 1

$$\begin{gathered} a_1 = 0, \quad a_2 = -\frac{\sqrt{(a_4^2 - a_7^2)^3}}{a_4^2 - a_7^2}, \quad a_3 = \frac{2a_7\sqrt{(a_4^2 - a_7^2)^3}}{a_6(-a_4^2 + a_7^2)}, \quad a_4 = a_4, \quad a_5 = a_5 \\ a_6 = a_6, \quad a_7 = a_7, \quad a_8 = \frac{2a_7^2}{a_6}, \quad a_9 = 0, \quad a_{10} = a_{10}, \quad a_{11} = \frac{a_7}{\sqrt{3}a_6} \\ a_{12} = \frac{a_7^2}{\sqrt{3}a_6^2}, \quad a_{13} = \frac{4a_7^3}{3\sqrt{3}a_6^3}, \quad a_{14} = 0, \quad a_{15} = a_{15}, \quad a_{16} = \frac{3a_6^4}{a_7^2} \end{gathered} \tag{19}$$

where $a_4, a_5, a_6, a_7, a_{10}, a_{15}$ are random parameters, which need to satisfy the determinant conditions:

$$a_2^2 - a_4^2 \ne 0, \quad a_6(a_4^4 - a_4^2 a_7^2) \ne 0 \tag{20}$$

By substituting eq. (19) into eq. (17) and using the transformation $u = -2(\ln f)_{xx}$, we obtain the lump-kink solution to eq. (1):

$$u = -2\left\{\frac{-[2a_6 h + a_{11}\exp(n)]^2 + [2a_6^2 + a_{11}^2 \exp(n)]f}{f^2}\right\} \tag{21}$$

where the functions *f*, *h*, and *n* are defined by eqs. (17) and (18).

Case 2

$$\begin{gathered} a_1 = \frac{a_6 a_9}{\sqrt{a_7^2 - a_9^2}}, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = -\sqrt{a_7^2 - a_9^2}, \quad a_5 = a_5, \quad a_6 = a_6 \\ a_7 = a_7, \quad a_8 = \frac{2(a_7^2 - a_9^2)}{a_6}, \quad a_9 = a_9, \quad a_{10} = a_{10}, \quad a_{11} = \frac{a_7^2 - a_9^2}{\sqrt{3}a_6 a_7} \\ a_{12} = \frac{(a_7^2 - a_9^2)^2}{\sqrt{3}a_6^2 a_7^2}, \quad a_{13} = \frac{4(a_7^2 - a_9^2)^3}{3\sqrt{3}a_6^3 a_7^3}, \quad a_{14} = 0, \quad a_{15} = a_{15}, \quad a_{16} = \frac{3a_6^4 a_7^4}{(a_7^2 - a_9^2)^3} \end{gathered} \tag{22}$$

where $a_5, a_6, a_7, a_9, a_{10}, a_{15}$ are random parameters, which need to satisfy the determinant conditions

$$\begin{cases} a_6 a_7 \neq 0, \quad a_4(a_7^2 - a_9^2) \neq 0, \quad a_4^2 + a_7^2 - a_9^2 \neq 0 \\ a_4^4 a_6 + 2a_4^2 a_6 a_7^2 + a_6 a_7^4 + 2a_4^2 a_6 a_9^2 - 2a_6 a_7^2 a_9^2 + a_6 a_9^4 \neq 0 \end{cases} \tag{23}$$

By substituting eq. (22) into eq. (17) and using the transformation $u = -2(\ln f)_{xx}$, we obtain the lump-kink solution to eq. (1).

$$u = -\frac{2\{[2a_{11}^2 + 2a_6^2 + a_{11}^2 \exp(n)]f - [a_{11}\exp(n) - 2a_1 g + 2a_6 h]^2\}}{f^2} \tag{24}$$

With $z = y$, $t = 0$, and assigning a value to a random parameter in forms:

$$a_5 = -1, \quad a_6 = \sqrt{3}, \quad a_7 = 2, \quad a_9 = 1, \quad a_{10} = 0, \quad a_{15} = -2 \tag{25}$$

we have the lump-kink solution eq. (24) as shown in fig. 4.

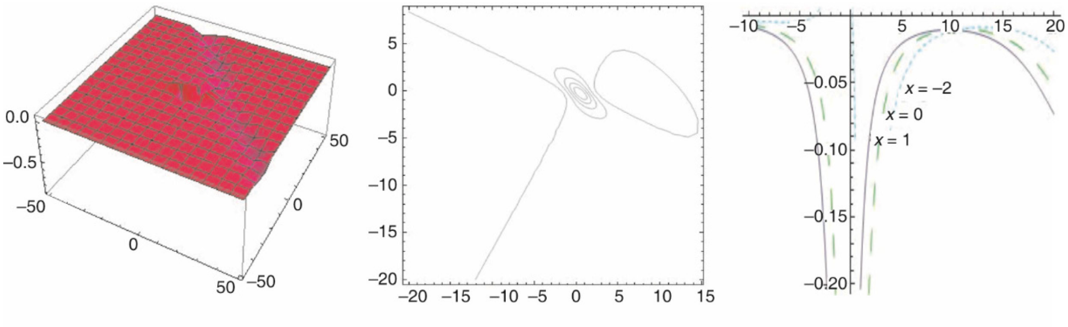

Figure 4. The 3-D plot, contour plot, and the profiles comparison of analytic approximation *y* with numerical results

Kinky breather-soliton solution of (3+1)-D KP equation

To search for kinky breather-soliton solution, we use the following function:

$$f = \exp(-pg) + b_0 \cos(p_1 h) + b_1 \exp(pg) \tag{26}$$

where b_0, b_1, p, p_1 are free constants and:

$$g = a_1 x + a_2 y + a_3 t + a_4 z + a_5, \quad h = a_6 x + a_7 y + a_8 t + a_9 z + a_{10} \tag{27}$$

where $a_i (1 \le i \le 10)$ are free real constants. By inserting eq. (26) into eq. (6), we obtain the following results.

Case 1

$$\begin{aligned} &a_1 = 1, \quad a_2 = 0, \quad a_3 = -\frac{2p}{\sqrt{3}}, \quad a_4 = \frac{4p}{\sqrt{3}}, \quad a_5 = a_5, \quad a_6 = a_6, \quad a_7 = \frac{16a_6 p^2}{3} \\ &a_8 = \frac{2a_6 p}{\sqrt{3}}, \quad a_9 = 0, \quad a_{10} = a_{10}, \quad b_0 = b_0, \quad b_1 = b_1, \quad p = p, \quad p_1 = \frac{\sqrt{-a_6 p^2}}{a_6^{3/2}} \end{aligned} \tag{28}$$

where $a_5, a_6, a_{10}, b_0, b_1, p$ are random parameters, which need to satisfy a determinant condition $a_6 > 0$. Substituting eq. (28) into eq. (26), we obtain:

$$f = \exp\left[-p\left(x - \frac{2py - 4pz}{\sqrt{3}} + a_5\right)\right] + b_1 \exp\left[p\left(x - \frac{2py - 4pz}{\sqrt{3}} + a_5\right)\right] + \\ + b_0 \cos\frac{\sqrt{-a_6 p^2}(3a_6 x + 16 a_6 p^2 y + 2\sqrt{3} a_6 pt + 3a_{10})}{3a_6^{3/2}} \tag{29}$$

Case 2

$$a_1 = 0, \quad a_2 = \frac{2(a_3 a_8 - a_4 a_9)}{a_6}, \quad a_3 = a_3, \quad a_4 = a_4, \quad a_5 = a_5, \quad a_6 = a_6 \\ a_7 = \frac{-a_3^2 p^2 + a_4^2 p^2 + a_8^2 p_1^2 - a_9^2 p_1^2 - a_6^4 p_1^4}{a_6 p_1^2}, \quad a_8 = a_8, \quad a_9 = a_9 \\ a_{10} = a_{10}, \quad b_0 = b_0, \quad b_1 = \frac{b_0^2 (a_3^2 p^2 - a_4^2 p^2 - 3a_6^4 p_1^4)}{4(a_3^2 - a_4^2) p^2}, \quad p = p, \quad p_1 = p_1 \tag{30}$$

where $a_3, a_4, a_5, a_6, a_8, a_9, a_{10}, b_0, p, p_1$ are random parameters, which need to satisfy the determinant conditions $a_6 p_1^2 \neq 0, (a_3^2 - a_4^2) p^2 \neq 0$. Substituting eq. (30) into eq. (26), we obtain:

$$f = \exp\left[-p\left(\frac{2a_3 a_8 - 2a_4 a_9}{a_6} y + a_3 t + a_4 z + a_5\right)\right] + \\ + \frac{b_0^2 (a_3^2 p^2 - a_4^2 p^2 - 3a_6^4 p_1^4) \exp\left[p\left(\frac{2a_3 a_8 - 2a_4 a_9}{a_6} y + a_3 t + a_4 z + a_5\right)\right]}{4(a_3^2 - a_4^2) p^2} + \\ + b_0 \cos\left[p_1\left(a_6 x + \frac{-a_3^2 p^2 + a_4^2 p^2 + a_8^2 p_1^2 - a_9^2 p_1^2 - a_6^4 p_1^4}{a_6 p_1^2} y + a_8 t + a_9 z + a_{10}\right)\right] \tag{31}$$

By the transformation $u = -2(\ln f)_{xx}$, we can obtain the kinky breather-soliton solution u of eq. (1). By using the variable $z = y$, and assigning a value to each random parameter:

$$a_3 = 4, \quad a_4 = 3, \quad a_5 = -1, \quad a_6 = -2, \quad a_8 = -2, \quad a_9 = 1, \\ a_{10} = 3, \quad b_0 = \frac{5}{4}, \quad p = 1, \quad p_1 = 2 \tag{32}$$

we obtain the kinky breather-soliton solution, which is:

$$u = -\frac{17920 \exp(\alpha)\{-560 \exp(\alpha) + [-448 e^2 + 19025 \exp(22t + 14y)] \cos\beta\}}{[448 e^2 - 19025 \exp(22t + 14y) + 560 \exp(\alpha) \cos\beta]^2} \tag{33}$$

where $\alpha = 1 + 11t + 7y, \beta = 6 + \dfrac{251t}{4} - 4x - 2y.$

When $t = 0$, the plots of the breather-soliton solution (33) are depicted in fig. 5.

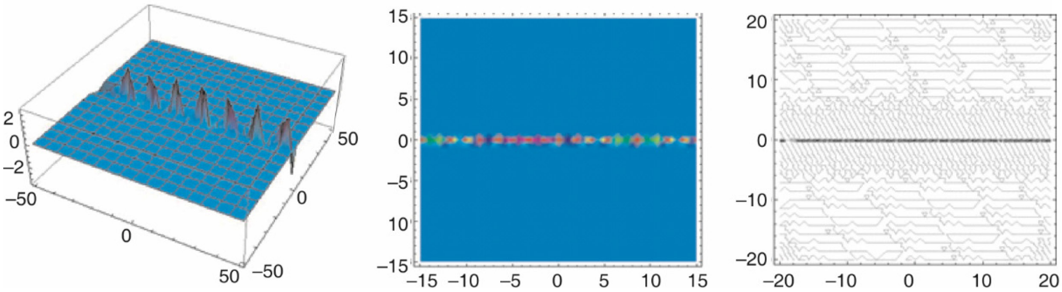

Figure 5. The 3-D plot, contour plot, and density plot

Conclusion

In summary, by the Hirota bilinear method and the symbolic computation with MATHEMATICA, the analyticity and localization of the resulting solutions are ensured by the nonzero determinant conditions. In this paper a class of lump solutions and two interaction solutions are derived, their dynamic properties are shown in figs. 1-3. We extend this method to combination of the positive quadratic function and the exponential function, and the dynamic properties of the interaction solutions are shown in fig. 4. Finally, we obtain the kinky breather-soliton solution of the (3+1)-D KP equation, as shown in fig. 5. The exp-function method is an alternative approach to the search for such solutions [19].

Acknowledgment

This work is supported by the National Natural Science Foundation of China (11661060, 11571008), and the Natural Science Foundation of Inner Mongolia Autonomous Region of China (2018LH01013).

References

[1] Manakov, S. V., Zakharov, V. E., Two-Dimensional Solitons of the Kadomtsev-Petviashvili Equation and their Interaction, *Phys. Lett., 63A* (1977), A3, pp. 205-206
[2] Kaup, D. J., The Lump Solutions and the Bäcklund Transformation for the Three-Dimensional Three-Wave Resonant Interaction, *J. Math. Phys., 22* (1981), 6, pp. 1176-1181
[3] Gilson, C. R., Nimmo, J. J. C., Lump Solutions of the BKP Equation, *Phys. Lett. A, 147* (1990), 8-9, pp. 472-476
[4] Ma, W. X., Zhou, Y., Lump Solutions to Non-Linear Partial Differential Equations via Hirota Bilinear Forms, *J. Diff. Eqn., 264* (2018), 4, pp. 2633-2659
[5] Lü, X., *et al.*, Constructing Lump Solutions to a Generalized Kadomtsev-Peviashvili-Boussineq Equation, *Non-Linear Dyn., 86* (2016), 1, pp. 523-534
[6] Ma, W. X., Lump Solutions to the Kadomtsev-Peviashvili Equation, *Phys. Lett. A, 379* (2015), 36, pp. 1975-1978
[7] Ma, W. X., *et al.*, Lump Solutions to Dimensionally Reduced p-gKP and p-gBKP Equations, *Non-Linear Dyn., 84* (2016), 2, pp. 923-931
[8] Wei, T., *et al.*, Dynamical Analysis of Lump Solution for the (2+1)-Dimensional Ito Equation, *Thermal Science, 21* (2017), 4, pp. 1673-1679
[9] Ma, H. C., *et al.*, Lump Solutions to the (2+1)-Dimensional Shallow Water Wave Equation, *Thermal Science, 21* (2017), 4, pp. 1765-1769
[10] Lv, J. Q., Bilige, S. D., Lump Solutions of a (2+1)-Dimensional bSK Equation, *Non-Linear Dyn., 90* (2017), 3, pp. 2119-2124

[11] Zhang, Y., *et al.*, Rational Solutions and Lump Solutions to the Generalized (3+1)-D Shallow Water-like Equation, *Comput. Math. Appl., 73* (2017), 2, pp. 246-252
[12] Yu, J. P., Sun, Y. L., Lump Solutions to Dimensionally Reduced Kadomtsev-Petviashvili-like Equations, *Non-Linear Dyn., 87* (2017), 2, pp. 1405-1412
[13] Zhang, J. B., Ma, W. X., Mixed Lump-Kink Solutions to the BKP Equation, *Comput. Math. Appl., 74* (2017), 3, pp. 591-596
[14] Yang, J. Y., Ma, W. X., Abundant Interaction Solutions of the KP Equation, *Non-Linear Dyn., 89* (2017), 2, pp. 1539-1544
[15] Huang, L. L., Chen, Y., Lump Solutions and Interaction Phenomenon for (2+1)-Dimensional Sawada-Kotera Equation, *Commun. Theor. Phys., 67* (2017), 5, pp. 473-478
[16] Lü, J. Q., *et al.*, The Study of Lump Solution and Interaction Phenomenon to (2+1)-Dimensional Generalized Fifth-Order KdV Equation, *Non-Linear Dyn., 91* (2018), 3, pp. 1669-1676
[17] Zhang, R. F., *et al.*, Interaction Phenomenon to Dimensionally Reduced p-gBKP Equation, *Mod. Phys. Lett. B, 32* (2018), 6, ID 1850074
[18] Senthilvelan, M., On the Extended Applications of Homogenous Balance Method, *Appl. Math. Lett., 123* (2001), 3, pp. 381-388
[19] He, J.-H. Asymptotic Methods for Solitary Solutions and Compactons, *Abstract and Applied Analysis, 2012* (2012), ID 916793

Paper submitted: May 8, 2018
Paper revised: October 20, 2018
Paper accepted: October 24, 2018