EXACT SOLUTIONS WITH ARBITRARY FUNCTIONS OF THE (4+1)-DIMENSIONAL FOKAS EQUATION

by

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In this paper, the (4+1)-dimensional Fokas equation is solved by the generalized F-expansion method, and new exact solutions with arbitrary functions are obtained. The obtained solutions include Jacobi elliptic function solutions, hyperbolic function solutions and trigonometric function solutions. It is shown that the generalized F-expansion method can be used for constructing exact solutions with arbitrary functions of some other high dimensional partial differential equations in fluids.

Key words: *(4+1)-dimensional Fokas equation, generalized F-expansion method, Jacobi elliptic function solution, hyperbolic function solution, trigonometric function solution*

Introduction

In the past several decades, the investigation of exact solutions of non-linear PDE has attached much attention [1-6] because of its direct connection with dynamical processes in some natural phenomena involved in many fields, for example fluid dynamics. Usually, researches restore to exact solutions of non-linear PDE to gain more insight into these non-linear phenomena for further applications. Since the celebrated KdV equation was exactly solved in 1967 [2], finding exact solutions of non-linear PDE has become one of the most important and significant tasks. In soliton theory, many effective methods [1-6] for constructing exact solutions of non-linear PDE have been proposed. Among these methods, the so-called F-expansion method [6] can be thought of as an over-all generalization of Jacobi elliptic function expansion method [7]. The F-expansion method has been widely used to a great many of non-linear PDE like those in [8, 9] and was improved in different manners [10-14]. In 2006, Zhang and Xia [14] generalized the F-expansion method by introducing a new and more general ansätz. The present paper is motivated by the desire to extend the generalized F-expansion method [14] the (4+1)-D Fokas equation [15]:

$$u_{tx_1} - \frac{1}{4}u_{x_1x_1x_1x_2} + \frac{1}{4}u_{x_2x_2x_2x_1} + \frac{3}{2}(u^2)_{x_1x_2} - \frac{3}{2}u_{y_1y_2} = 0 \tag{1}$$

for constructing its new exact solutions.

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Transformations

Firstly, we suppose that eq. (1) has exact solutions of the form:

$$u = a_0 + \sum_{i=1}^{n}[a_i F^i(\xi) + b_i F^{-i}(\xi) + c_i F^{i-1}(\xi)F'(\xi) + d_i F^{-i}(\xi)F'(\xi)] \tag{2}$$

where $\xi = k_1x_1 + k_2x_2 + \eta$, the integer n and the constants k_1 and k_2 are to be determined, while $a_0 = a_0(y_1, y_2, t)$, $a_i = a_i(y_1, y_2, t)$, $b_i = b_i(y_1, y_2, t)$, $c_i = c_i(y_1, y_2, t)$, $d_i = d_i(y_1, y_2, t)$, and $\eta = \eta(y_1, y_2, t)$ are all undetermined functions of the indicated variables, and $F(\xi)$ satisfies:

$$F'^2(\xi) = PF^4(\xi) + QF^2(\xi) + R \tag{3}$$

and hence holds

$$\begin{cases} F''(\xi) = 2PF^3(\xi) + QF(\xi) \\ F^{(3)}(\xi) = \left[6PF^2(\xi) + Q\right]F'(\xi) \\ F^{(4)}(\xi) = 24P^2F^5(\xi) + 20PQF^3(\xi) + (Q^2 + 12PR)F(\xi) \\ \qquad \cdots \end{cases} \tag{4}$$

where *P, Q,* and *R* are parameters. In [14], Jacobi elliptic function solutions and their degenerated solutions of eq. (3) are listed, which depend on the values of parameters *P, Q,* and *R*.

Secondly, substituting eq. (2) along with eqs. (3) and (4) into eq. (1) and then balancing the highest order partial derivative $u_{x_1x_1x_1x_2}$ and the highest order non-linear term $(u^2)_{x_1x_2}$ yields the integer $n + 4 = 2n + 2$ which gives $n = 2$.

Thirdly, we substitute eq. (2) given the value of $n = 2$ along with eqs. (3) and (4) into eq. (1) and collect all terms with the same order of $F^j(\xi)F'^s(\xi)$ $(j = 0, \pm 1, \pm 2, \cdots; s = 0, 1)$ together. Then setting each coefficient of $F^j(\xi)F'^s(\xi)$ to zero, a set of non-linear PDE for $a_0, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, \eta, k_1$, and k_2 are derived.

$$F'(\xi)F^{-6}(\xi): -30k_1^3k_2R^2d_2 + 30k_1k_2^3R^2d_2 + 60k_1k_2Rb_2d_2 = 0 \tag{5}$$

$$F'(\xi)F^{-5}(\xi): -6k_1^3k_2R^2d_1 + 6k_1k_2^3R^2d_1 + 36k_1k_2Rb_2d_1 + 36k_1k_2Rb_1d_2 = 0 \tag{6}$$

$$\begin{aligned} F'(\xi)F^{-4}(\xi): &18k_1k_2Rb_2c_1 + 18k_1k_2Rb_1d_1 - 15k_1^3k_2QRd_2 + 15k_1k_2^3QRd_2 + 18k_1k_2Ra_0d_2 + \\ &+27k_1k_2Qb_2d_2 + 6k_1Rd_2\eta_t - 9Rd_2\eta_{y_1}\eta_{y_2} = 0 \end{aligned} \tag{7}$$

$$\begin{aligned} F'(\xi)F^{-3}(\xi): &6k_1k_2Rb_1c_1 + 6k_1k_2Rb_2c_2 - 2k_1^3k_2QRd_1 + 2k_1k_2^3QRd_1 + 6k_1k_2Ra_0d_1 + \\ &+12k_1k_2Qb_2d_1 + 6k_1k_2Ra_1d_2 + 12k_1k_2Qb_1d_2 - 2k_1b_{2t} + 2k_1Rd_1\eta_t + 3b_{2y_1}\eta_{y_2} + \\ &+3b_{2y_2}\eta_{y_1} - 3Rd_1\eta_{y_1}\eta_{y_2} + 3b_2\eta_{y_1y_2} = 0 \end{aligned} \tag{8}$$

$$F'(\xi)F^{-2}(\xi): 3k_1k_2Qb_2c_1 + 3k_1k_2Qb_1d_1 - \frac{1}{4}k_1^3k_2Q^2d_2 + \frac{1}{4}k_1k_2^3Q^2d_2 - 3k_1^3k_2PRd_2 + \\ +3k_1k_2^3PRd_2 + 3k_1k_2Qa_0d_2 + 6k_1k_2Pb_2d_2 - k_1b_{1t} + k_1Qd_2\eta_t + \frac{3}{2}b_{1y_1}\eta_{y_2} + \frac{3}{2}b_{1y_2}\eta_{y_1} + \\ + \frac{3}{2}Qd_2\eta_{y_1}\eta_{y2} - \frac{3}{2}d_{2y_1y_2} + \frac{3}{2}b_1\eta_{y_1y_2} = 0 \tag{9}$$

$$F'(\xi)F^{-1}(\xi): -\frac{3}{2}d_{1y_1y_2} = 0 \tag{10}$$

$$F'(\xi): -\frac{1}{4}k_1^3k_2Q^2c_1 + \frac{1}{4}k_1k_2^3Q^2c_1 - 3k_1^3k_2PRc_1 + 3k_1k_2^3PRc_1 + 3k_1k_2Qa_0c_1 + 6k_1k_2Ra_2c_1 + \\ +6k_1k_2Ra_1c_2 + 3k_1k_2Rb_1c_2 + 3k_1k_2Qa_1d_1 + 3k_1k_2Qa_2d_2 + k_1a_{1t} + k_1Qc_1\eta_t - \\ -\frac{3}{2}a_{1y_1}\eta_{y_2} - \frac{3}{2}a_{1y_2}\eta_{y_1} - \frac{3}{2}Qc_1\eta_{y_1}\eta_{y_2} - \frac{3}{2}c_{1y_1y_2} - \frac{3}{2}a_1\eta_{y_1y_2} = 0 \tag{11}$$

$$F'(\xi)F(\xi): 12k_1k_2Qa_1c_1 + 6k_1k_2Pb_1c_1 - 4k_1^3k_2Q^2c_2 + 4k_1k_2^3Q^2c_2 - 18k_1^3k_2PRc_2 + 18k_1k_2^3PRc_2 + \\ +12k_1k_2Qa_0c_2 + 18k_1k_2Ra_2c_2 + 6k_1k_2Pb_2c_2 - 2k_1^3k_2PQd_1 + 2k_1k_2^3PQd_1 + \\ +6k_1k_2Pa_0d_1 + 12k_1k_2Qa_2d_1 + 6k_1k_2Pa_1d_2 + 2k_1a_{2t} + 4k_1Qc_2\eta_t + \\ +2k_1Pd_1\eta_t - 3a_{2y_1}\eta_{y_2} - 3a_{2y_2}\eta_{y_1} - 6c_2Q\eta_{y_1}\eta_{y_2} - \\ -3d_1P\eta_{y_1}\eta_{y_2} - \frac{3}{2}c_{2y_1y_2} - 3a_2\eta_{y_1y_2} = 0 \tag{12}$$

$$F'(\xi)F^2(\xi): -15k_1^3k_2PQc_1 + 15k_1k_2^3PQc_1 + 18k_1k_2Pa_0c_1 + 27k_1k_2Qa_2c_1 + 27k_1k_2Qa_1c_2 + \\ + 18k_1k_2Pb_1c_2 + 18k_1k_2Pa_1d_1 + 18k_1k_2Pa_2d_2 + 6k_1Pc_1\eta_t - 9c_1P\eta_{y_1}\eta_{y_2} = 0 \tag{13}$$

$$F'(\xi)F^3(\xi): 36k_1k_2Pa_1c_1 - 60k_1^3k_2PQc_2 + 60k_1k_2^3PQc_2 + 36k_1k_2Pa_0c_2 + 48k_1k_2Qa_2c_2 - \\ -6k_1^3k_2P^2d_1 + 6k_1k_2^3P^2d_1 + 36k_1k_2Pa_2d_1 + 12k_1Pc_2\eta_t - 18Pc_2\eta_{y_1}\eta_{y_2} = 0 \tag{14}$$

$$F'(\xi)F^4(\xi): -30k_1^3k_2P^2c_1 + 30k_1k_2^3P^2c_1 + 60k_1k_2Pa_2c_1 + 60k_1k_2Pa_1c_2 = 0 \tag{15}$$

$$F'(\xi)F^5(\xi): -90k_1^3k_2P^2c_1 + 90k_1k_2^3P^2c_2 + 90k_1k_2Pa_2c_2 = 0 \tag{16}$$

$$F^{-6}(\xi): -30k_1^3k_2R^2b_2 + 30k_1k_2^3R^2b_2 + 30k_1k_2Rb_2^2 + 30k_1k_2R^2d_2^2 = 0 \tag{17}$$

$$F^{-5}(\xi): -6k_1^3k_2R^2b_1 + 6k_1k_2^3R^2b_1 + 36k_1k_2Rb_1b_2 + 36k_1k_2R^2d_1d_2 = 0 \tag{18}$$

$$F^{-4}(\xi): 9k_1k_2Rb_1^2 - 30k_1^3k_2QRb_2 + 30k_1k_2^3QRb_2 + 18k_1k_2Ra_0b_2 + 24k_1k_2Qb_2^2 + 9k_1k_2R^2d_1^2 + \\ +18k_1k_2R^2c_1d_2 + 33k_1k_2QRd_2^2 + 6k_1Rb_2\eta_t - 9Rb_2\eta_{y_1}\eta_{y_2} = 0 \tag{19}$$

$$F^{-3}(\xi): -5k_1^3k_2QRb_1 + 5k_1k_2^3QRb_1 + 6k_1k_2Ra_0b_1 + 6k_1k_2Ra_1b_2 + 27k_1k_2Qb_1b_2 + $$
$$+6k_1k_2R^2c_1d_1 + 6k_1k_2R^2c_2d_2 + 33k_1k_2QRd_1d_2 - 2k_1Rd_{2t} + 2k_1Rb_1\eta_t +$$
$$+3Rd_{2y_1}\eta_{y_2} + 3Rd_{2y_2}\eta_{y_1} - 3Rb_1\eta_{y_1}\eta_{y_2} + 3Rd_2\eta_{y_1y_2} = 0 \quad (20)$$

$$F^{-2}(\xi): 6k_1k_2Qb_1^2 - 4k_1^3k_2Q^2b_2 + 4k_1k_2^3Q^2b_2 - 18k_1^3k_2PRb_2 + 18k_1k_2^3PRb_2 + 12k_1k_2Qa_0b_2 +$$
$$+18k_1k_2Rb_2^2 + 6k_1k_2QRd_1^2 + 12k_1k_2QRc_1d_2 + 6k_1k_2Q^2d_2^2 + 18k_1k_2PRd_2^2 -$$
$$-k_1Rd_{1t} + 4k_1Qb_2\eta_t + \frac{3}{2}Rd_{1y_1}\eta_{y_2} + \frac{3}{2}Rd_{1y_2}\eta_{y_1} -$$
$$-6Qb_2\eta_{y_1}\eta_{y_2} - \frac{3}{2}b_{2y_1y_2} + \frac{3}{2}Rd_1\eta_{y_1y_2} = 0 \quad (21)$$

$$F^{-1}(\xi): -\frac{1}{4}k_1^3k_2Q^2b_1 + \frac{1}{4}k_1k_2^3Q^2b_1 - 3k_1^3k_2PRb_1 + 3k_1k_2^3PRb_1 + 3k_1k_2Qa_0b_1 + 3k_1k_2Qa_1b_2 +$$
$$+18k_1k_2Pb_1b_2 + 3k_1k_2QRc_1d_1 + 3k_1k_2QRc_2d_2 + 3k_1k_2Q^2d_1d_2 + 18k_1k_2PRd_1d_2 -$$
$$-k_1Qd_{2t} + k_1Qb_1\eta_t + \frac{3}{2}Qd_{2y_1}\eta_{y_2} + \frac{3}{2}Qd_{2y_2}\eta_{y_1} - \frac{3}{2}Qb_1\eta_{y_1}\eta_{y_2} -$$
$$-\frac{3}{2}b_{1y_1y_2} + \frac{3}{2}Qd_2\eta_{y_1y_2} = 0 \quad (22)$$

$$F^0(\xi): 3k_1k_2Ra_1^2 - 2k_1^3k_2QRa_2 + 2k_1k_2^3QRa_2 + 6k_1k_2Ra_0a_2 + 3k_1k_2Pb_1^2 - 2k_1^3k_2PQb_2 +$$
$$+ 2k_1k_2^3PQb_2 + 6k_1k_2Pa_0b_2 + 3k_1k_2QRc_1^2 + 3k_1k_2R^2c_2^2 +$$
$$+6k_1k_2QRc_2d_1 + 6k_1k_2PRd_1^2 + 12k_1k_2PRc_1d_2 + 3k_1k_2PQd_2^2 + k_1Rc_{2t} +$$
$$+2k_1Ra_2\eta_t + 2k_1Pb_2\eta_t - \frac{3}{2}Rc_{2y_1}\eta_{y_2} - \frac{3}{2}Rc_{2y_2}\eta_{y_1} - 3Ra_2\eta_{y_1}\eta_{y_2} -$$
$$-3Pb_2\eta_{y_1}\eta_{y_2} - \frac{3}{2}a_{0y_1y_2} - \frac{3}{2}Rc_2\eta_{y_1y_2} = 0 \quad (23)$$

$$F(\xi): -\frac{1}{4}k_1^3k_2Q^2a_1 + \frac{1}{4}k_1k_2^3Q^2a_1 - 3k_1^3k_2PRa_1 + 3k_1k_2^3PRa_1 + 3k_1k_2Qa_0a_1 + 18k_1k_2Ra_1a_2 +$$
$$+3k_1k_2Qa_2b_1 + 21k_1k_2QRc_1c_2 + 3k_1k_2Q^2c_1d_1 + 18k_1k_2PRc_1d_1 + 3k_1k_2Q^2c_2d_2 +$$
$$+18k_1k_2PRc_2d_2 + 3k_1k_2PQd_1d_2 + k_1Qc_{1t} + k_1Qa_1\eta_t - \frac{3}{2}Qc_{1y_1}\eta_{y_2} -$$
$$-\frac{3}{2}Qc_{1y_2}\eta_{y_1} - \frac{3}{2}Qa_1\eta_{y_1}\eta_{y_2} - \frac{3}{2}a_{1y_1y_2} - \frac{3}{2}Qc_1\eta_{y_1y_2} = 0 \quad (24)$$

$$F^2(\xi): 6k_1k_2Qa_1^2 - 4k_1^3k_2Q^2a_2 + 4k_1k_2^3Q^2a_2 - 18k_1^3k_2PRa_2 + 18k_1k_2^3PRa_2 + 12k_1k_2Qa_0a_2 +$$
$$+18k_1k_2Ra_2^2 + 6k_1k_2Q^2c_1^2 + 18k_1k_2PRc_1^2 + 24k_1k_2QRc_2^2 + 12k_1k_2Q^2c_2d_1 +$$
$$+36k_1k_2PRc_2d_1 + 6k_1k_2PQd_1^2 + 12k_1k_2PQc_1d_2 + 2k_1Qc_{2t} + k_1Pd_{1t} +$$
$$+4k_1Qa_2\eta_t - 3Qc_{2y_2}\eta_{y_1} - \frac{3}{2}Pd_{1y_2}\eta_{y_1} - 3Qc_{2y_2}\eta_{y_1} - \frac{3}{2}Pd_{1y_2}\eta_{y_1} -$$
$$-6Qa_2\eta_{y_1}\eta_{y_2} - \frac{3}{2}a_{2y_1y_2} - 3Qc_2\eta_{y_1y_2} - \frac{3}{2}Pd_1\eta_{y_1y_2} = 0 \tag{25}$$

$$F^3(\xi): -5k_1^3k_2PQa_1 + 5k_1k_2^3PQa_1 + 6k_1k_2Pa_0a_1 + 27k_1k_2Qa_1a_2 + 6k_1k_2Pa_2b_1 + 27k_1k_2Q^2c_1c_2 +$$
$$+66k_1k_2PRc_1c_2 + 33k_1k_2PQc_1d_1 + 33k_1k_2PQc_2d_2 + 6k_1k_2P^2d_1d_2 + 2k_1Pc_{1t} +$$
$$+2k_1Pa_1\eta_t - 3Pc_{1y_1}\eta_{y_2} - 3Pc_{1y_2}\eta_{y_1} - 3Pa_1\eta_{y_1}\eta_{y_2} - 3Pc_1\eta_{y_1y_2} = 0 \tag{26}$$

$$F^4(\xi): 9k_1k_2Pa_1^2 - 30k_1^3k_2PQa_2 + 30k_1k_2^3PQa_2 + 18k_1k_2Pa_0a_2 + 24k_1k_2Qa_2^2 + 33k_1k_2PQc_1^2 +$$
$$+24k_1k_2Q^2c_2^2 + 54k_1k_2PRc_2^2 + 66k_1k_2PQc_2d_1 + 9k_1k_2P^2d_1^2 + 18k_1k_2P^2c_1d_2 + 3k_1Pc_{2t} +$$
$$+6k_1Pa_2\eta_t - \frac{9}{2}Pc_{2y_1}\eta_{y_2} - \frac{9}{2}Pc_{2y_2}\eta_{y_1} - 9Pa_2\eta_{y_1}\eta_{y_2} - \frac{9}{2}Pc_2\eta_{y_1y_2} = 0 \tag{27}$$

$$F^5(\xi): -6k_1^3k_2P^2a_1 + 6k_1k_2^3P^2a_1 + 36k_1k_2Pa_1a_2 + 111k_1k_2PQc_1c_2 +$$
$$+36k_1k_2P^2c_1d_1 + 36k_1k_2P^2c_2d_2 = 0 \tag{28}$$

$$F^6(\xi): -30k_1^3k_2R^2a_2 + 30k_1k_2^3P^2a_2 + 30k_1k_2Pa_2^2 + 30k_1k_2P^2c_1^2 +$$
$$+84k_1k_2PQc_2^2 + 60k_1k_2P^2c_2d_1 = 0 \tag{29}$$

$$F^7(\xi): 90k_1k_2P^2c_1c_2 = 0 \tag{30}$$

$$F^8(\xi): 63k_1k_2P^2c_2^2 = 0 \tag{31}$$

Exact solutions

Under the condition that:

$$\eta(y_1, y_2, t) = f_1(t)y_1 + f_2(t)g_1(y_2) + f_3(t) \tag{32}$$

or

$$\eta(y_1, y_2, t) = f_4(t)g_2(y_1) + f_5(t)y_2 + f_6(t) \tag{33}$$

where $f_1(t)$, $f_2(t)$, $f_3(t)$, $f_4(t)$, $f_5(t)$, and $f_6(t)$ are differentiable functions with respective to t, while $g_1(y_2)$ and $g_2(y_1)$ are differentiable functions of y_2 and y_1 respectively, solving eqs. (5)-(31), we have six cases:

Case 1:

$$a_2 = \frac{1}{2}P(k_1^2 - k_2^2), \quad b_2 = \frac{1}{2}R(k_1^2 - k_2^2), \quad c_1 = \pm\frac{1}{2}\sqrt{P}(k_1^2 - k_2^2), \quad d_2 = \pm\frac{1}{2}\sqrt{R}(k_1^2 - k_2^2) \tag{34}$$

$$a_0 = \frac{k_1 k_2 (k_1^2 - k_2^2)(Q \pm 6\sqrt{PR}) - 4k_1\eta_t + 6\eta_{y_1}\eta_{y_2}}{12k_1k_2}, \quad a_1 = b_1 = c_2 = d_1 = 0 \tag{35}$$

Case 2:

$$a_2 = \frac{1}{2}P(k_1^2 - k_2^2), \quad c_1 = \pm\frac{1}{2}\sqrt{P}(k_1^2 - k_2^2) \tag{36}$$

$$a_0 = \frac{k_1 k_2 Q(k_1^2 - k_2^2) - 4k_1\eta_t + 6\eta_{y_1}\eta_{y_2}}{12k_1k_2}, \quad a_1 = b_1 = b_2 = c_1 = d_1 = d_2 = 0 \tag{37}$$

Case 3:

$$b_2 = \frac{1}{2}R(k_1^2 - k_2^2), \quad d_2 = \pm\frac{1}{2}\sqrt{R}(k_1^2 - k_2^2) \tag{38}$$

$$a_0 = \frac{k_1 k_2 Q(k_1^2 - k_2^2) - 4k_1\eta_t + 6\eta_{y_1}\eta_{y_2}}{12k_1k_2}, \quad a_1 = a_2 = b_1 = c_1 = c_2 = d_1 = 0 \tag{39}$$

Case 4:

$$a_2 = P(k_1^2 - k_2^2), \quad b_2 = R(k_1^2 - k_2^2) \tag{40}$$

$$a_0 = \frac{2k_1 k_2 Q(k_1^2 - k_2^2) - 2k_1\eta_t + 3\eta_{y_1}\eta_{y_2}}{6k_1k_2}, \quad a_1 = b_1 = c_1 = c_2 = d_1 = d_2 = 0 \tag{41}$$

Case 5:

$$a_2 = P(k_1^2 - k_2^2), \quad a_0 = \frac{2k_1 k_2 Q(k_1^2 - k_2^2) - 2k_1\eta_t + 3\eta_{y_1}\eta_{y_2}}{6k_1k_2} \tag{42}$$

$$a_1 = b_1 = b_2 = c_1 = c_2 = d_1 = d_2 = 0 \tag{43}$$

Case 6:

$$b_2 = R(k_1^2 - k_2^2), \quad a_0 = \frac{2k_1 k_2 Q(k_1^2 - k_2^2) - 2k_1\eta_t + 3\eta_{y_1}\eta_{y_2}}{6k_1k_2} \tag{44}$$

$$a_1 = a_2 = b_1 = c_1 = c_2 = d_1 = d_2 = 0 \tag{45}$$

where k_1 and k_2 are arbitrary constants, the sign " $\pm$ " in eqs. (34) and (35) means that all possible combinations of " $+$ " and " $-$ " can be taken. If it is taken the same sign in c_1 and d_2, then it must be taken " – " in a_0. If it is taken the different signs in c_1 and d_2, then it must be taken " + " in a_0.

Finally, from eqs. (34)-(45) we obtain six formulae of solutions of eq. (1):

$$u=\frac{k_1k_2(k_1^2-k_2^2)(Q\pm 6\sqrt{PR})-4k_1\eta_t+6\eta_{y_1}\eta_{y_2}}{12k_1k_2}+\frac{1}{2}P(k_1^2-k_2^2)F^2(\xi)+\frac{1}{2}R(k_1^2-k_2^2)F^{-2}(\xi)\pm$$

$$\pm\frac{1}{2}\sqrt{P}(k_1^2-k_2^2)F'(\xi)\pm\frac{1}{2}\sqrt{R}(k_1^2-k_2^2)F^{-2}(\xi)F'(\xi) \tag{46}$$

$$u=\frac{k_1k_2Q(k_1^2-k_2^2)-4k_1\eta_t+6\eta_{y_1}\eta_{y_2}}{12k_1k_2}+\frac{1}{2}P(k_1^2-k_2^2)F^2(\xi)\pm\frac{1}{2}\sqrt{P}(k_1^2-k_2^2)F'(\xi) \tag{47}$$

$$u=\frac{k_1k_2Q(k_1^2-k_2^2)-4k_1\eta_t+6\eta_{y_1}\eta_{y_2}}{12k_1k_2}+\frac{1}{2}R(k_1^2-k_2^2)F^{-2}(\xi)\pm$$

$$\pm\frac{1}{2}\sqrt{R}(k_1^2-k_2^2)F^{-2}(\xi)F'(\xi) \tag{48}$$

$$u=\frac{2k_1k_2Q(k_1^2-k_2^2)-2k_1\eta_t+3\eta_{y_1}\eta_{y_2}}{6k_1k_2}+P(k_1^2-k_2^2)F^2(\xi)+R(k_1^2-k_2^2)F^{-2}(\xi) \tag{49}$$

$$u=\frac{2k_1k_2Q(k_1^2-k_2^2)-2k_1\eta_t+3\eta_{y_1}\eta_{y_2}}{6k_1k_2}+P(k_1^2-k_2^2)F^2(\xi) \tag{50}$$

$$u=\frac{2k_1k_2Q(k_1^2-k_2^2)-2k_1\eta_t+3\eta_{y_1}\eta_{y_2}}{6k_1k_2}+R(k_1^2-k_2^2)F^{-2}(\xi) \tag{51}$$

where

$$\xi=k_1x_1+k_2x_2+f_1(t)y_1+f_2(t)g_1(y_2)+f_3(t) \tag{52}$$

or

$$\xi=k_1x_1+k_2x_2+f_4(t)g_2(y_1)+f_5(t)y_2+f_6(t) \tag{53}$$

With the help of eqs. (46)-(51) and Appendices A, B, and C [14], we obtain many exact solutions of eq. (1). For example, selecting $P=1$, $Q=-(1+m^2)$, $R=m^2$, $F(\xi)=\text{ns}\xi$, from eq. (46) we obtain Jacobi elliptic function solutions of eq. (1):

$$u=\frac{k_1k_2(k_1^2-k_2^2)(-1-m^2\pm 6m)-4k_1\eta_t+6\eta_{y_1}\eta_{y_2}}{12k_1k_2}+\frac{1}{2}(k_1^2-k_2^2)\text{ns}^2(\xi)+\frac{1}{2}m^2(k_1^2-k_2^2)\text{sn}^2(\xi)\mp$$

$$\mp\frac{1}{2}(k_1^2-k_2^2)\text{cs}\xi\text{ds}\xi\mp\frac{1}{2}m(k_1^2-k_2^2)\text{sn}^2\xi\text{cs}\xi\text{ds}\xi \tag{54}$$

In the limit at $m \to 1$, the Jacobi elliptic function solutions in eq. (54) degenerate into hyperbolic function solutions:

$$u = \frac{k_1 k_2 (k_1^2 - k_2^2)(-2 \pm 6) - 4k_1\eta_t + 6\eta_{y_1}\eta_{y_2}}{12k_1k_2} + \frac{1}{2}(k_1^2 - k_2^2)\coth^2(\xi) + \frac{1}{2}(k_1^2 - k_2^2)\tanh^2(\xi) \mp$$

$$\mp \frac{1}{2}(k_1^2 - k_2^2)\text{csch}^2\xi \mp \frac{1}{2}(k_1^2 - k_2^2)\text{sech}^2\xi \quad (55)$$

In the limit at $m \to 0$, the Jacobi elliptic function solutions in eq. (54) degenerate into trigonometric function solutions:

$$u = \frac{-k_1 k_2 (k_1^2 - k_2^2) - 4k_1\eta_t + 6\eta_{y_1}\eta_{y_2}}{12k_1k_2} + \frac{1}{2}(k_1^2 - k_2^2)\csc^2(\xi) + \frac{1}{2}(k_1^2 - k_2^2)\sin^2(\xi) \mp$$

$$\mp \frac{1}{2}(k_1^2 - k_2^2)\cot\xi\csc\xi \mp \frac{1}{2}(k_1^2 - k_2^2)\cos\xi \quad (56)$$

Conclusion

This paper studies the (4+1)-D Fokas equation by the generalized F-expansion method, which provides an effective tool to the search for exact solutions of non-linear equations. This paper obtains some exact solutions with arbitrary functions, which might reveal some hidden mechanism for some fascinating phenomena for the Fokas equation. Our theory is suitable for other spatio-temporal dynamical systems.

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