INTEGRABILITY AND LUMP-TYPE SOLUTIONS TO THE 3-D KADOMTSEV-PETVIASHVILI-BOUSSINESQ-LIKE EQUATION

by

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The (3+1)-D Kadomtsev-Petviashvili-Boussinesq-like equation is studied, and its bilinear form, Backlund transformation and Lax pairs are elucidated. Lump-type solutions are obtained, which include periodic lump and interaction lump solutions, through the three-wave method and the ansatz method. The dynamic evolution mechanisms of solutions are illustrated graphically.

Key words: *bilinear form, Backlund transformation, Lax pairs, periodic lump solutions, interaction lump solutions*

Introduction

Non-linear evolution equations (NLEE) can be used to express significant phenomena and dynamical processes in many fields, such as marine engineering, fluid dynamics, plasma physics, solid-state materials and so on [1-5]. Looking for the exact solutions is a particularly significant measure to study NLEE, and researching the integrability of NLEE is a preparatory work to obtain their exact solutions. The Bell-polynomials method was first put forward by Li *et al.* [6] and Lambeet and Springael [7], and employed triumphantly to achieve bilinear form, Backlund transformations, and Lax pairs to NLEE. Many papers have shown the efficiency of this method [8-12]. To obtain the exact solutions, recently, many methods have been proposed, such as the inverse scattering method, the Hirota direct method, the Darboux transformation method, the homogeneous balance method, the homotopy perturbation method, the exp-function method, the three-wave approach and so on [13-19]. Through symbolic computation with the help of MAPLE and MATHEMATICA, a series of intricate and cockamamie calculations can be rigorously solved.

In our paper, we mainly discuss the following (3+1)-D Kadomtsev-Petviashvili-Boussinesq-like equation (KPBL) [20]:

$$u_{xxxy} + 3(u_x u_y)_x + u_{tx} + u_{ty} + u_{tt} - u_{zz} = 0 \tag{1}$$

where u is a function with respect to x, y, z, and t. Equation (1) is first proposed by Wazwaz *et al.* [19]. This equation has the properties of Kadomtsev-Petviashvili and Boussinesq equations. Its Backlund transformation and traveling wave solutions have been studied in [21].

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Bilinear form, Backlund transformations, Lax pairs to the KPBL

In this section, the integrability of eq. (1) is constructed through the Bell polynomials approach. A few utmost vital properties, for instance, Bilinear form, Backlund transformation and the Lax pair are presented.

In order to establish the linearizable bilinear equation of eq. (1), we introduce the transformation $u = cq_x$, where c is a real constant determined later. Inserting the transformation into eq. (1) yields:

$$cq_{4xy} + 3c^2(q_{xx}q_{xy})_x + cq_{xxt} + cq_{txy} + cq_{xtt} - cq_{xzz} = 0 \tag{2}$$

Integrating eq. (2) with respect to x once, we obtain:

$$E(q) = cq_{xxyy} + 3c^2q_{xx}q_{xy} + cq_{xt} + cq_{ty} + cq_{tt} - cq_{zz} = 0 \tag{3}$$

Setting $c = 1$ and combining with P-polynomials, eq. (3) turns into:

$$E(q) = p_{xxxy}(q) + p_{xt}(q) + p_{ty}(q) + p_{tt}(q) - p_{zz}(q) = 0 \tag{4}$$

Through the equation given in [8] and $q = 2\ln(g)$, eq. (4) provides the bilinear form of eq. (1):

$$(D_x^3D_y + D_tD_x + D_tD_y + D_t^2 - D_z^2)gg = 0 \tag{5}$$

In the following, the Backlund transformation and Lax pairs of eq. (1) are presented by the theorems and its unambiguous proof procedures.

Theorem 1. (Backlund transformation): Supposing $f(x, y, z, t)$ and $g(x, y, z, t)$ are two distinct solutions of the bilinear equation, then they satisfy:

$$(D_x^2 - \lambda)f\,g = 0, \quad (D_xD_y - \alpha D_x)fg = 0$$

$$[\partial_x(D_x^2D_y + D_t) + \partial_t(D_t + D_y) - \partial_z(D_z)]fg = 0 \tag{6}$$

with α a free constant. Equation (6) is called the bilinear Backlund transformation of eq. (1), which can be paraded as a relation of the two different solutions $f(x, y, z, t)$ and $g(x, y, z, t)$.

Proof: Making $q = 2\ln(g)$ and $q' = 2\ln(f)$ are two different solutions of eq. (5), we have:

$$E(q') - E(q) = (q' - q)_{xxxy} + 3q'_{xx}q'_{xy} - 3q_{xx}q_{xy} + (q' - q)_{tx} +$$
$$+ (q' - q)_{ty} + (q' - q)_{tt} - (q' - q)_{zz} \tag{7}$$

We introduce some auxiliary variables $2v = q' - q$, $2v = q' + q$, $w = \ln(fg)$, and $v = \ln(fg^{-1})$, and eq. (7) becomes:

$$E(q') - E(q) = 2v_{xxxy} + 6w_{xx}v_{xy} + 6w_{xy}v_{xx} + 2v_{tx} + 2v_{ty} + 2v_{tt} - 2v_{zz} =$$
$$= 2\partial_x[y_{xxy}(v,w) + y_t(v,w)] + 2\partial_t[y_y(v,w) + y_t(v,w)] - 2v_{zz} + R(v,w) \tag{8}$$

where

$$R(v,w) = -2\partial_x[(w_{xx} + v_x^2)v_y] + 2\text{Wronskian}[y_{xy}(v,w), y_y(v,w)] -$$

$$-2w_{xxy}v_x + 2v_x^2 + 6w_{xx}v_{xy} \tag{9}$$

Then, $R(v, w)$ in eq. (9) can be rewritten:

$$R(v,w) = -2\partial_x[(w_{xx} + v_x^2)v_y] + 2\text{Wronskian}[y_{xy}(v,w), y_y(v,w)] + 6\lambda v_{xy} \tag{10}$$

Combination of eqs. (10)-(12), we generate the system of y-polynomials:

$$y_{xx}(v,w) - \lambda = 0, \quad y_{xy}(v,w) - \alpha y_x(v,w) = 0$$

$$\partial_x[y_{xxy}(v,w) + y_t(v,w)] + \partial_t[y_t(v,w) + y_y(v,w)] - \partial_z[y_z(v,w)] = 0 \tag{11}$$

Taking advantage of the expression in [8], the system eq. (11) can be translated into the Backlund transformation eq. (6). In the next *Theorem*, we will find the Lax pairs of eq. (1) via the system of y-polynomials eq. (11).

Theorem 2. (Lax pairs): eq. (1) satisfies the following Lax pairs:

$$\psi_{xx} + (u_{xx} - \lambda)\psi = 0, \quad \psi_{xy} + u_{xy}\psi - \alpha\psi_x = 0$$

$$\partial_x(u_{xx}\psi_y + 2u_{xy}\psi_x + \psi_{xxy}) + \partial_t(\psi_t + \psi_y) - \partial_z(\psi_z) = 0 \tag{12}$$

where u is a solution of eq. (1).

Proof: Using the Cole-Hopf transformation $v = \ln(\psi)$ and the expressions [8-10], we have the following formulas:

$$y_t(v,w) = \frac{\psi_t}{\psi}, \quad y_x(v,w) = \frac{\psi_x}{\psi}, \quad y_{xx}(v,w) = q_{xx} + \frac{\psi_{xx}}{\psi}$$

$$y_{xy}(v,w) = q_{xy} + \frac{\psi_{xy}}{\psi}, \quad y_{xxy}(v,w) = \frac{q_{xx}\psi_y}{\psi} + \frac{2q_{xy}\psi_x}{\psi} + \frac{\psi_{xxy}}{\psi} \tag{13}$$

Through eq. (13), the system eq. (11) can be linearized into Lax pairs:

$$\psi_{xx} + (u_{xx} - \lambda)\psi = 0, \quad \psi_{xy} + u_{xy}\psi - \alpha\psi_x = 0$$

$$\partial_x(u_{xx}\psi_y + 2u_{xy}\psi_x + \psi_{xxy}) + \partial_t(\psi_t + \psi_y) - \partial_z(\psi_z) = 0 \tag{14}$$

Periodic lump and interaction lump solutions to the KPBL

In this section, we employ the three-wave method [22] and the ansatz method [23, 24] which are a combination of quadratic and exponential function to establish periodic lump solutions and interaction lump solutions for eq. (1).

First of all, for the sake of finding the periodic lump solutions, extending the bilinear form equation is highly necessary. So, eq. (5) is extended as the following form:

$$\begin{aligned} H_{\text{KPBL}} &= (D_x^3 D_y + D_t D_x + D_t D_y + D_t^2 - D_z^2)gg = \\ &= 2(g_{xxxxy}g - g_{xxx}g_y + 3g_{xx}g_{xy} - 3g_x g_{xxy} + g_{xt}g - g_x g_t + \\ &\quad + g_{yt}g - g_y g_t + g_{tt}g - g_t^2 - g_{zz}g + g_z^2) = 0 \end{aligned} \tag{15}$$

Making an assumption that the solution of eq. (15) has the following composition:

$$g = a_1 \cosh(c_1 t + k_1 x + l_1 y + m_1 z) + a_2 \cos(c_2 t + k_2 x + l_2 y + m_2 z) +$$
$$+ a_3 \cosh(c_3 t + k_3 x + l_3 y + m_3 z) \tag{16}$$

Substituting eq. (16) into eq. (15) and collecting the coefficients about hyperbolic functions to zero by means of symbolic computation. We select the following two typical solutions:

– Case 1:

$$a_2 = a_2, \quad a_3 = a_3, \quad c_1 = c_1, \quad k_1 = k_1, \quad k_2 = k_2, \quad l_1 = l_1, \quad m_1 = m_1, \quad a_1 = m_3 = 0$$
$$c_2 = 4k_2^3, \quad c_3 = -k_3 = -\frac{1}{2}, \quad l_2 = -\frac{4k_2}{3(1+4k_2^2)}, \quad l_3 = \frac{8k_2^2}{3(1+4k_2^2)}, \quad m_2 = -2\sqrt{1+4k_2^2} \tag{17}$$

where a_2, a_3, c_1, k_1, k_2, l_1, m_1 are arbitrary constants. Substituting the identified variables into eq. (16), one reads:

$$g_1 = a_2 \cos(k_2 x - \frac{4k_2}{3(1+4k_2^2)} y + 2\sqrt{1+4k_2^2} z + 4k_2^3 t) +$$
$$+ a_3 \cosh\left[-\frac{1}{2} x - \frac{8k_2^2}{3(1+4k_2^2)} y + \frac{1}{2} t \right] \tag{18}$$

Hence, the first periodic lump solution is given by:

$$u_2 = 2\left[-a_2 k_2 \sin(k_2 x - \frac{4k_2}{3(1+4k_2^2)} y + 2\sqrt{1+4k_2^2} z + 4k_2^3 t \right] - \frac{1}{2} a_3 \cdot$$
$$\cdot \sinh\left[-\frac{1}{2} x - \frac{8k_2^2}{3(1+4k_2^2)} y + \frac{1}{2} t \right] \div \left[a_2 \cos(k_2 x - \frac{4k_2}{3(1+4k_2^2)} y + \right.$$
$$\left. +2\sqrt{1+4k_2^2} z + 4k_2^3 t) + a_3 \cosh\left(-\frac{1}{2} x - \frac{8k_2^2}{3(1+4k_2^2)} y + \frac{1}{2} t \right) \right] \tag{19}$$

– Case 2:

$$a_1 = a_1, \quad a_2 = a_2, \quad c_1 = c_1, \quad c_3 = c_3, \quad k_2 = k_2, \quad k_3 = k_3, \quad l_3 = l_3, \quad m_3 = m_3$$
$$a_3 = k_1 = l_2 = m_2 = 0, \quad c_2 = -k_2, \quad l_1 = -\frac{c_1}{1+k_2^2}, \quad m_1 = -\frac{c_1 k_2}{\sqrt{1+k_2^2}} \tag{20}$$

in which a_1, a_2, c_1, c_3, k_2, k_3, l_3, m_3 are free parameters. So, the solution of eq. (15) is:

$$g_2 = a_1 \cosh\left(-\frac{c_1}{1+k_2^2} y + \frac{c_1 k_2}{\sqrt{1+k_2^2}} z + c_1 t \right) + a_2 \cos(k_2 t - k_2 x) \tag{21}$$

The second periodic lump solution is obtained:

$$u_2 = \frac{2a_2 k_2 \sin(k_2 t - k_2 x)}{a_1 \cosh\left(-\dfrac{c_1}{1+k_2^2} y + \dfrac{c_1 k_2}{\sqrt{1+k_2^2}} z + c_1 t\right) + a_2 \cos(k_2 t - k_2 x)} \tag{22}$$

Similarly, in order to find interaction lump solutions, we suppose that the solution of eq. (15) has the following form:

$$g = (a_1 x + a_2 y + a_3 z + a_4 t + a_5)^2 + (a_6 x + a_7 y + a_8 z + a_9 t + a_{10})^2 + \\ + \mathrm{e}^{a_{11}x + a_{12}y + a_{13}z + a_{14}t + a_{15}} + a_{16} \tag{23}$$

where a_i $(1 \le i \le 16)$ are unknown parameters determined later. Substituting eq. (20) into eq. (15) and isolating the coefficients about exponential functions to zero, we obtained:

$$a_1 = a_1,\quad a_5 = a_5,\quad a_6 = a_6,\quad a_{10} = a_{10},\quad a_{14} = a_{14},\quad a_{16} = a_{16},\quad a_7 = -a_6 \\ a_{12} = -a_{14},\quad a_2 = \frac{a_6^2}{a_1},\quad a_4 = -\frac{a_1^2 + a_6^2}{a_1},\quad a_3 = a_8 = a_9 = a_{11} = a_{13} = 0 \tag{24}$$

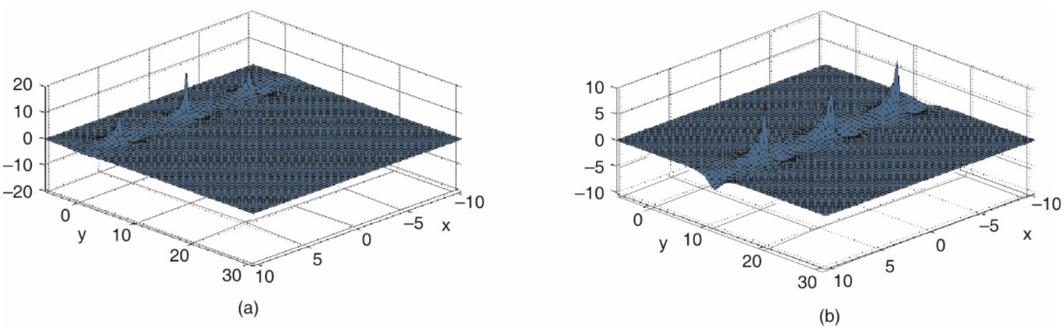
Substituting eq. (24) into eq. (23), we get the solution of eq. (15):

$$g = \left(a_1 x + \frac{a_6^2}{a_1} y + \frac{(a_1^2 + a_1 a_4 + a_6^2)a_4}{\sqrt{a_1 a_4 (a_1^2 + a_1 a_4 + a_6^2)}} z + a_4 t + a_5\right)^2 + (a_6 x - a_6 y + a_{10})^2 + \\ + \mathrm{e}^{\frac{a_{14}(a_1^2 + a_6^2)}{a_1 a_4} y + \frac{\sqrt{a_1 a_4 (a_1^2 + a_1 a_4 + a_6^2)}}{a_1 a_4} z + a_{14} t + a_{15}} + a_{16} \tag{25}$$

Consequently, the interaction lump solution of eq. (1) is successfully found:

$$u = 4\left[\left(a_1 x + \frac{a_6^2}{a_1} y + \frac{(a_1^2 + a_1 a_4 + a_6^2)a_4}{\sqrt{a_1 a_4 (a_1^2 + a_1 a_4 + a_6^2)}} z + a_4 t + a_5\right)a_1 + (a_6 x - a_6 y + a_{10})a_6\right] \div \\ \div \left[\left(a_1 x + \frac{a_6^2}{a_1} y + \frac{(a_1^2 + a_1 a_4 + a_6^2)a_4}{\sqrt{a_1 a_4 (a_1^2 + a_1 a_4 + a_6^2)}} z + a_4 t + a_5\right)^2 + (a_6 x - a_6 y + a_{10})^2 + \right. \\ \left. + \mathrm{e}^{\frac{a_{14}(a_1^2 + a_6^2)}{a_1 a_4} y + \frac{\sqrt{a_1 a_4 (a_1^2 + a_1 a_4 + a_6^2)}}{a_1 a_4} z + a_{14} t + a_{15}} + a_{16}\right] \tag{26}$$

The obtained solutions, eqs. (22) and (26) are illustrated in figs. 1 and 2 with some given values of the parameters.



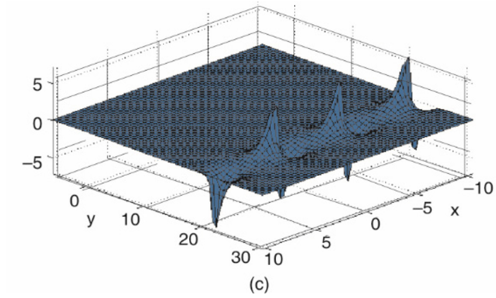
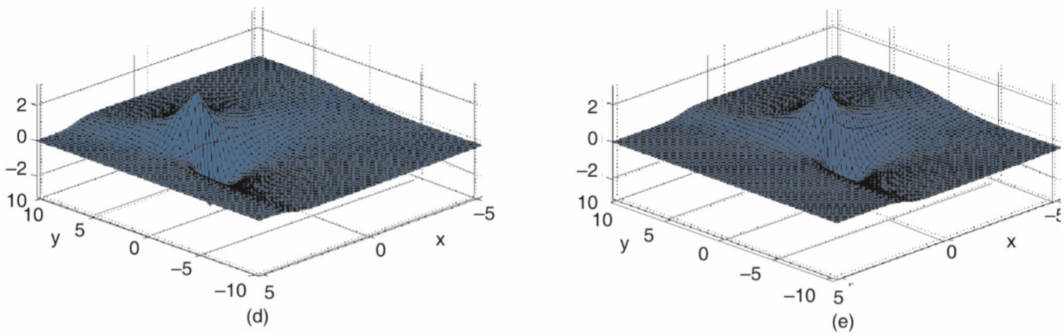


Figure 1. The dynamic evolution mechanisms respect time of the interaction lump solution eq. (22) with $a_1 = 1$, $a_2 = -1$, c_1 0 2, $k_2 = 1$, $z = 1$, when $t = 0$, 5, and 10, respectively



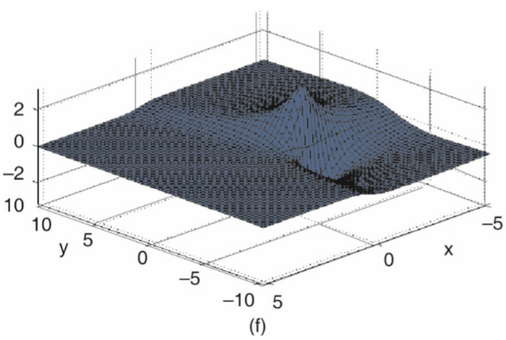


Figure 2. The dynamic evolution mechanisms respect time of the interaction lump solution eq. (26) with $a_1 = a_6 = 2$, $a_4 = a_{10} = a_{14} = 1$, $a_5 = a_{15} = -1$, $z = 0$, when $t = -6$, 0, and 6, respectively

Conclusions

In the present work, we systematically investigated the (3+1)-D KPBL equation. According to the binary Bell polynomial method, we derived the Bilinear form, Backlund transformations, and Lax pairs of eq. (1). Furthermore, periodic lump and interaction lump solutions are obtained by the three-wave method and the ansatz method which are combination of quadratic and exponential function, respectively. The wave propagations are presented in figs. 1 and 2 by selecting the appropriate parameters.

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Nomenclature

x, y, z – space co-ordinate, [m]

t – time co-ordinate, [s]

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