EXACT SOLUTIONS OF THE SPACE-TIME FRACTIONAL EQUAL WIDTH EQUATION

by

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A class of fractional differential equations is investigated in this paper. By the use of modified Remann-Liouville derivative and the tanh-sech method, the exact bright soliton solutions for the space-time fractional equal width are obtained.

Key words: *fractional complex transformation, fractional differential equations, tanh-sech, homogeneous balance methods*

Introduction

In recent years, fractional calculus is used to study differentiation and integration of arbitrary order and is used in many areas such as fluid dynamics, turbulence, image processing, finance, non-linear control theory, astrophysics, stochastic dynamical systems, plasma physics, non-linear biological systems, nanotechnology, and textile engineering. A newly developed method has drawn the attention of many researchers in science and engineering for the exact solution of a PDE, and many actual problems can be modeled by differential equations involving the derivatives of fractional order. According to the best possible method of equation, the exact solutions of most of fractional PDE may be found difficultly. Soliton type solutions have great importance in fluid dynamics, propagation of surface waves, and many other fields of physics and some engineering fields. Krishan and Biswas [1] has recommended an effective ansatz method to find the travel wave solution of space-time fractional modified equal width equation. Ansatz methods have been applied to obtain exact solution such as the fractional biological population model. In this paper, the exact solutions obtained by tanh-sech method are compared with dark soliton solution obtained by using the ansatz method for space-time fractional equal width equation.

This paper also bases on the homogeneous balance method which is a powerful technique for finding exact solutions of fractional PDE introduced by Zhang and Zhang [2]. Various methods for the exact solutions of fractional PDE can be found in [3-7]. As a result, many effective methods have been established to solve PDE exactly. For example, lumped Galerkin methods based on B-splines, and also these methods were implemented for fractional differential-difference equations in quoted equation [8-10].

In this paper, we aim to get the exact solution of the space-time fractional equal width equation. The fractional equal width equation is transformed to ODE by the method of fractional complex transform and some useful formulas of modified Riemann-Liouville derivative.

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Fractional derivative and integration

Recently, several effective methods were utilized to resolve the derivatives of fractional order. Riemann-Liouville fractional derivative has introduced to look for solutions of PDE. Jumarie has obtained some useful formulas by proposing a modified Riemann-Liouvville derivative. In order to investigate space-time fractional modified equal width equation, we give some properties and formula of the modified Riemann-Liouville derivative which is used further in this paper. Assume that $f: R \rightarrow R$, $x \rightarrow f(x)$ denotes a continuous function. The Riemann-Liouville derivative is introduced in [4].

$$\mathrm{D}_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_0^\alpha (x-\xi)^{-\alpha} \left[f(\xi) - f(0) \right] \mathrm{d}\xi, \quad 0 \le \alpha \le 1 \tag{1}$$

where

$$\mathrm{D}_x^\alpha f(x) = [f^{\alpha-n}(x)]^{(n)}, \quad n \le \alpha < n+1, \quad n \ge 1 \tag{2}$$

Some properties of Riemann-Liouville fractional derivative are defined:

$$\mathrm{D}_x^\alpha x^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}, \quad \gamma > 0 \tag{3}$$

$$\mathrm{D}_x^\alpha [cf(x)] = c\mathrm{D}_x^\alpha f(x), \quad c = cons\tan t \tag{4}$$

$$\mathrm{D}_x^\alpha [af(x) + bf(x]) = a\mathrm{D}_x^\alpha f(x) + b\mathrm{D}_x^\alpha f(x) \tag{5}$$

$$\mathrm{D}_x^\alpha f[g(x)] = f_g^{'}[g(x)]\mathrm{D}_x^\alpha g(x) \tag{6}$$

Applications

Consider the following fractional PDE of the type:

$$M(u, u_x, u_t, \mathrm{D}_t^\alpha u, \mathrm{D}_x^\alpha u, ...) = 0, \quad 0 \le \alpha \le 1 \tag{7}$$

where x and t are two independent variables, u – an unknown function, $\mathrm{D}_t^\alpha u$ and $\mathrm{D}_x^\alpha u$ – the Riemann-Liouville of fractional derivatives of u. Using the fraction complex transform which is proposed by Li and He [11].

$$u(x,t) = U(\xi)$$

$$\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{t^\alpha}{\Gamma(1+\alpha)} \tag{8}$$

where k is non-zero arbitrary constant, eq. (7) can be converted to an integer order PDE [12]. The chain rule can be calculated:

$$\mathrm{D}_t^\alpha u = \sigma_t \frac{\mathrm{d}U}{\mathrm{d}\xi} \mathrm{D}_t^\alpha \xi$$

$$\mathrm{D}_x^\alpha u = \sigma_x \frac{\mathrm{d}U}{\mathrm{d}\xi} \mathrm{D}_x^\alpha \xi \tag{9}$$

where σ_x and σ_t are called the sigma indexes [13]. Without loss of generality we can take $\sigma_x = \sigma_x = l$, where l is constant.

Substitute eq. (8) into eq. (7) and use chain rule defined by eq. (9), the eq. (7) can be transformed into an ODE equation of the following form:

$$G\left(U, \frac{\mathrm{d}U}{\mathrm{d}\xi}, \frac{\mathrm{d}^2U}{\mathrm{d}\xi^2}, ...\right) = 0 \tag{10}$$

where G is a polynomial in the variable U and its derivatives.

We consider the following fractional modified equal width equation [14]:

$$\mathrm{D}_t^{\alpha} u + \varepsilon u^2 \mathrm{D}_x^{\alpha} u - \delta \mathrm{D}_{xxt}^{3\alpha} u = 0 \tag{11}$$

where α is a parameter describing the order of the fractional space and time derivative, ε and δ are real parameters. Among them when $\alpha = 1$, eq. (11) is called the modified equal width equation. This equation has been solved by Raslan [15] with the first integral method. Additionally, Taghizadeh and Mirzazaden [16] and his colleagues have implemented the modified simple equation method to get the exact solutions of the equal width equation.

By using fractional complex transform:

$$u(x,t) = U(\xi), \quad \xi = \frac{kx^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{\alpha}}{\Gamma(1+\alpha)} \tag{12}$$

eq. (11) can be converted to:

$$lU' + kl\varepsilon U^2 U' - l^3 k^2 \delta U''' = 0 \tag{13}$$

where $U' = \mathrm{d}U/\mathrm{d}\xi$. By once integration and setting the constants of integration to zero we get:

$$lU + \frac{kl\varepsilon}{3} U^3 - l^3 k^2 \delta U'' = 0 \tag{14}$$

Substituting:

$$U(\xi) = a_0 + \sum_{i=1}^{n} a_i Y^i$$

into eq. (14) and balancing U'' with U^3 of (14), we can get $n = 1$. Hence, we have the following ansatz:

$$U(\xi) = a_0 + a_1 Y \tag{15}$$

where $Y = \tanh(\xi)$.

We also have:

$$\frac{\mathrm{d}U}{\mathrm{d}\xi} = (1-Y^2)\frac{\mathrm{d}U}{\mathrm{d}Y} = (1-Y^2)a_1$$

$$\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2} = (1-Y^2)\left[-2Y\frac{\mathrm{d}U}{\mathrm{d}Y} + (1-Y^2)\frac{\mathrm{d}^2U}{\mathrm{d}Y^2}\right] = -2Ya_1 + 2Y^3 a_1 \tag{16}$$

Submitting eq. (15) along with eq. (16) into eq. (14), and collecting all terms of the same power of Y^i (i = 0, 1, 2, …, n) and simultaneously equating to zero, we obtain the following system of non-linear algebraic equation:

$$\text{Coefficient of } Y^0: \ la_0 + \frac{kl\varepsilon}{3} a_0{}^3 = 0$$

$$\text{Coefficient of } Y^1: \ la_1 + kl\varepsilon a_0{}^2 a_1 + 2\delta l^3 k^2 a_1 = 0$$

$$\text{Coefficient of } Y^2: \ lk\varepsilon a_0 a_1{}^2 = 0$$

$$\text{Coefficient of } Y^3: \ \frac{l\varepsilon k}{3} a_1{}^3 \_ 2\delta l^3 k^2 a_1 = 0$$

Finally, we can obtain:

$$k = \pm\sqrt{\frac{-1}{2\delta l^2}}, \quad a_0 = 0, \quad a_1 = \pm\sqrt{\frac{6\delta l^2 k}{\varepsilon}} \tag{17}$$

Substituting eq. (17) into eq. (15), we get:

$$u(x,t) = U(\xi) = a_0 + a_1 Y = \pm\sqrt{\frac{6\delta l^2 k}{\varepsilon}} \tanh\left[\pm\sqrt{\frac{-1}{2\delta l^2}} \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{t^\alpha}{\Gamma(1+\alpha)}\right] \tag{18}$$

where $\delta < 0$ and $k/\varepsilon < 0$.

In order to solve the proposed problem, we make the following hypothesis. This method depends on Mizazadeh [17] and Guner [18]. Substituting eq. (9) with eq. (19) and eq. (20) into eq. (11), it can reduce into an ODE:

$$lcU + \frac{kl\varepsilon}{3} U^3 - \delta l^3 ck^2 U'' = 0 \tag{19}$$

We introduce

$$U(\xi) = A \tanh^p \xi \tag{20}$$

with

$$\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{ct^\alpha}{\Gamma(1+\alpha)} \tag{21}$$

where the parameters A and k are the free parameters and c – the velocity of the solution. The exponent p will be determined later.

It is calculated by the derivative:

$$\frac{\mathrm{d}^2 U(\xi)}{\mathrm{d}\xi^2} = Ap[(p-1)\tanh^{p-2}\xi - 2p\tanh^p \xi + (p+1)\tanh^{p+2}\xi] \tag{22}$$

and

$$U^3(\xi) = A^3 \tanh^{3p} \xi \tag{23}$$

Substituting eqs. (20)-(23) into eq. (19), we obtain:

$$lcA\tanh^{p}\xi+\frac{lk\varepsilon}{3}A^{3}\tanh^{3p}\xi-l^{3}ck^{2}\delta Ap(p-1)\tanh^{p-2}\xi-$$
$$-l^{3}ck^{2}\delta Ap2p\tanh^{p}\xi+l^{3}ck^{2}\delta Ap(p+1)\tanh^{p+2}\xi=0 \tag{24}$$

and equating the exponents $p + 2$ and $3p$, we have:

$$p = 1 \tag{25}$$

By balancing the coefficient of $\tanh^{3p}\xi$ and $\tanh^{p+2}\xi$ terms in eq. (24), we can get:

$$A=\pm l\sqrt{6\delta kc} \tag{26}$$

Meanwhile, setting the coefficients of $\tanh^{p}\xi$ terms in eq. (24) to zero, we can obtain:

$$k=\pm\sqrt{\frac{-1}{2\delta l^{2}}} \tag{27}$$

The exact solution of the space-time fractional eq. (19) is given by:

$$U=\pm\sqrt{6\delta kc}\tanh\left[\pm\sqrt{\frac{-1}{2\delta l^{2}}}\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right] \tag{28}$$

From eqs. (26) and (27), we know $\delta < 0$ and $kc < 0$.

Assume that eq. (19) has a solution of the form:

$$U(\xi)=A\frac{1}{\cosh^{p}\xi} \tag{29}$$

with
$$\xi=\frac{kx^{\alpha}}{\Gamma(1+\alpha)}+\frac{ct^{\alpha}}{\Gamma(1+\alpha)} \tag{30}$$

where the parameters A and k are free parameters. The exponent p will be determined later. Substituting eq. (29) with eq. (30) into eq. (19), we get:

$$\frac{\mathrm{d}^{2}U(\xi)}{\mathrm{d}\xi^{2}}=Ap^{2}\frac{1}{\cosh^{p}\xi}-Ap^{2}\frac{1}{\cosh^{p+2}\xi}-Ap\frac{1}{\cosh^{p+2}\xi} \tag{31}$$

and
$$U^{3}(\xi)=A^{3}\frac{1}{\cosh^{3p}\xi} \tag{32}$$

Substituting eqs. (29)-(32) and eq. (30) into eq. (19):

$$lcA\frac{1}{\cosh^{p}\xi}+\frac{lk\varepsilon}{3}\frac{1}{\cosh^{3p}\xi}-l^{3}ck^{2}\delta Ap^{2}\frac{1}{\cosh^{p}\xi}-$$
$$-l^{3}ck^{2}\delta Ap^{2}\frac{1}{\cosh^{p+2}\xi}-l^{3}ck^{2}\delta Ap\frac{1}{\cosh^{p+2}\xi}=0 \tag{33}$$

and equating the exponents $3p$ and $p + 2$ leads to:

$$p = 1 \tag{34}$$

By balancing the coefficients of $1/\cosh^{3p}\xi$ and $1/\cosh^{p+2}\xi$ terms in eq. (33):

$$lcA - l^3ck^2\delta Ap^2 = 0 \tag{35}$$

$$\frac{kl\varepsilon}{3}A^3 + Al^3ck^2\delta + Al^3ck^2\delta = 0 \tag{36}$$

then we get

$$k = \pm\frac{1}{l\sqrt{\delta}}, \quad A = \pm\sqrt{\frac{-6l^2kc\delta}{\varepsilon}} \tag{37}$$

Therefore, we obtain the exact solution of the space-time eq. (11):

$$u(x,t) = U(\xi) = \pm\sqrt{\frac{-6l^2kc\delta}{\varepsilon}}\frac{1}{\cosh\left[\pm\frac{1}{l\delta}\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{t^\alpha}{\Gamma(1+\alpha)}\right]} \tag{38}$$

Conclusions

In this paper, the PDE of space-time fractional order is studied based on symbolic computation method of fractional equation. We have obtained the exact solution of fractional differential equations by the tanh-sech method. The results show that this method is accurate and effective, and it can be used for many other non-linear fractional differential equations with real world applications.

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