# SYMMETRY REDUCTION
# A Promising Method for Heat Conduction Equations

by

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*Though there are many approximate methods, e. g., the variational iteration method and the homotopy perturbation, for non-linear heat conduction equations, exact solutions are needed in optimizing the heat problems. Here we show that the Lie symmetry and the similarity reduction provide a powerful mathematical tool to searching for the needed exact solutions. Lie algorithm is used to obtain the symmetry of the heat conduction equations and wave equations, then the studied equations are reduced by the similarity reduction method.*

Key words: *Lie algorithm, similarity reduction method, wave equations, heat conduction equations*

## Introduction

Heat conduction arises everywhere in engineering, especially in nanotechnology, for example, the solvent evaporation in the process of fabrication of nanofibers plays an important role in control the morphology of the nanofibers and their mechanical, chemical and electronic properties [1-7]. It is difficult to be solved for non-linear heat equations with complex boundary conditions. In order to find the solution properties of a complex heat conduction problem, there are useful analytical methods, for examples, the homotopy perturbation method [8-10], the variational iteration method [11-13], and the variational principle-based methods [14-17]. It is well known, the Lie group method and the symmetry reduction are two powerful approaches to constructing exact solutions of PDE [18-21]. Furthermore, based on this method, many exact solutions can be obtained. The symmetry reduction has attracted much attention as it is a useful mathematical tool to solving non-linear heat problems.

In this paper, symmetry analysis and exact solutions of a class of heat conduction equations and wave equations are considered. Firstly, Lie algorithm is used to get the infinitesimal generator of symmetry for the given equation, then similarity reduction method is used to complete the reduction of the heat conduction equations and wave equations.

## Lie algorithm

Consider the following PDE:

$$G_i[u] = G_i(x, u, u_{(1)}, \cdots) = 0, \qquad i = 1, 2, \cdots \tag{1}$$

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the one-parameter Lie group of point transformations:

$$\begin{cases} x^* = f(x,u,\varepsilon) = x + \varepsilon\xi(x,u) + O(\varepsilon^2) \\ u^* = g(x,u,\varepsilon) = u + \varepsilon\eta(x,u) + O(\varepsilon^2) \end{cases} \tag{2}$$

leaves the PDE (1) invariant, if and only if its $k^{\text{th}}$ extension leaves (1) invariant.

Let

$$X = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u} \tag{3}$$

is the infinitesimal generator of the Lie group of point transformations (2), and:

$$\begin{aligned} X^{(k)} = {} & \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u} + \eta_i^{(1)}(x,u,\partial u)\frac{\partial}{\partial u_i} + \cdots \\ & + \eta_{i_1 i_2 \cdots i_k}^{(k)}(x,u,\partial u,\partial^2 u,\cdots,\partial^k u)\frac{\partial}{\partial u_{i_1 i_2 \cdots i_k}} \end{aligned} \tag{4}$$

is the $k^{\text{th}}$ extended infinitesimal generator of (3). Then one-parameter Lie group of point transformations (2) is admitted by PDE (1), if and only if:

$$X^{(k)} G_i[u]|_{G_i[u]=0} = 0, \qquad i = 1,2,\cdots \tag{5}$$

## Similarity reduction and exact solutions to heat conduction equations

Consider the following heat conduction equations:

$$u_t = a u_{xx} \tag{6}$$

and suppose that the infinitesimal generator admitted by eq. (6) is:

$$X = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u \tag{7}$$

The determining equations of $X$ derived from eq. (5) are:

$$\begin{gathered} \eta_t - a\eta_{xx} = 0 \\ 2a\xi_{xu} - a\eta_{uu} = 0 \\ -\xi_t + a\xi_{xx} - 2a\eta_{xu} = 0 \\ \tau_u = \tau_x = 0, \ \xi_u = 0 \\ 2\xi_x - \tau_t + a\tau_{xx} = 0 \end{gathered} \tag{8}$$

The general solution of determining eqs. (8) is:

$$\xi = (-2c_3 t + c_7)x - 2tc_1 a + c_8$$

$$\tau = -2c_3 t^2 + 2c_7 t + c_9$$

$$\eta = (c_3 t + \frac{c_3}{2a} x^2 + c_1 x + c_2)u + \mathrm{e}^{\sqrt{c_1} x} c_6 \mathrm{e}^{c_1 a t} c_4 + \frac{c_6 \mathrm{e}^{c_1 a t} c_5}{\mathrm{e}^{\sqrt{c_1} x}} \tag{9}$$

where $c_i (i = 1, 2, \cdots, 9)$ are arbitrary constants. Further, the Lie algebra of infinitesimal symmetries of eq. (6) is spanned by the vector field:

$$X_1 = \partial_x$$

$$X_2 = \partial_t$$

$$X_3 = u\partial_u$$

$$X_4 = x\partial_x + 2t\partial_t$$

$$X_5 = 2ta\partial_x - ux\partial_u ,\ X_6 = 4txa\partial_x + 4t^2 a\partial_t + (-2uta - ux^2)\partial_u \tag{10}$$

– For the generator $X_4 = x\partial_x + 2t\partial_t$, we have the following similarity variables:

$$U = \frac{t}{x^2}, \quad V = u \tag{11}$$

and the group-invariant solution is $V = G(U)$, that is:

$$u = G(U) \tag{12}$$

Substituting eq. (12) into eq. (6),we obtain the following reduction equation:

$$(1 - 6aU)G'(U) - 4aU^2 G''(U) = 0 \tag{13}$$

– For the generator $X_5 = 2ta\partial_x - ux\partial_u$, we have the following similarity variables:

$$U = t, \quad V = \frac{u}{\mathrm{e}^{-\frac{x^2}{4ta}}} \tag{14}$$

and the group-invariant solution is $V = G(U)$, that is:

$$u = G(U)\mathrm{e}^{-\frac{x^2}{4ta}} \tag{15}$$

Substituting eq. (15) into eq. (6),we obtain the following reduction equation:

$$\frac{G(U) + 2UG'(U)}{2U} = 0 \tag{16}$$

Therefore, eq. (6) has a solution:

$$u = \frac{c_1}{\sqrt{t}} \mathrm{e}^{-\frac{x^2}{4td}}$$

where $c_1$ is an arbitrary constant.

– For the generator $X_6 = 4txa\partial_x + 4t^2 a\partial_t + (-2uta - ux^2)\partial_u$, we have the following similarity variables:

$$U = \frac{t}{x}, \quad V = \frac{u\sqrt{x}}{\mathrm{e}^{-\frac{x^2}{4ta}}} \tag{17}$$

and the group-invariant solution is $V = G(u)$, that is:

$$u = \frac{G(U)}{\sqrt{x}} \mathrm{e}^{-\frac{x^2}{4ta}} \tag{18}$$

Substituting eq. (18) into eq. (6),we obtain the following reduction equation:

$$3G(U) + 4U[3G'(U) + UG''(U)] = 0 \tag{19}$$

Therefore, eq. (6) has a solution:

$$u = \frac{1}{\sqrt{x}} \mathrm{e}^{-\frac{x^2}{4ta}} \left[ \frac{c_1}{\sqrt{\frac{t}{x}}} + \frac{c_2}{\left(\frac{t}{x}\right)^{3/2}} \right]$$

where $c_1, c_2$ are arbitrary constants.

*Remark:* In most literature, an infinitesimal generator like (10) is used in similarity reduction method, *e. g.* [7-9], but as far as the author knows, there are few papers using infinitesimal generator like eq. (7) to reduce PDE directly. In the next section, we give a special case of similarity reduction method by using infinitesimal generator like eq. (7) to reduce PDE.

## Similarity reduction and exact solutions of a class of wave equations

Consider the following wave equations:

$$\frac{u_{xx}}{u_x^2} = u_{tt} - u_t \tag{20}$$

and suppose the infinitesimal generator admitted by eq. (20) is:

$$X = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u \tag{21}$$

The determining equations of $X$ derived from eq. (5) are:

$$\begin{gathered} \xi_u = \xi_t = \xi_{xx} = 0 \\ \tau_x = \tau_u = 0 \\ \eta_t - \eta_{tt} = 0 \\ -\eta_u + \tau_t = 0 \\ -2\eta_{tu} + \tau_t + \tau_{tt} = 0 \\ \eta_x = \eta_{uu} = 0 \end{gathered} \tag{22}$$

The general solution of determining eqs. (22) is:

$$\xi = c_1 x + c_2$$

$$\tau = c_3 + c_4 \mathrm{e}^t$$

$$\eta = (c_4 u + c_6)\mathrm{e}^t + c_5 \tag{23}$$

where $c_i (i = 1, 2, \cdots, 6)$ are arbitrary constants. Further, eq. (20) admits six parameter symmetry, the infinitesimal generator is:

$$X = (c_1 x + c_2)\partial_x + (c_3 + c_4 \mathrm{e}^t)\partial_t + [(c_4 u + c_6)\mathrm{e}^t + c_5]\partial_u \tag{24}$$

Thus, we know that the Lie algebra of infinitesimal symmetries of eq. (20) is spanned by the vector field:

$$X_1 = \partial_x$$
$$X_2 = \partial_t$$
$$X_3 = \partial_u$$
$$X_4 = x\partial_x$$
$$X_5 = \mathrm{e}^t \partial_u$$
$$X_6 = \mathrm{e}^t \partial_t + u\mathrm{e}^t \partial_u \tag{25}$$

The one-parameter groups $G_i$ generated by the $X_i = (i = 1,\ldots, 6)$ are:

$$G_1 : (x, t, u) \to (x + \varepsilon, t, u) \tag{26}$$

$$G_2 : (x, t, u) \to (x, t + \varepsilon, u) \tag{27}$$

$$G_3 : (x, t, u) \to (x, t, u + \varepsilon) \tag{28}$$

$$G_4 : (x, t, u) \to (\mathrm{e}^{\varepsilon} x, t, u) \tag{29}$$

$$G_5 : (x, t, u) \to (x, t, u + \mathrm{e}^t \varepsilon) \tag{30}$$

$$G_6 : (x, t, u) \to [x, -\ln(-\mathrm{e}^t \varepsilon + 1) + t, u(\mathrm{e}^t \varepsilon + 1)] \tag{31}$$

Since each $G_i = (i = 1,\ldots, 6)$ is a symmetry group, it implies that if $u = f(x, t)$ is a solution of eq. (20), then $u^{(1)},\ldots, u^{(6)}$ given below are solutions of eq. (20) as well:

$$u^{(1)} = (x - \varepsilon, t) \tag{32}$$

$$u^{(2)} = (x, t - \varepsilon) \tag{33}$$

$$u^{(3)} = f(x, t) + \varepsilon \tag{34}$$

$$u^{(4)} = f(\mathrm{e}^{-\varepsilon} x, t) \tag{35}$$

$$u^{(5)} = f(x,t) + \mathrm{e}^t \varepsilon \tag{36}$$

$$u^{(6)} = f[x, -\ln(\mathrm{e}^t \varepsilon + 1) + t](\mathrm{e}^t \varepsilon + 1) \tag{37}$$

The characteristic equation corresponding with eq. (24) is:

$$\frac{\mathrm{d}u}{(c_4 u + c_6)\mathrm{e}^t + c_5} = \frac{\mathrm{d}t}{c_3 + c_4 \mathrm{e}^t} = \frac{\mathrm{d}x}{c_1 x + c_2} \tag{38}$$

Solving eq. (38), we have

*Case 1.* $c_1 = 0, \quad c_3 c_4 \neq 0.$
We have the following invariants of the symmetry $X$:

$$U = \frac{-c_2 \ln(\mathrm{e}^t) + c_2 \ln(c_3 + c_4 \mathrm{e}^t)}{c_3} + x \tag{39}$$

$$V = \frac{c_4 c_5 (c_3 + c_4 \mathrm{e}^t) \ln(c_3 + c_4 \mathrm{e}^t) - c_5 t c_4{}^2 \mathrm{e}^t - \{[(-c_3 u + c_5 (t+1)]c_4 - c_6 c_3\} c_3}{c_3{}^2 c_4 (c_3 + c_4 \mathrm{e}^t)} \tag{40}$$

Therefore, eq. (20) has the following invariant solution:

$$u = \frac{-c_4 c_5 (c_3 + c_4 \mathrm{e}^t) \ln(c_3 + c_4 \mathrm{e}^t) + c_4^2 [G(U) c_3{}^2 + c_5 t] \mathrm{e}^t + \{[G(U) c_3{}^2 + c_5 (t+1)] c_4 - c_6 c_3\} c_3}{c_3{}^2 c_4} \tag{41}$$

Inserting eq. (41) into eq. (20), we arrive at the reduced equation:

$$c_5 - c_2 c_3 G'(U) - c_2^2 G''(U) + \frac{G''(U)}{G'(U)^2} = 0 \tag{42}$$

Consequently, eq. (20) has invariant solution of the form of eq. (41), where $G(U)$ satisfies eq. (42).

*Case 2.* $c_3 = 0, \quad c_1 c_4 \neq 0.$
We have the following invariants of the symmetry $X$:

$$U = \frac{-[c_4 \mathrm{e}^t \ln(c_1 x + c_2) + c_1]\mathrm{e}^{-t}}{c_1 c_4} \tag{43}$$

$$V = \frac{c_5 \ln^2(c_1 x + c_2) c_4 + 2c_1 (c_5 \mathrm{e}^{-t} + c_6) \ln(c_1 x + c_2) - 2u c_1^2 \mathrm{e}^{-t}}{2 c_1 c_4} \tag{44}$$

It is obvious that eq. (20) has the following invariant solution:

$$u = -\frac{\mathrm{e}^t \left[ -\frac{1}{2} c_4 c_5 \ln^2(c_1 x + c_2) - c_1 (c_5 \mathrm{e}^{-t} + c_6) \ln(c_1 x + c_2) + G(U) c_1 c_4 \right]}{c_1{}^2} \tag{45}$$

Substituting eq. (45) into eq. (20) results in an ODE which reads:

$$\frac{-c_1^4c_4c_6+c_1^3c_4[(U+c_4)c_5-c_4G'(U)]+c_1^2(-c_4^2+c_6^2)G''(U)}{c_1c_4[-Uc_5+c_1c_6+c_4G'(U)]^2}+$$

$$+\frac{[Uc_5-c_4G'(U)]^2G''(U)+2c_1c_6[-Uc_5+c_4G'(U)]G''(U)}{c_1c_4[-Uc_5+c_1c_6+c_4G'(U)]^2}=0 \tag{46}$$

Consequently, eq. (20) has invariant solution of the form eq. (45), where $G(U)$ satisfies eq. (46).

*Case 3.* $c_4=0, \quad c_1c_3\neq 0.$

Equation (20) has the following invariant solution:

$$u=\frac{c_6e^t+c_5\ln\left[(c_1x+c_2)^{c_3/c_1}\right]}{c_3}+G(U), \quad U=-\frac{c_3\ln(c_1x+c_2)}{c_1}+t \tag{47}$$

in which $G(U)$ satisfies:

$$G'(U)-G''(U)+\frac{c_1[-c_5+c_3G'(U)]+c_3^2G''(U)}{[c_5-c_3G'(U)]^2}=0 \tag{48}$$

*Case 4.* $c_1=c_3=0, \quad c_2c_4\neq 0.$

Equation (20) has the following invariant solution:

$$u=-\frac{[2G(U)c_2c_4-c_4c_5x^2-2xc_2c_5e^{-t}-2c_6c_2x]e^t}{2c_2^2}, \quad U=-\frac{(e^tc_4x+c_2)e^{-t}}{c_4} \tag{49}$$

in which $G(U)$ satisfies:

$$\frac{U^2c_4^2c_5^2G''(U)+c_2^2\{c_6^2+2c_4c_6G'(U)+c_4^2[-1+G'^2(U)]\}G''(U)}{c_4\{c_2c_6+c_4[-Uc_5+c_2G'(U)]\}^2}+$$

$$+\frac{c_2c_4c_5\{-2Uc_6G''(U)+c_4[1-2UG'(U)G''(U)]\}}{c_4\{c_2c_6+c_4[-Uc_5+c_2G'(U)]\}^2}=0 \tag{50}$$

*Case 5.* $c_1=c_4=0, \quad c_2c_3\neq 0.$

Equation (20) has the following invariant solution:

$$u=\frac{c_5x}{c_2}+\frac{c_6e^t}{c_3}+G(U), \quad U=t-\frac{c_3x}{c_2} \tag{51}$$

in which $G(U)$ satisfies:

$$\frac{c_5^2[G'(U)-G''(U)]+2c_5c_3G'(U)[-G'(U)+G''(U)]+c_3^2[G'^3(U)+G''(U)-G'^2(U)G''(U)]}{[c_5-c_3G'(U)]^2}=0 \tag{52}$$

*Case 6.* $c_3=c_4=0, \quad c_1\neq 0.$

Equation (20) has the following invariant solution:

$$u = \frac{(c_6 \mathrm{e}^t + c_5)\ln(c_1 x + c_2)}{c_1} + G(U), \quad U = t \tag{53}$$

in which $G(U)$ is the solution of the following equation:

$$-\frac{c_1}{c_5 + c_6 \mathrm{e}^t} + G'(t) - G''(t) = 0 \tag{54}$$

Solving the eq (54), we obtain:

$$G(t) = \frac{\mathrm{e}^t c_5{}^2 C_1 + c_1\{[t - \ln(c_5 + c_6 \mathrm{e}^t)]c_5 - e^t \ln(\mathrm{e}^{-t} c_5 + c_6)c_6\}}{c_5{}^2} + C_2 \tag{55}$$

where $C_1$ and $C_2$ are arbitrary constants, and $c_5 \neq 0$, so the invariant solution of eq. (20) is:

$$u = \frac{(c_6 \mathrm{e}^t + c_5)\ln(c_1 x + c_2)}{c_1} + \frac{\mathrm{e}^t c_5^2 C_1 + c_1\{[t - \ln(c_5 + c_6 \mathrm{e}^t)]c_5 - \mathrm{e}^t \ln(\mathrm{e}^{-t} c_5 + c_6)c_6\}}{c_5{}^2} + C_2 \tag{56}$$

*Case 7.* $c_1 = c_3 = c_4 = 0, \quad c_2 \neq 0.$

Equation (20) has the following invariant solution:

$$u = \frac{c_6 x \mathrm{e}^t + c_5 x}{c_2} + G(U), \quad U = t \tag{57}$$

in which $G(U)$ is the solution of the following equation:

$$G'(U) - G''(U) = 0 \tag{58}$$

which leads to $G(U) = C_1 \mathrm{e}^t + C_2$, thus we have:

$$u = \frac{c_6 x \mathrm{e}^t + c_5 x}{c_2} + C_1 \mathrm{e}^t + C_2 \tag{59}$$

where $C_1$ and $C_2$ are arbitrary constants.

## Conclusion

In this paper, firstly, we use Lie algorithm to determine symmetry of a class of heat conduction equations, then similarity reduction method is used to complete the reduction of the given equations. Secondly, by using a special case of similarity reduction method, a class of wave equations are reduced, the example shows the effective of the method.

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