A CLASS OF LUMP SOLUTIONS FOR ITO EQUATION

by

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In this paper, we investigate the exact solutions for the (1+1)-D Ito equation. Some lump solutions are obtained by using Hirota's bilinear method, and the conditions to guarantee analytical and rational localization of the lump solutions are presented. Suitable choices of the involved parameters guaranteeing analyticity of the solution are given. The 3-D plots with particular choices of the involved parameters are illustrated.

Key words: *Ito equation, lump solution, solitons, Hirota's bilinear method*

Introduction

The exact solution of PDE is one of the hot trends in the field of non-linear science. In order to obtain the solutions, some methods have been investigated by scientists and physicists, for example the inverse scattering method [1], Backlund transformation, the exp-function method, variational iteration method, homotopy perturbation method, and Hirota bilinear methods. In recent years, the determination of exact soliton solutions to non-linear wave equations is of great value to understand widely different physical phenomena [2]. There has been a renewed and growing interest in rational solutions to non-linear PDE such as the rational solutions to the Toda lattice equation in Casoratian form [3], KdV-like equation [4], extended Kadomtsev-Petviashvili-like equation with symbolic computation [5]. Particularly important are rationally localized solutions, called lump solutions, and examples of lump solutions are found for many integrable equations. Lump solutions have great importance in fluid dynamics, propagation of surface waves, and many other fields of physics and some engineering fields. Recently, Xu *et al.* [6] proposed the periodic solitary wave and doubly periodic solutions through using the bilinear method and extended homoclinic test approach. Fu *et al.* [7] acquired the periodic solutions by using the Jacobi elliptic function expansion method. Wang *et al.* [8] discussed the traveling wave solutions of single variable by using Riccati equation method. Among these methods, it is well known that the Hirota bilinear formulation is a powerful tool to find explicit solutions to the non-linear PDE. The Hirota bilinear method also plays an important role in presenting lump solutions owing to its simplicity and intelligibility. Lump and rational solution become more and more important and attractive [9-12]. In contract to the soliton solution, lump solution is a different kind of rational function solutions, decayed polynomially in all directions in the space. The lump solution which is called the vortex and anti-vortex solution for Ito equation was firstly put forward by Zakharov [13] and later by Craik [14]. Some lump solutions are explored through homoclinic test technique [15]. There are many other types of equations possessing lump solutions: the 3-D three-wave reso-

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nant interaction [16], the Davey-Stewartson equation II [17]. In this paper, we attend to construct lump solutions of the (1+1)-D Ito equation. One class of lump solutions are obtained through analysis and symbolic computations with maple, which supplement the existing literature on soliton solutions.

Lump solution of Ito equation

In this section, we construct a bilinear representation of an infinite hierarchy related to the Ito equation [18]:

$$u_{2t}+u_{3x,t}+6u_x u_t+3uu_{xt}+3u_{2x}\int_{-\infty}^{x}u_t\mathrm{d}x'=0 \tag{1}$$

which is an extension of the K-dV (mK-dV) type to higher orders. Ito equation also can be used to predict the rolling behavior of ships in irregular sea. It is also used to describe the interaction process of two internal long waves. A lump solution for (1+1)-D Ito equation are obtained by using Hirota's bilinear method. According to Bell polynomial theories of soliton equations, we assume that the solution of eq. (1):

$$u(x,t)=2(\ln f)_{xx} \tag{2}$$

where $f(x, t)$ is unknown real function. By substituting eq. (2) into eq. (1), under the convert:

$$u=2(\ln f)_{xx}=\frac{2(f_{xx}f-f_x^{\,2})}{f^2} \tag{3}$$

the following bilinear equation is obtained:

$$\left[\mathrm{D}_t^2+\mathrm{D}_x^3\mathrm{D}_t\right]ff=0 \tag{4}$$

where D_t, D_x are all the bilinear derivative operators which is defined by [19]:

$$\mathrm{D}_x^m\mathrm{D}_t^n a(x,t)b(x,t)=(\partial_t-\partial_{t'})^n(\partial_x-\partial_{x'})^m a(x,t)b(x',t')|_{(x,t)=(x,t')} \tag{5}$$

Using the dependent variable transformation, we can transform the dimensional reduced bilinear eq. (4) into:

$$\left[\mathrm{D}_t^2+\mathrm{D}_x^3\mathrm{D}_t\right]ff=(2f_{tt}f-2f_t^2)+2ff_{txxx}-2f_tf_{xxx}-6f_{txx}f_x+6f_{xx}f_{tx}=0 \tag{6}$$

Soliton solution

By constructing an assume function that is different from [20], and using the homoclinic assume technique [21], we assume that:

$$f(x,t)=1+b_1(\mathrm{e}^{ipx}+\mathrm{e}^{-ipx})\mathrm{e}^{\Omega t+\lambda}+b_2\mathrm{e}^{2(\Omega t+\lambda)} \tag{7}$$

where p, Ω, λ, b_1, and b_2 are parameters to be determined later. Substituting eq. (7) into eq. (2), yields the exact solution of eq. (1):

$$u(x,t)=2(\ln f)_{xx}=$$

$$=\frac{2b_1(-p^2\mathrm{e}^{ipx}-p^2\mathrm{e}^{-ipx})\mathrm{e}^{\Omega t+\lambda}}{1+b_1\mathrm{e}^{\Omega t+\lambda}(\mathrm{e}^{ipx}+\mathrm{e}^{-ipx})+b_2\mathrm{e}^{2(\Omega t+\lambda)}}-\frac{2b_1^2(ip\mathrm{e}^{ipx}-ip\mathrm{e}^{-ipx})^2(\mathrm{e}^{\Omega t+\lambda})^2}{[1+b_1\mathrm{e}^{\Omega t+\lambda}(\mathrm{e}^{ipx}+\mathrm{e}^{-ipx})+b_2\mathrm{e}^{2(\Omega t+\lambda)}]^2} \tag{8}$$

Substituting eq. (7) into eq. (4) leads to:

$$[-4\sin(px)\Omega b_1 b_2 p^3 + 4\cos(px)\Omega^2 b_1 b_2]\mathrm{e}^{3\Omega t+3\lambda} + 8b_2\Omega^2\mathrm{e}^{2\Omega t+2\lambda} + \\ +[4\sin(px)\Omega b_1 p^3 + 4\cos(px)\Omega^2 b_1]\mathrm{e}^{\Omega t+\lambda} = 0 \tag{9}$$

Equating all coefficients of different powers of $\mathrm{e}^{\Omega t+\lambda}$, $\mathrm{e}^{2(\Omega t+\lambda)}$, $\mathrm{e}^{3(\Omega t+t)}$, to zero, we get:

$$-4\sin(px)\Omega b_1 b_2 p^3 + 4\cos(px)\Omega^2 b_1 b_2 = 0 \tag{10}$$

$$8b_2\Omega^2 = 0 \tag{11}$$

$$4\sin(px)\Omega b_1 p^3 + 4\cos(px)\Omega^2 b_1 = 0 \tag{12}$$

Solving previous system of algebraic eqs. (10)-(12) with MAPLE we have:

$$\Omega = 0, \quad b_1 = b_2 \tag{13}$$

or

$$\Omega = -\frac{\sin(px)p^3}{\cos(px)}, \quad b_2 = 0 \tag{14}$$

Substituting eqs. (13) and (14) into eq. (8), we get an exact periodically breather solitary solution of (1):

$$u = -\frac{4b_1 p^2\mathrm{e}^{\lambda}[b_2\mathrm{e}^{2\lambda}\cos(px) + 2b_1\mathrm{e}^{\lambda} + \cos(px)]}{4b_1^2\mathrm{e}^{2\lambda}\cos(px)^2 + 4b_1\cos(px)[b_2\mathrm{e}^{3\lambda} + \mathrm{e}^{\lambda}] + b_2^2\mathrm{e}^{4\lambda} + 2b_2\mathrm{e}^{2\lambda} + 1} \tag{15}$$

and

$$u = -\frac{4b_1 p^2\mathrm{e}^{-\frac{\sin(px)p^3 t}{\cos(px)}+\lambda}\left[2b_1\mathrm{e}^{-\frac{\sin(px)p^3 t}{\cos(px)}+\lambda} + \cos(px)\right]}{4\mathrm{e}^{-2\frac{\sin(px)p^3 t}{\cos(px)}+2\lambda}\cos(px)^2 b_1^2 + 1 + 4b_1\cos(px)\mathrm{e}^{-\frac{\sin(px)p^3 t}{\cos(px)}+\lambda}} \tag{16}$$

We get two different forms of soliton solutions. As follows, contour plots and 3-D plots of the soliton solution u at $b_1 = 1$, $p = 1$, $b_2 = 0$, $\lambda = 0$, and at $b_1 = -1$, $p = 1$, $b_2 = 2$, $\lambda = 0$ are shown in figs. 1 and 2, respectively.

Lump solution

In order to search for the lump solutions of eq. (1), we introduce the following assumption:

$$f = (a_1 x + a_2 t + a_3)^2 + (a_4 x + a_5 t + a_6)^2 + k\mathrm{e}^{2k_1 x + 2k_2 t} + a_7 \tag{17}$$

where a_1, a_2, a_3, a_4, a_5, a_6, a_7, k_1, k_2, and k are real parameters to be determined later.

Substituting eq. (17) into eq. (2) we have:

$$u = \frac{2(2a_1^2 + 2a_4^2 + 4kk_1^2\mathrm{e}^{2k_1 x+2k_2 t})}{(a_1 x + a_2 t + a_3)^2 + (a_4 x + a_5 t + a_6)^2 + k\mathrm{e}^{2k_1 x+2k_2 t} + a_7} - \\ -\frac{2[2(a_1 x + a_2 t + a_3)a_1 + 2(a_4 x + a_5 t + a_6)a_4 + 2kk_1\mathrm{e}^{2k_1 x+2k_2 t}]^2}{[(a_1 x + a_2 t + a_3)^2 + (a_4 x + a_5 t + a_6)^2 + k\mathrm{e}^{2k_1 x+2k_2 t} + a_7]^2} \tag{18}$$

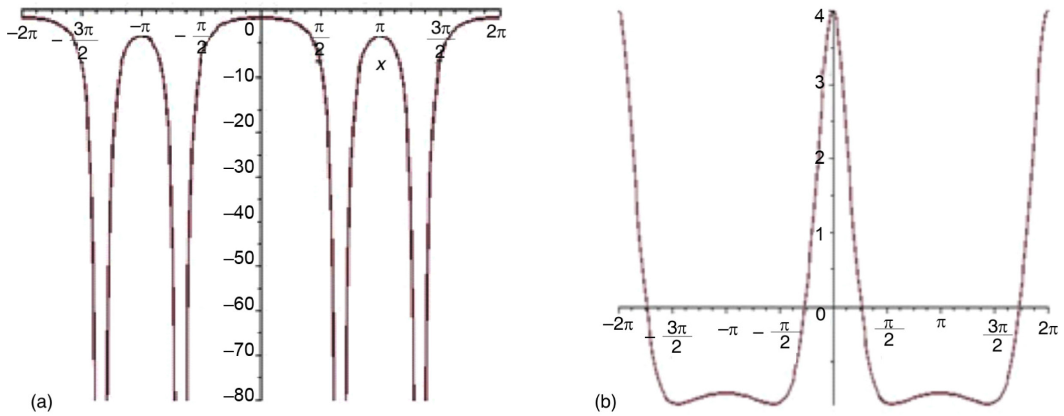

Figure 1. Plots of eq. (15) with (a) $b_1 = 1$, $p = 1$, $b_2 = 0$, $\lambda = 0$, (b) $b_1 = -1$, $p = 1$, $b_2 = 2$, $\lambda = 0$

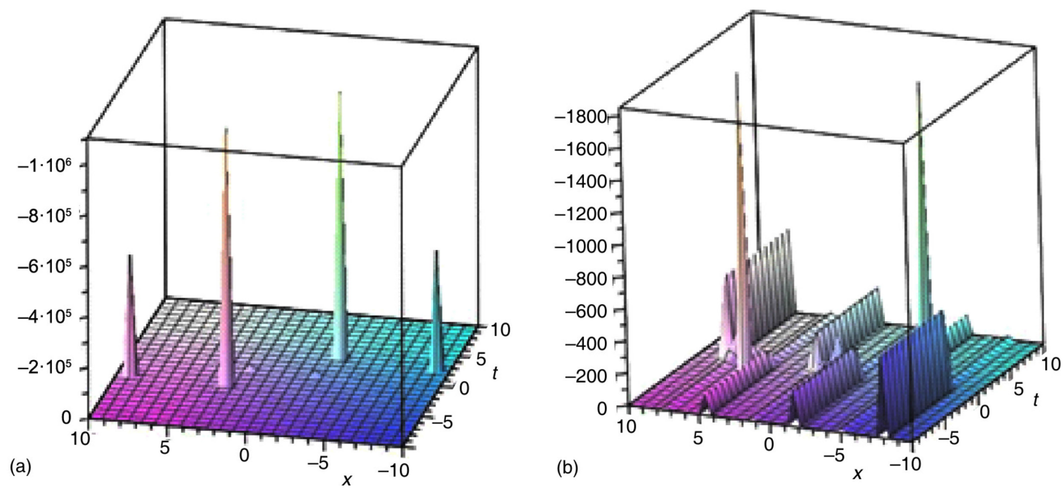

Figure 2. Plots of eq. (16) with (a) $b_1 = 1$, $p = 1$, $b_2 = 0$, $\lambda = 0$, (b) $b_1 = -1$, $p = 1$, $b_2 = 0$, $\lambda = 0$
(*for color image see journal web site*)

Substituting eq. (17) into eq. (4) leads to:

$$[(32{a_2}^2 kk_1^3 - 32{a_5}^2 kk_1^3)t - 32a_2a_3kk_1^3 + 48a_1a_2kk_1^2 + 48a_4a_5kk_1^2 + 4a_2^2k + 4a_5^2k + (-32a_1a_2kk_1^3 - 32a_4a_5kk_1^3)x]\mathrm{e}^{2k_1x+2k_2t} + (-4a_2^4 - 8a_2^2a_5^2 - 4a_5^4)t^2 + [-8a_2^2a_5a_6 - 8a_2a_3a_5^2 - 8a_2^3a_3 - 8a_5^3a_6 + (-8a_1a_2^3 - 8a_1a_2a_5^2 - 8a_2^2a_4a_5 - 8a_4a_5^3)x]t - 16a_2a_3a_5a_6 + 24a_1a_2a_4^2 + 24a_1^3a_2 - 4a_2^2a_3^2 + 4a_2^2a_6^2 + 4a_3^2a_5^2 + 24a_4^3a_5 - 4a_5^2a_6^2 + 4a_2^2a_7 + 4a_5^2a_7 + (-8a_1a_2^2a_3 - 16a_1a_2a_5a_6 + 8a_1a_3a_5^2 + 8a_2^2a_4a_6 - 16a_2a_3a_4a_5 - 8a_4a_5^2a_6)x + (-4a_1^2a_2^2 + 4a_1^2a_5^2 - 16a_1a_2a_4a_5 + 4a_2^2a_4^2 - 4a_4^2a_5^2)x^2 = 0 \quad (19)$$

Equating the coefficients of the eq. (19) to zero we can get:

$$a_2 = 0, \quad a_3 = a_3, \quad a_5 = 0, \quad a_6 = a_6 \tag{20}$$

Substituting eq. (20) into eq. (18) leads to a class of positive solution, analytical functions, f. Meanwhile, through transformation $u(x, t) = 2(\ln f)_{xx}$ one can generate a class of lump solutions of eq. (4):

$$u(x,t) = \frac{2(2a_1^2 + 2a_4^2 + 4kk_1^2 e^{2k_1x+2k_2t})}{(a_1x+a_3)^2 + (a_4x+a_6)^2 + ke^{2k_1x+2k_2t} + a_7} - \\ - \frac{2[2(a_1x+a_3)a_1 + 2(a_4x+a_6)a_4 + 2kk_1e^{2k_1x+2k_2t}]^2}{[(a_1x+a_3)^2 + (a_4x+a_6)^2 + ke^{2k_1x+2k_2t} + a_7]^2} \tag{21}$$

The particular lump solutions with specific values of the included parameters are plotted. Figure 3 represents the spatial structures of the lump solutions by choosing $a_1 = -1/2$, $a_3 = 3/2$, $a_4 = 1/2$, $a_6 = 2/3$, $a_7 = -1$, $k = 0$, $k_1 = -1$, $k_2 = 1$.

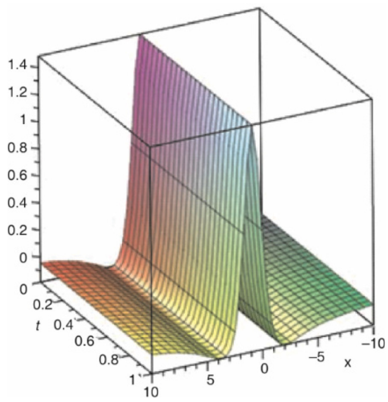

Figure 3. Lump structure of eq. (21) with $a_1 = -1/2$, $a_3 = 3/2$, $a_4 = 1/2$, $a_6 = 2/3$, $a_7 = -1$, $k = 0$, $k_1 = -1$, $k_2 = 1$ (*for color image see journal web site*)

Conclusion

Based on the Hirota bilinear form of the (1+1)-D Ito equation, we presented two classes of solutions including soliton and lump solutions through MAPLE symbolic computation. It is becoming more and more popular to search for lump solutions to non-linear PDE by Hirota bilinear forms. The result is guaranteed by some valid parameters. According to the equation of the constructed solution, the parameter selection in the equation is also crucial. Different parameter selection may result in the diversity of the structure of the lump solution. All this provides abundant structure supplementing existing lump and soliton solutions. We also can investigate different interaction solutions of (1+1)-D Ito equation and other high order equation in the future. This kind of solutions could describe complicated non-linear physical phenomena.

Authors' contributions

All authors have made the same contribution and finalized the current version of this manuscript.

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