NUMERICAL METHOD FOR FRACTIONAL ZAKHAROV-KUZNETSOV EQUATIONS WITH HE'S FRACTIONAL DERIVATIVE

by

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In this paper, a fractional Zakharov-Kuznetsov equation with He's fractional derivative is studied by the fractional complex transform and He's homotopy perturbation method. The solution process is elucidated step by step to show its simplicity and effectiveness of the proposed method.

Key words: *He's fractional derivative, homotopy perturbation method, fractional complex transform, fractional Zakharov-Kuznetsov equation*

Introduction

The Zakharov-Kuznetsov (ZK) equation plays an important role in physics, thermal science, and mathematics, it describes an ion-acoustic wave in a strongly magnetized lossless plasma [1]. An ion-acoustic wave is a longitudinal oscillation of ions and electrons in a plasma, and it can be best described by a fractional model, which reads [1]:

$$\mathrm{D}_{\mathrm{t}}^{\alpha} u + a(u^s)_x + b(u^m)_{xxx} + c(u^s)_{xyy} = 0$$

where $\mathrm{D}_{\mathrm{t}}^{\alpha} = \partial^{\alpha} / \partial t^{\alpha}$ is He's fractional derivative defined as $(0 < \alpha \leq 1)$ [2, 3].

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} (s-t)^{n-\alpha-1} [u_0(s) - u(s)] \mathrm{d}s$$

He's fractional derivative is proposed by He [2-5]. It has been adopted to described a variety of natural phenomena, for examples, polar hair's thermal property [6], silkworm cocoon's permeability [7], nanofiber membrane's permeability [8, 9], heat transfer in Mongolian Yurt [10], fractional Fornberg-Whitham wave [11], and fractional tsunami waves [12, 13].

He's fractional derivative has been a powerful tool in fractional differential equations. In this paper, we successfully use He's fractional derivative to describe the behavior of weakly non-linear ion-acoustic waves in plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field. The fractional partner of the ZK is first derived in this paper for describing weakly non-linear ion-acoustic waves in strongly magnetized loss less plasma in two dimensions [14-16].

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In this paper, we will use He's homotopy perturbation method (HPM) [17-26] and the fractional complex transform (FCT) [2-5] to solve the fractional ZK equations based on He's fractional derivative. The FCT was first proposed by He and Li [27, 28]. The FCT can convert fractional differential equation into its differential partner, therefore the HPM can be effectively applied [29, 30].

The He's HPM

Consider the following differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{1}$$

with the following boundary conditions:

$$B\left(u, \frac{\partial u}{n}\right) = 0, \quad r \in \Gamma \tag{2}$$

where A is a general differential operator, B – a boundary operator, $f(r)$ – a known analytical function, Γ – the boundary of the domain Ω.

We can divide operator A into N and L, where N is a non-linear operator, and L – a linear operator.

Therefore, eq. (1) can be written in the following form:

$$L(u) + N(u) - f(r) = 0 \tag{3}$$

According to the homotopy technique, we can construct a homotopy as $\mu(r,q): \Omega \times [0,1] \to R$ which satisfies:

$$H(\mu,q) = (1-q)[L(\mu) - L(u_0)] + q[A(\mu) - f(r)] = 0 \tag{4}$$

or

$$H(\mu,q) = L(\mu) - L(u_0) + qL(u_0) + q[N(\mu) - f(r)] = 0 \tag{5}$$

where $q \in [0,1]$ is an embedding parameter, u_0 – an initial approximation of eq. (1), which satisfies the boundary conditions. Using eqs. (4) and (5), we can obtain:

$$H(\mu,0) = L(\mu) - L(u_0) = 0 \tag{6}$$

$$H(\mu,1) = A(\mu) - f(r) = 0 \tag{7}$$

The changing process of q from zero to unity is just that of $\mu(r, q)$ from $u_0(r)$ to $u(r)$. This is called deformation in topology. The $L(\mu) - L(\mu_0)$ and $A(\mu) - f(f)$ are called homotopy. Using the HPM, we can first apply the embedding parameter q as a small parameter and assume that the solution of eqs. (4) and (5) can be written into a power series in term of q:

$$\mu = \mu_0 + q\mu_1 + q^2\mu_2 + q^3\mu_3 + q^4\mu_4 + \cdots \tag{8}$$

Setting $q = 1$ in eq. (8), we obtain:

$$u = \lim_{q \to 1} \mu = \mu_0 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \cdots \tag{9}$$

Numerical applications

Consider the fractional ZK (2,2,2) equation in the following form:

$$D_t^\alpha u+(u^2)_x+\frac{1}{8}(u^2)_{xxx}+\frac{1}{8}(u^2)_{yyx}=0,\quad u(x,y,0)=\frac{4}{3}\sigma\sinh^2(x+y) \tag{10}$$

where σ is an arbitrary constant. The first step to solve eq. (10) by HPM is to convert the equation into its differential partner by the fractional complex transform [3, 4, 27, 28]:

$$T=\frac{t^\alpha}{\Gamma(1+\alpha)} \tag{11}$$

We can easily convert eq. (10) into a differential equation, which is the following form:

$$u_T+(u^2)_x+\frac{1}{8}(u^2)_{xxx}+\frac{1}{8}(u^2)_{yyx}=0,\quad u(x,y,0)=\frac{4}{3}\sigma\sinh^2(x+y) \tag{12}$$

To solve eq. (12) by HPM, we construct a homotopy in the following form:

$$H(\mu,q)=(1-q)\left[\frac{\partial\mu}{\partial T}-\frac{\partial u_0}{\partial T}\right]+q\left[\frac{\partial\mu}{\partial T}+\frac{\partial}{\partial x}\mu^2+\frac{1}{8}\frac{\partial^3}{\partial x^3}\mu^2+\frac{1}{8}\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2}\mu^2\right]=0 \tag{13}$$

Assume $u(x, y, 0) = 4/3\sigma\sinh^2(x + y)$ as an initial approximation that satisfies the initial condition. By HPM, we have the following results:

$$q^0:\frac{\partial\mu_0}{\partial T}=\frac{\partial u_0}{\partial T}$$

$$\mu_0(x,y,0)=\frac{4}{3}\sigma\sinh^2(x+y) \tag{14}$$

$$q^1:\frac{\partial\mu_0}{\partial T}=\frac{\partial}{\partial T}u_0-\frac{\partial}{\partial x}\mu_0^2-\frac{1}{8}\frac{\partial^3}{\partial x^3}\mu_0^2-\frac{1}{8}\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2}\mu_0^2$$

$$\mu_1(x,y,0)=0 \tag{15}$$

$$q^2:\frac{\partial\mu_2}{\partial T}=-2\frac{\partial}{\partial T}\mu_0\mu_1-\frac{1}{4}\frac{\partial^3}{\partial x^3}\mu_0\mu_1-\frac{1}{4}\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2}\mu_0\mu_1$$

$$\mu_2(x,y,0)=0 \tag{16}$$

$$q^3:\frac{\partial\mu_3}{\partial T}=-2\frac{\partial}{\partial T}\mu_0\mu_3-\frac{\partial}{\partial x}\mu_1^2-\frac{1}{4}\frac{\partial^3}{\partial x^3}\mu_0\mu_2-\frac{1}{8}\frac{\partial^3}{\partial x^3}\mu_1^2-\frac{1}{4}\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2}\mu_0\mu_2-\frac{1}{8}\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2}\mu_1^2$$

$$\mu_3(x,y,0)=0 \tag{17}$$

$$\cdots\cdots$$

According to eqs. (14)-(17), we obtain:

$$\mu_0(x,y,T)=\frac{4}{3}\sigma\sinh^2(x+y)$$

$$\mu_1(x,y,T)=-\frac{224}{9}\sigma\sinh^3(x+y)\cosh(x+y)T-\frac{32}{3}\sigma^2\sinh(x+y)\cosh(x+y)^3T$$

$$\mu_2(x,y,T)=-\frac{64}{27}\sigma^3[1200\cosh^6(x+y)-2080\cosh^4(x+y)+968\cosh^2(x+y)-79]T^2$$

$$\mu_3(x,y,T)=-\frac{4096}{243}\sinh(x+y)\cosh(x+y)[23800\cosh^6(x+y)-42900\cosh^4(x+y)+$$
$$+22665\cosh^2(x+y)-3142]\sigma^4T^3$$

$$\cdots\cdots$$

So, when $q\to 1$, the solution of eq. (12) is given by:

$$\mu(x,y,T)=\frac{4}{3}\sigma\sinh^2(x+y)-\frac{224}{9}\sigma\sinh^3(x+y)\cosh(x+y)T-$$
$$-\frac{32}{3}\sigma^2\sinh(x+y)\cosh(x+y)^3T-\frac{64}{27}\sigma^3[1200\cosh^6(x+y)-$$
$$-2080\cosh^4(x+y)+968\cosh^2(x+y)-79]T^2-\frac{4096}{243}\sinh(x+y)\cdot$$
$$\cdot\cosh(x+y)[23800\cosh^6(x+y)-42900\cosh^4(x+y)+22665\cosh^2(x+y)-3142]\sigma^4T^3$$

Substituting eq. (11) into previous equation, we have the solution of eq. (10):

$$\mu(x,y,T)=\frac{4}{3}\sigma\sinh^2(x+y)-\frac{224}{9}\sigma\sinh^3(x+y)\cosh(x+y)\frac{t^\alpha}{\Gamma(1+\alpha)}-$$
$$-\frac{32}{3}\sigma^2\sinh(x+y)\cosh(x+y)^3\frac{t^\alpha}{\Gamma(1+\alpha)}-\frac{64}{27}\sigma^3[1200\cosh^6(x+y)-$$
$$-2080\cosh^4(x+y)+968\cosh^2(x+y)-79]\left[\frac{t^\alpha}{\Gamma(1+\alpha)}\right]^2-\frac{4096}{243}\sinh(x+y)\cdot$$
$$\cdot\cosh(x+y)[23800\cosh^6(x+y)-42900\cosh^4(x+y)+22665\cosh^2(x+y)-$$
$$-3142]\sigma^4\left[\frac{t^\alpha}{\Gamma(1+\alpha)}\right]^3$$

Remark 5.1. When $\alpha=1$, the exact solution of eq. (10) is given:

$$u(x,y,t)=\frac{4}{3}\sigma\sinh^2(x+y\text{-}\sigma t).$$

Consider the fractional ZK (3,3,3) equation in the following form:

$$\mathrm{D}_\mathrm{t}^\alpha u+(u^3)_x+2(u^3)_{xxx}+2(u^3)_{yyx}=0 \tag{18}$$

subject to the following initial condition:

$$u(x,y,0)=\frac{3}{2}\sigma\sinh\left[\frac{1}{6}(x+y)\right]$$

We convert the eq. (11) into its differential partner by the fractional complex transform [3, 4, 27, 28]:

$$T=\frac{t^{\alpha}}{\Gamma(1+\alpha)} \tag{19}$$

Therefore, eq. (18) can be written in the following form:

$$u_T+(u^3)_x+2(u^2)_{xxx}+2(u^3)_{yyx}=0 \tag{20}$$

subject to the following initial condition:

$$u(x,y,0)=\frac{3}{2}\sigma\sinh\left[\frac{1}{6}(x+y)\right]$$

To solve eq. (20) applying HPM, we construct a homotopy as follows:

$$H(\mu,q)=(1-q)\left[\frac{\partial\mu}{\partial T}-\frac{\partial u_0}{\partial T}\right]+q\left[\frac{\partial\mu}{\partial T}+\frac{\partial}{\partial x}\mu^3+2\frac{\partial^3}{\partial x^3}\mu^2+2\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2}\mu^3\right]=0 \tag{21}$$

Consider $u(x, y, 0) = 3/2\sigma\sinh[1/6(x + y)]$ as an initial approximation that satisfies the initial condition. Using HPM, we have:

$$q^0:\frac{\partial\mu_0}{\partial T}=\frac{\partial u_0}{\partial T}$$

$$\mu_0(x,y,0)=\frac{3}{2}\sigma\sinh\left[\frac{1}{6}(x+y)\right] \tag{22}$$

$$q^1:\frac{\partial\mu_1}{\partial T}=-\frac{\partial}{\partial T}\mu_0^3-2\frac{\partial^3}{\partial x^3}\mu_0^2-2\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2}\mu_0^2$$

$$\mu_1(x,y,0)=0 \tag{23}$$

$$q^2:\frac{\partial\mu_2}{\partial T}=-3\frac{\partial}{\partial T}\mu_0^2\mu_1-6\frac{\partial^3}{\partial x^3}\mu_0^2\mu_1-6\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2}\mu_0^2\mu_1$$

$$\mu_2(x,y,0)=0 \tag{24}$$

$$q^3:\frac{\partial\mu_3}{\partial T}=-2\frac{\partial}{\partial T}\mu_0^2\mu_2-3\frac{\partial}{\partial x}\mu_0\mu_1^2-6\frac{\partial^3}{\partial x^3}\mu_0^2\mu_2-6\frac{\partial^3}{\partial x^3}\mu_0\mu_1^2-$$

$$-6\frac{\partial^2}{\partial y^2}\frac{\partial}{\partial x}\mu_0^2\mu_2-6\frac{\partial^2}{\partial y^2}\frac{\partial}{\partial x}\mu_0\mu_1^2$$

$$\mu_3(x,y,0)=0 \tag{25}$$

$$\cdots\cdots$$

By iteration eqs. (22)-(25), we obtain:

$$\mu_0(x,y,0)=\frac{3}{2}\sigma\sinh\left[\frac{1}{6}(x+y)\right]$$

$$\mu_1(x,y,T)=-3\sigma^3\sinh^2\left[\frac{1}{6}(x+y)\right]\cosh\left[\frac{1}{6}(x+y)\right]T-\frac{3}{8}\sigma^3\cosh^3\left[\frac{1}{6}(x+y)\right]^3 T$$

$$\mu_2(x,y,T)=\frac{9}{128}T\sigma^3\left\{135T\sigma^2\sinh\left[\frac{1}{6}(x+y)\right]\cosh^4\left[\frac{1}{6}(x+y)\right]+24T\sigma^2\sinh\left[\frac{1}{6}(x+y)\right]-\right.$$
$$\left.-72\cosh^3\left[\frac{1}{6}(x+y)\right]+56\cosh\left[\frac{1}{6}(x+y)\right]\right\}$$

$$\mu_3(x,y,T)=-\frac{9}{1024}T^2\sigma^5\left\{287469T\sigma^2\cosh^7\left[\frac{1}{6}(x+y)\right]-587592T\sigma^2\cosh^5\left[\frac{1}{6}(x+y)\right]+\right.$$
$$\left.+362511T\sigma^2\cosh^3\left[\frac{1}{6}(x+y)\right]-61784\cosh\left[\frac{1}{6}(x+y)\right]-50220\sinh\left[\frac{1}{6}(x+y)\right]\right\}\cdot$$
$$\cdot\cosh^4\left[\frac{1}{6}(x+y)\right]+45360\sinh\left[\frac{1}{6}(x+y)\right]-5220\sinh\left[\frac{1}{6}(x+y)\right]$$

$$\cdots\cdots$$

So, when $q\to 1$, the solution of eq. (13) is given by:

$$\mu(x,y,T)=\frac{3}{2}\sigma\sinh\left[\frac{1}{6}(x+y)\right]-3\sigma^3\sinh^2\left[\frac{1}{6}(x+y)\right]\cosh\left[\frac{1}{6}(x+y)\right]T-$$
$$-\frac{3}{8}\sigma^3\cosh^3\left[\frac{1}{6}(x+y)\right]T+\frac{9}{128}T\sigma^3\left\{135T\sigma^2\sinh\left[\frac{1}{6}(x+y)\right]\cdot\right.$$
$$\cdot\cosh^4\left[\frac{1}{6}(x+y)\right]-153T\sigma^2\sinh\left[\frac{1}{6}(x+y)\right]\cosh^2\left[\frac{1}{6}(x+y)\right]+$$
$$\left.+24T\sigma^2\sinh\left[\frac{1}{6}(x+y)\right]-72\cosh^3\left[\frac{1}{6}(x+y)\right]+56\cosh\left[\frac{1}{6}(x+y)\right]\right\}-$$
$$-\frac{1}{1024}T^2\sigma^5(287469T\sigma^2\cosh^7\left[\frac{1}{6}(x+y)\right]-587592T\sigma^2\cosh^5\left[\frac{1}{6}(x+y)\right]+$$
$$+362511T\sigma^2\cosh^3\left[\frac{1}{6}(x+y)\right]-61784T\sigma^2\cosh\left[\frac{1}{6}(x+y)\right]-50220\sinh\left[\frac{1}{6}(x+y)\right]\cdot$$
$$\cdot\cosh^4\left[\frac{1}{6}(x+y)\right]+45360\sinh\left[\frac{1}{6}(x+y)\right]\cosh^2\left[\frac{1}{6}(x+y)\right]-5220\sinh\left[\frac{1}{6}(x+y)\right]$$

Substituting eq. (19) into previous equation, we have the following solution of eq. (18):

$$\mu(x,y,T)=\frac{3}{2}\sigma\sinh\left[\frac{1}{6}(x+y)\right]-3\sigma^3\sinh^2\left[\frac{1}{6}(x+y)\right]\cosh\left[\frac{1}{6}(x+y)\right]\frac{t^\alpha}{\Gamma(1+\alpha)}-$$

$$-\frac{3}{8}\sigma^3\cosh^3\left[\frac{1}{6}(x+y)\right]\frac{t^\alpha}{\Gamma(1+\alpha)}+\frac{9t^\alpha\sigma^3}{128\Gamma(1+\alpha)}\left\{\frac{135t^\alpha\sigma^2}{\Gamma(1+\alpha)}\sinh\left[\frac{1}{6}(x+y)\right]\cdot\right.$$

$$\cdot\cosh^4\left[\frac{1}{6}(x+y)\right]-\frac{153t^\alpha}{\Gamma(1+\alpha)}\sigma^2\sinh\left[\frac{1}{6}(x+y)\right]\cosh^2\left[\frac{1}{6}(x+y)\right]+$$

$$\left.+\frac{24t^\alpha\sigma^2}{\Gamma(1+\alpha)}\sinh\left[\frac{1}{6}(x+y)\right]-72\cosh^3\left[\frac{1}{6}(x+y)\right]+56\cosh\left[\frac{1}{6}(x+y)\right]\right\}-$$

$$-\frac{1}{1024}\left[\frac{t^\alpha}{\Gamma(1+\alpha)}\right]^2\sigma^5\left\{\frac{287469t^\alpha\sigma^2}{\Gamma(1+\alpha)}\cosh^7\left[\frac{1}{6}(x+y)\right]\right\}-\frac{587592t^\alpha\sigma^2}{\Gamma(1+\alpha)}\cosh^5\left[\frac{1}{6}(x+y)\right]+$$

$$+\frac{362511t^\alpha}{\Gamma(1+\alpha)}\sigma^2\cosh^3\left[\frac{1}{6}(x+y)\right]-\frac{61784t^\alpha}{\Gamma(1+\alpha)}\sigma^2\cosh\left[\frac{1}{6}(x+y)\right]-50220\sinh\left[\frac{1}{6}(x+y)\right]\cdot$$

$$\cdot\cosh^4\left[\frac{1}{6}(x+y)\right]+45360\sinh\left[\frac{1}{6}(x+y)\right]\cosh^2\left[\frac{1}{6}(x+y)\right]-5220\sinh\left[\frac{1}{6}(x+y)\right]$$

Remark 5.2. When $\alpha=1$, the exact solution of eq. (18) is given as:

$$u(x,y,t)=\frac{3}{2}\sigma\sinh^2\frac{1}{6}(x+y-\sigma t) \tag{26}$$

Conclusion

In this paper, based on He's fractional derivative, we have successfully used fractional complex transform and He's homotopy perturbation method for finding the approximate solutions of fractional ZK equations. The results show that the proposed method is very efficient, powerful and easy mathematical method for solving the non-linear fractional differential equations in science and engineering.

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References

[1] Kumar, D., *et al.*, Numerical Computation of Nonlinear Fractional Zakharov-Kuznetsov Equation Arising in Ion-Acoustic Waves, *J. Egypt. Math. Soc. 22* (2014), Oct. pp. 373-378
[2] He, J.-H., A Tutorial Review on Fractal Spacetime and Fractional Calculus, *International Journal of Theoretical Physics, 53* (2014), 11, pp. 3698-3718
[3] Li, X-X., *et al.*, A Fractal Modification of the Surface Coverage Model for an Electrochemical Arsenic Sensor, *Electrochimica Acta*, *296* (2019), Feb., pp. 491-493
[4] He, J.-H., Fractal Calculus and its Geometrical Explanation, *Results in Physics*, *10* (2018), Sept., pp. 272-276
[5] He, J.-H., *et al.*, Geometrical Explanation of the Fractional Complex Transform and Derivative Chain Rule for Fractional Calculus, *Phys. Lett. A*, *376* (2012), 4, pp. 257-259
[6] Wang, Q. L., *et al.*, Fractal Calculus and its Application to Explanation of Biomechanism of Polar Bear Hairs, *Fractals*, *26* (2018), 6, ID 1850086

[7] Fei, D. D., *et al.*, Fractal Approach to Heat Transfer in Silkworm Cocoon Hierarchy, *Thermal Science, 17* (2013), 5, pp. 1546-1548
[8] Wang, F. Y., *et al.*, Improvement of Air Permeability of Bubbfil Nanofiber Membrane, *Thermal Science, 22,* (2018), 1A, pp. 17-218
[9] Yu, D. N., *et al.*, Snail-Based Nanofibers, *Mater. Lett., 220* (2018), June, pp. 5-7
[10] Liu, H. Y., *et al.*, A Fractional Model for Heat Transfer in Mongolian Yurt, *Thermal Science*, *21* (2017), 4, pp. 1861-1866
[11] Wang, K. L., Liu, S. Y. He's Fractional Derivative and its Application for Fractional Fornberg-Whitham Equation, *Thermal Science*, *21* (2017), 5, pp. 2049-2055
[12] Wang, Y., Deng, Q., Fractal Derivative Model for Tsunami Travelling, *Fractals*, On-line first, https://doi.org/10.1142/ S0218348X19500178
[13] Wang, Y., An, J.Y., Amplitude-Frequency Relationship to a Fractional Duffing Oscillator Arising in Microphysics and Tsunami Motion, *Journal of Low Frequency Noise, Vibration & Active Control*, On-line first, https://doi.org/10.1177/1461348418795813
[14] Molita, Y. M., Moorani, M. S. M., *et al.*, Approximate Solutions of Fractional Zakharov-Kuznetsov Equations by VIM. *J. Comput. Appl. Math. 233* (2009), Nov., pp. 103-108
[15] Jiao, H. W., Liu, S. Y., An Efficient Algorithm for Quadratic Sum-of-Ratios Fractional Programs Problem, *Numerical Functional Analysis and Optimization, 38* (2017), 11, pp. 1426-1445
[16] Wang, K. L., Wang. K. J., A Modification of the Reduced Differential Transform Method for Fractional Calculus, *Thermal Science, 22* (2018), 4, pp. 1871-1875
[17] He, J.-H., Homotopy Perturbation Technique, *Comput. Method. Appl. M., 178* (1999), 3-4, pp. 257-262
[18] He, J.-H., A Coupling Method of a Homotopy Technique and a Perturbation Technique for Nonlinear Problems, *Int. J. Nonl. In. Mech*., *35* (2000), 1, pp. 37-43
[19] He, J.-H., Some Asymptotic Methods for Strongly Nonlinear Equations, *International Journal of Modern Physics B*, *20* (2006), 10, 1141-1199
[20] He, J.-H., Homotopy Perturbation Method with an Auxiliary Term, *Abstract and Applied Analysis, 2012* (2012), ID 857612
[21] He, J.-H., Homotopy Perturbation Method with Two Expanding Parameters, *Indian Journal of Physics, 88* (2014), 2, pp. 193-196
[22] Wu, Y., He, J.-H., Homotopy Perturbation Method for Nonlinear Oscillators with Coordinate Dependent Mass, *Results in Physics, 10* (2018), Sept., pp. 270-271
[23] Liu, Z. J., *et al.* Hybridization of Homotopy Perturbation Method and Laplace Transformation for the Partial Differential Equations, *Thermal Science, 21* (2017), 5, pp. 1843-1846
[24] Adamu, M. Y., Ogenyi, P., Parameterized Homotopy Perturbation Method, *Nonlinear Sci. Lett. A, 8* (2017), 2, pp. 240-243.
[25] Yu, D. N., *et al.*, Homotopy Perturbation Method with an Auxiliary Parameter for Nonlinear Oscillators, *Journal of Low Frequency Noise, Vibration & Active Control,* On-line first, https://doi.org/10.1177/ 1461348418811028
[26] Li, X. X., He, C. H. Homotopy Perturbation Method Coupled with the Enhanced Perturbation Method, *Journal of Low Frequency Noise, Vibration & Active Control,* On-line first, https://doi.org/10.1177/ 1461348418800554
[27] He, J.-H., Li, Z. B., Converting Fractional Differential Equations into Partial Differential Equations, *Thermal Science, 16* (2012), 2, pp. 331-334
[28] Li, Z. B., He, J.-H., Fractional Complex Transform for Fractional Differential Equations, *Math. Comput. Appl., 15* (2010), 5, pp. 970-973
[29] Yao, S. W., Chen, Z. B., The Homotopy Perturbation Method for a Nonlinear Oscillator with a Damping, *Journal of Low Frequency Noise, Vibration and Active Control*, 2019, On-line first, https://doi.org10.1177/ 1461348419836344
[30] Wei, C. F., Solving Time-Space Fractional Fitzhugh-Nagumo Equation by Using He-Laplace Decomposition Method, *Thermal Science, 22* (2018), 4, pp.1723-1728

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