DIFFUSION-CONVECTION EQUATIONS AND CLASSICAL SYMMETRY CLASSIFICATION

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In this paper, Lie algorithm is used to classify the classical symmetry of a general diffusion-convection equation. The solution process is elucidated for different conditions, and the obtained symmetries can be used to study the solution properties of the diffusion-convection equation.

Key words: *Lie algorithm, diffusion-convection equations, symmetry classification*

Introduction

Diffusion-convection equations are widely studied in thermal science, however, their symmetries were rarely appeared in open literature. Symmetries of PDE play an important role in thermal science, mechanics, mathematics, and physics, because symmetries can provide important information on the solution properties [1-5], conservation laws [6, 7], and variational principle or Hamilton principle [8-11]. Finding symmetries of an PDE has been caught much attention in various fields.

Consider the following PDE:

$$F(x,u,\partial u,\partial^2 u,\cdots,\partial^k u)=0 \tag{1}$$

where $x=(x_1,x_2,\cdots,x_n)$, u denotes the coordinate corresponding to its dependent variable.

The one-parameter Lie group of point transformations:

$$\begin{cases} x^* = X(x,u,\varepsilon) \\ u^* = U(x,u,\varepsilon) \end{cases} \tag{2}$$

leaves an invariant of eq. (1) , *i. e.*, a point symmetry, if and only if its k^{th} extension leaves an invariant surface (1).

Let:

$$X=\xi_i(x,u)\frac{\partial}{\partial x_i}+\eta(x,u)\frac{\partial}{\partial u} \tag{3}$$

be the infinitesimal generator of the Lie group of point transformations (2).

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Let:

$$X^{(k)}=\xi_i(x,u)\frac{\partial}{\partial x_i}+\eta(x,u)\frac{\partial}{\partial u}+\eta_i^{(1)}(x,u,\partial u)\frac{\partial}{\partial u_i}+\cdots$$
$$+\eta^{(k)}_{i_1i_2\cdots i_k}(x,u,\partial u,\partial^2 u,\cdots,\partial^k u)\frac{\partial}{\partial u_{i_1i_2\cdots i_k}} \tag{4}$$

be the k^{th}-extended infinitesimal generator of (3). Then one-parameter Lie group of point transformations (2) is admitted by PDE (1), if and only if:

$$X^{(k)}F(x,u,\partial u,\partial^2 u,\cdots,\partial^k u)=0, \quad \text{when} \quad F(x,u,\partial u,\partial^2 u,\cdots,\partial^k u)=0 \tag{5}$$

In this paper, symmetry analysis of the diffusion-convection equations is considered, the problems of determining symmetry requires to solve an over-determined linear PDE (determining equations), and the solution process is complex and requires some special technologies, which will be addressed in the forthcoming section.

Classical symmetry classification

Consider the following general class of non-linear diffusion-convection equations in the form:

$$g(x)u_t=[f(u)u_x]_x+h(u)u_x \tag{6}$$

which can model a wide variety of phenomena in thermal science, physics, engineering, chemistry, and biology. In the case $g(x)=1$, eq. (6) describes the vertical 1-D transport of water in homogeneous no-deformable porous media. When $h(u)=0$, eq. (6) can describe the stationary motion of a boundary-layer of fluid over a flat plat, and a vortex of incompressible fluid in a porous medium for polytropic relations of gas density and pressure.

Suppose the infinitesimal generator admitted by eq. (6) is:

$$X=\xi(x,t,u)\frac{\partial}{\partial x}+\tau(x,t,u)\frac{\partial}{\partial t}+\eta(x,t,u)\frac{\partial}{\partial u} \tag{7}$$

From eq. (5), the determining equations for X are:

$$\begin{cases}\xi_u=0,\tau_u=\tau_x=0\\ f(u)g(x)\eta_t-f(u)h(u)\eta_x-f(u)^2\eta_{xx}=0\\ 2f(u)g(x)\xi_x-f(u)g(x)\tau_t-f'(u)g(x)\eta+f(u)g'(x)\xi=0\\ -g(x)f(u)\xi_t-f(u)h(u)\xi_x+f(u)^2\xi_{xx}-2f(u)^2\eta_{xu}+f'(u)h(u)\eta\\ -2f(u)f'(u)\eta_x-f(u)h'(u)\eta=0\\ -f(u)^2\eta_{uu}-f(u)f'(u)\eta_u+f'(u)^2\eta-f(u)f''(u)\eta=0\end{cases} \tag{8}$$

When $h(u)=0$, eq. (6) becomes:

$$g(x)u_t=[f(u)u_x]_x \tag{9}$$

Solving eq. (8), we summarize the results of group classification as follows (c_1, c_2, …, c_6 are arbitrary constants in Cases 1 to 6).

Case 1.

$$g'''(x)=\frac{2g''(x)^2 g(x)-g''(x)g'(x)^2}{g(x)g'(x)},\quad f'''(u)=\frac{f''(u)[-f'(u)^2+2f''(u)f(u)]}{f(u)f'(u)}$$

which leads to $g(x)=(-x+c_5)^{c_4}c_6, f(u)=(-u+c_2)^{c_1}c_3$, the given PDE (9) admits:

$$X_1=(c_2c_4+2c_2-c_4u-2u)\frac{\partial}{\partial u}+(-c_1x+c_1c_5)\frac{\partial}{\partial x}$$

$$X_2=c_1\frac{\partial}{\partial t}+c_6(c_2c_4+2c_2-c_4u-2u)\frac{\partial}{\partial u}+c_6c_1(c_5-x)\frac{\partial}{\partial x}$$

$$X_3=c_1t\frac{\partial}{\partial t}+(2c_2c_6-u+c_2c_4c_6+c_2-c_4c_6u-2c_6u)\frac{\partial}{\partial u}+c_6c_1(c_5-x)\frac{\partial}{\partial x} \tag{10}$$

Case 2.

$$g''(x)=\frac{g'(x)^2}{g(x)},\quad f'''(u)=\frac{f''(u)[-f'(u)^2+2f''(u)f(u)]}{f(u)f'(u)}$$

which leads to $g(x)=\mathrm{e}^{c_1x}c_2,\quad f(u)=(-u+c_4)^{c_3}c_5,$ the given PDE (9) admits:

$$X_1=\frac{\partial}{\partial t},\quad X_2=(c_4-u)\frac{\partial}{\partial u}+c_3t\frac{\partial}{\partial t},\quad X_3=c_1(c_4-u)\frac{\partial}{\partial u}-c_3\frac{\partial}{\partial x} \tag{11}$$

Case 3.

$$g'(x)=0,\quad f'''(u)=\frac{f''(u)[-f'(u)^2+2f''(u)f(u)]}{f(u)f'(u)}$$

i. e. $g(x)=c_1, f(u)=(-u+c_3)^{c_2}c_4,$ the given PDE (9) admits:

$$X_1=\frac{\partial}{\partial t},\quad X_2=\frac{\partial}{\partial x},\quad X_3=(c_3-u)\frac{\partial}{\partial u}+c_2t\frac{\partial}{\partial t},\quad X_4=2(u-c_3)\frac{\partial}{\partial u}+c_2x\frac{\partial}{\partial x} \tag{12}$$

Case 4.

$$g'''(x)=\frac{3g(x)g'(x)g''(x)-2g'(x)^3}{g(x)^2},\quad f''(u)=\frac{2f'(u)^2}{f(u)}$$

which lead to $g(x)=\mathrm{e}^{c_3x}\mathrm{e}^{\frac{1}{2}c_4x^2}c_5,\quad f(u)=-\left[1/(c_1u+c_2)\right],$ the given PDE (9) admits:

$$X_1=\frac{\partial}{\partial t},\quad X_2=(c_1u+c_2)\frac{\partial}{\partial u}+c_1t\frac{\partial}{\partial t},\quad X_3=(c_1c_3u+c_1c_4ux+c_2c_4x+c_2c_3)\frac{\partial}{\partial u}-c_1\frac{\partial}{\partial x} \tag{13}$$

Case 5.

The $g(x)$ is arbitrary, $f'(u)=0,$ *i. e.* $f(u)=c_1,$ the given PDE (9) admits:

$$X_1=\frac{\partial}{\partial t},\quad X_2=u\frac{\partial}{\partial u},\quad X_3=F(x,t)\frac{\partial}{\partial u},\quad c_1F_{xx}(x,t)=g(x)F_t(x,t) \tag{14}$$

Case 6.

The $g''(x) = 5g'(x)^2/4g(x)$, $f'(u) = 0$, which leads to $g(x) = 256/(c_2x + c_3)^4$, $f(u) = c_1$, the given PDE (9) admits:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = u\frac{\partial}{\partial u}, \quad X_3 = F(x,t)\frac{\partial}{\partial u}$$

$$c_1x^4c_2{}^4F_{xx}(x,t) + 4c_1c_3x^3c_2{}^3F_{xx}(x,t) + 6c_1c_3{}^2x^2c_2{}^2F_{xx}(x,t) +$$

$$+ 4c_1c_3{}^3xc_2F_{xx}(x,t) + c_1c_3{}^4F_{xx}(x,t) - 256F_t(x,t) = 0 \tag{15}$$

When $g(x) = 1$, eq. (6) becomes:

$$u_t = [f(u)u_x]_x + h(u)u_x \tag{16}$$

Solving eq. (8), we summarize the results of group classification as follows ($c_1, c_2, \ldots, c_6$ are arbitrary constants in Cases 1 to 7).

Case1.

$$f'''(u) = \frac{-f'(u)^2 f''(u) + 2f''(u)^2 f(u)}{f(u)f'(u)}, \quad h'(u) = 0$$

which leads to $f(u) = (-u + c_3)^{c_2} c_4, h(u) = c_1$, the given PDE (16) admits:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = c_2t\frac{\partial}{\partial t} + c_2x\frac{\partial}{\partial x} + (u - c_3)\frac{\partial}{\partial u} \tag{17}$$

Case 2.

$$f'(u) = \frac{f(u)h'(u)}{h(u)}, \quad h''(u) = \frac{3}{2}\frac{h'(u)^2}{h(u)}$$

which leads to $f(u) = c_1h(u)$, $h(u) = 4/[(c_2u + c_3)^2]$, the given PDE (16) admits:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = 2c_2t\frac{\partial}{\partial t} + (c_2u + c_3)\frac{\partial}{\partial u} \tag{18}$$

Case 3.

$$f'(u) = 0, \quad h''''(u) = \frac{-h''(u)^2 h'''(u) + 2h'(u)h'''(u)^2}{h'(u)h''(u)}$$

which lead to $f(u) = c_1,\ h(u) = -\dfrac{(u + c_4)\left(-\dfrac{c_2}{u + c_4}\right)^{c_2} c_3}{c_2 - 1} + c_5$, the given PDE (16) admits:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t} + c_6\frac{\partial}{\partial x}$$

$$X_3 = (u + c_4)\frac{\partial}{\partial u} + (2c_2 t - 2t)\frac{\partial}{\partial t} + (c_2 x - x - c_2 c_5 t + c_5 t + c_6)\frac{\partial}{\partial x} \tag{19}$$

Case 4.

$$f'(u) = \frac{f(u)h'(u)}{h(u)}, \quad h'''(u) = \frac{h''(u)[2h(u)h''(u) - h'(u)^2]}{h(u)h'(u)}$$

which lead to $f(u) = c_1 h(u), h(u) = (-u + c_3)^{c_2} c_4$, the given PDE (16) admits:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = (-u + c_3)\frac{\partial}{\partial u} + c_2 t\frac{\partial}{\partial t} \tag{20}$$

Case 5.

$$f'''(u) = \frac{-f'(u)^2 f''(u) + 2f''(u)^2 f(u)}{f(u)f'(u)}$$

$$h'''(u) = \frac{f(u)f'(u)h''(u)^2 + f(u)h''(u)f''(u)h'(u) - f'(u)^2 h'(u)h''(u)}{f(u)f'(u)h'(u)}$$

which leads to $f(u) = (-u + c_5)^{c_4} c_6$, $h(u) = \int f(u)^{c_1} c_2 \, du + c_3$, the given PDE (16) admits:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t} + c_4\frac{\partial}{\partial x}$$

$$X_3 = (-2tc_4 c_1 + tc_4)\frac{\partial}{\partial t} + (-c_5 + u)\frac{\partial}{\partial u} + c_4(x - c_1 x + c_3 c_1 t + c_4 - c_4 c_1)\frac{\partial}{\partial x} \tag{21}$$

Case 6.

$$f''(u) = \frac{f'(u)h''(u)}{h'(u)}, \quad h'''(u) = \frac{2f(u)h''(u)^2 - f(u)h''(u)h'(u)}{f(u)h'(u)}$$

which lead to $f(u) = c_1 u + c_2, h(u) = c_3 u + c_4$, the given PDE (16) admits:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = -tc_1\frac{\partial}{\partial t} + (c_1 u + c_2)\frac{\partial}{\partial u} + (-c_3 c_2 t + tc_4 c_1)\frac{\partial}{\partial x} \tag{22}$$

Case 7.

$$f''(u) = \frac{3}{2}\frac{f'(u)^2}{f(u)}, h''(u) = \frac{3}{2}\frac{f'(u)h'(u)}{f(u)}, \quad i.e. \quad f(u) = \frac{4}{(c_3 u + c_4)^2}$$

$$h(u) = c_1 + \left\lfloor \int f(u)^{3/2} \mathrm{d}u \right\rfloor c_2$$

the given PDE (16) admits:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = -4c_3 t\frac{\partial}{\partial t} + (-c_3 u - c_4)\frac{\partial}{\partial u} + (-c_3 x + 3c_3 c_1 t)\frac{\partial}{\partial x} \tag{23}$$

Conclusions

In this paper, we use Lie algorithm to determine classical symmetry classification of the diffusion-convection equations, the obtained symmetries can be used to study the solution properties of the diffusion-convection equation as that discussed in [12] and our results can be easily extended to fractal calculus [12-17].

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References

[1] Hao, X. Z., *et al.*, The Residual Symmetry And Exact Solutions Of The Davey-Stewartson III Equation, *Computers & Mathematics with Applications, 73* (2017), 11, pp. 2404-2414
[2] Liu, Y. K., Li, B., Nonlocal Symmetry and Exact Solutions of the (2+1)-Dimensional Gardner Equation, *Chinese Journal of Physics, 54* (2016), 5, pp. 718-723
[3] Cheng, W. G., *et al.*, Nonlocal Symmetry And Exact Solutions Of The (2+1)-Dimensional Breaking Soliton Equation, *Communications in Nonlinear Science and Numerical Simulation*, *29* (2015), 1-3, pp.198-207
[4] Feng, L. L., *et al.*, Lie Symmetries, Conservation Laws and Analytical Solutions for Two-Component Integrable Equations, *Chinese Journal of Physics, 55* (2017), 3, pp. 996-1010
[5] Tian, S. F., *et al.*, Lie Symmetry Analysis, Conservation Laws and Analytical Solutions for the Constant Astigmatism Equation, *Chinese Journal of Physics, 55* (2017), 5, pp. 1938-1952
[6] Wei, G. M., *et al.*, Lie Symmetry Analysis and Conservation Law of Variable-Coefficient Davey-Stewartson Equation, *Computers & Mathematics with Applications, 79* (2018), 5, pp. 3420-3430
[7] Zhang, Z. Y., Conservation Laws of Partial Differential Equations: Symmetry, Adjoint Symmetry and Nonlinear Self-Adjointness, *Computers & Mathematics with Applications, 74* (2017), 12, pp. 3129-3140
[8] He, J.-H., Hamilton's Principle for Dynamical Elasticity, *Applied Mathematics Letters, 72* (2017), Oct., pp. 65-69
[9] He, J.-H., Generalized Equilibrium Equations for Shell Derived from a Generalized Variational Principle, *Applied Mathematics Letters, 64* (2017), Feb., pp. 94-100
[10] Wu, Y., He, J.-H., A Remark on Samuelson's Variational Principle in Economics, *Applied Mathematics Letters 84* (2018), Oct., pp.143-147
[11] He, J.-H., An Alternative Approach to Establishment of a Variational Principle for the Torsional Problem of Piezoelastic Beams, *Applied Mathematics Letters, 52* (2016), Feb., pp. 1-3
[12] Wang, Y., Dong, Z. Z., Symmetry of a 2+1-D System, *Thermal Science, 22* (2018), 4, pp. 1811-1822
[13] He, J.-H., Fractal Calculus and its Geometrical Explanation, *Results in Physics, 10* (2018), Sept., pp. 272-276
[14] Li, X. X., *et al.*, A Fractal Modification of the Surface Coverage Model for an Electrochemical Arsenic Sensor, *Electrochimica Acta, 296* (2019), Feb., pp. 491-493
[15] Wang, Q. L., *et al.*, Fractal Calculus and its Application to Explanation of Biomechanism of Polar Bear Hairs, *Fractals, 26* (2018), 6, ID 1850086
[16] Wang, Y., Deng, Q., Fractal Derivative Model for Tsunami Travelling, *Fractals*, On-line first, https://doi.org/10.1142/ S0218348X19500178
[17] Wang, Y., An, J. Y., Amplitude-Frequency Relationship to a Fractional Duffing Oscillator Arising in Microphysics and Tsunami Motion, *Journal of Low Frequency Noise, Vibration & Active Control*, On-line first, https://doi.org/10.1177/1461348418795813

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