COMPARISONS OF SIX DIFFERENT ESTIMATION METHODS FOR LOG-KUMARASWAMY DISTRIBUTION

by

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In this paper, it is considered the problem of estimation of unknown parameters of log-Kumaraswamy distribution via Monte-Carlo simulations. Firstly, it is described six different estimation methods such as maximum likelihood, approximate bayesian, least-squares, weighted least-squares, percentile, and Cramer-von-Mises. Then, it is performed a Monte-Carlo simulation study to evaluate the performances of these methods according to the biases and mean-squared errors of the estimators. Furthermore, two real data applications based on carbon fibers and the gauge lengths are presented to compare the fits of log-Kumaraswamy and other fitted statistical distributions.

Key words*: log-Kumaraswamy distribution, maximum likelihood estimation, Cramer-von-Mises estimation method, least-squares estimation, percentile estimation, Monte-Carlo simulation*

Introduction

Log-Kumaraswamy (LKw) distribution is a special case of log-exponentiated Kumaraswamy distribution proposed by Lemonte *et al.* [1]. They have generated LKw distribution by using a log-transform in cumulative distribution function (cdf) of Kumaraswamy (Kw) distribution suggested by Kumaraswamy [2]. Let Y be a random variable having Kw distribution with parameters a and b. The LKw distribution is obtained by $X = -\log(1 - Y)$ transformation. The cdf, the probability density function (pdf), and hazard rate function (hf):

$$f(x;a,b) = abe^{-x}\left(1-\mathrm{e}^{-x}\right)^{a-1}\left[1-\left(1-\mathrm{e}^{-x}\right)^{a}\right]^{b-1} \tag{1}$$

$$F(x,a,b) = 1-\left[1-\left(1-\mathrm{e}^{-x}\right)^{a}\right]^{b} \tag{2}$$

$$h(x;a,b) = \frac{abe^{-x}\left(1-\mathrm{e}^{-x}\right)^{a-1}}{\left[1-\left(1-\mathrm{e}^{-x}\right)^{a}\right]} \tag{3}$$

where $a > 0$, $b > 0$, and $x > 0$. The LKw(a, b) distribution can be useful in order to model real data in areas such as hydrology, engineering, science, medicine, agriculture*, etc.*

Some studies on LKw distribution can be listed as follows. Mohammed [3] studied inference on the log-exponentiated Kw distribution. Chacko and Mohan [4] investigated the

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problem of the estimation of parameters for Kw-exponential distribution under progressive type-II censoring. Akinsete *et al.* [5] proposed Kw-geometric distribution. Moreover, Jose and Varghese [6] introduced Wrapped LKw distribution. Korkmaz and Genc [7] suggested two-sided generalized exponential distribution. Korkmaz and Genc [8] introduced a new lifetime distribution based on a transformation of two sided power variate. The problem of parameter estimation for many distributions is very popular. In recent years, there are many extensive studies on parameter estimation for various distributions in literature. Ramos and Louzada [9] have introduced a new distribution called as the generalized weighted lindley distribution and studied different methods of estimation for this distribution. Dey *et al.* [10] have compared the methods of estimation for Nadarajah and Haghighi distribution. Dey *et al.* [11] have studied different estimation methods for Kw distribution. Also, in [12] they estimated the parameters of Gompertz distribution using different estimation methods. Ramos *et al.* [13] have considered the problem of estimation of parameters for Frechet distribution. Balakrishnan and Kundu [14] presented an extensive study including new estimation methods and extensions for Birnbaum-Saunders distribution.

The aim of this article is to compare the performances of methods of estimation for LKw(a, b) distribution via Monte-Carlo simulations and real data applications. For this reason, six different estimation methods such as the maximum likelihood, approximate bayesian, least-squares, weighted least-squares, percentile and Cramer-von-Mises are considered.

Estimation methods

Maximum likelihood estimates

Let $X_1, X_2,...,X_n$ be a random sample taken from LKw(a, b) distribution. The log-likelihood function:

$$\ell(a,b\,|\,\mathbf{x}) = n(\log a + \log b) - \sum_{i=1}^{n} x_i + (a-1)\sum_{i=1}^{n}\log\left(1-e^{-x_i}\right) + \\ +(b-1)\sum_{i=1}^{n}\log\left[1-\left(1-\mathrm{e}^{-x_i}\right)^a\right] \tag{4}$$

The maximum likelihood estimators (MLE) of unknown parameters are derived by maximizing the log-likelihood function in eq. (4). The likelihood equations are also obtained from the partial derivatives of log-likelihood function with respect to a and b parameters. These derivates are:

$$\frac{\partial\ell(a,b\,|\,\mathbf{x})}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n}\log\left(1-\mathrm{e}^{-x_i}\right) - (b-1)\sum_{i=1}^{n}\frac{\left(1-\mathrm{e}^{-x_i}\right)^a \log\left(1-\mathrm{e}^{-x_i}\right)}{1-\left(1-\mathrm{e}^{-x_i}\right)^a} \tag{5}$$

$$\frac{\partial\ell(a,b\,|\,\mathbf{x})}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log\left[1-\left(1-\mathrm{e}^{-x_i}\right)^a\right] \tag{6}$$

The MLE, $\hat{a}_{\mathrm{MLE}}$ and $\hat{b}_{\mathrm{MLE}}$, can be obtained by solving of likelihood equations, in eqs. (5) and (6). These non-linear equations can be solved by some numerical methods.

Approximate bayesian estimates

Let $X_1, X_2,...,X_n$ be a random sample with size n taken from LKw(a, b) distribution. In this study, the independent gamma priors for a and b parameters are used:

$$\pi(a) \propto a^{d_1-1}\mathrm{e}^{-ae_1}, \quad a, e_1, d_1 > 0 \tag{7}$$

$$\pi(b) \propto b^{d_2-1}\mathrm{e}^{-be_2}, \quad b, e_2, d_2 > 0 \tag{8}$$

The joint priors and posterior distributions of a and b parameters, are given by, respectively:

$$\pi(a,b) = \pi(a)\pi(b) \propto a^{d_1-1}b^{d_2-1}\mathrm{e}^{-(ae_1+be_2)} \tag{9}$$

$$\pi(a,b \mid \mathbf{x}) = \frac{f(\mathbf{x} \mid a,b)\pi(a,b)}{f_x(x)} = \frac{w(\mathbf{x};a,b)\pi(a,b)}{\int_0^\infty\int_0^\infty w(\mathbf{x};a,b)\pi(a,b)\mathrm{d}a\mathrm{d}b} \tag{10}$$

where

$$w(\mathbf{x};a,b) = (ab)^n \mathrm{e}^{-\sum_{i=1}^{n} x_i} \prod_{i=1}^{n}\left(1-\mathrm{e}^{-x_i}\right)^{a-1}\left[1-\left(1-\mathrm{e}^{-x_i}\right)^a\right]^{b-1}$$

Thus, Bayes estimator (BE) under squared loss function for any function of a and b, say $u(a, b)$:

$$\hat{u}_B(a,b) = E\left[u(a,b) \mid \mathbf{x}\right] = \frac{\int_0^\infty\int_0^\infty u(a,b \mid \mathbf{x})\mathrm{e}^{\left[\ell(a,b|\mathbf{x})+\rho(a,b)\right]}\mathrm{d}a\mathrm{d}b}{\int_0^\infty\int_0^\infty \mathrm{e}^{\left[\ell(a,b|\mathbf{x})+\rho(a,b)\right]}\mathrm{d}a\mathrm{d}b} \tag{11}$$

where $\ell(a, b|x)$ is log-likelihood function, $\rho(a, b)$ – the logarithm of joint prior distribution. It is difficult to get the integral presented in eq. (11) in closed form. Some approximate methods to get the integrals are used. One of these methods is Tierney Kadane's approximation method.

Bayes estimates with Tierney and Kadane's method

Tierney and Kadane's approximation introduced by Tierney and Kadane [15] to compute integral ratios. In Bayes analysis has been studied by many authors such as Danish and Aslam [16], Gencer and Saracoglu [17], Kumar [18], Kinaci *et al.* [19], Tanis and Saracoglu [20], Jung and Chung [21]. Tierney and Kadane approximation can be summarized:

$$I(a,b) = \frac{1}{n}\left\{\rho(a,b) + \ell(a,b \mid \mathbf{x})\right\} \tag{12}$$

$$I^*(a,b) = \frac{1}{n}\log u(a,b) + I(a,b) \tag{13}$$

where $\rho(a, \mathrm{b})$ is defined:

$$\rho(a,b) = (d_1-1)\log(a) + (d_2-1)\log(b) - (ae_1+be_2) \tag{14}$$

The BE with Tierney and Kadane approximation of $u(a, b)$ under squared error loss function for LKw(a, b):

$$\hat{u}_b(a,b) = E\left[u(a,b) \mid \mathbf{x}\right] = \frac{\int_0^\infty\int_0^\infty e^{nI^*(a,b)}\mathrm{d}a\mathrm{d}b}{\int_0^\infty\int_0^\infty e^{nI(a,b)}\mathrm{d}a\mathrm{d}b} = \left[\left(\frac{\det\Sigma^*}{\det\Sigma}\right)^{1/2}\exp\left\{n\left[I^*\left(\hat{a}_{I^*},\hat{b}_{I^*}\right) - I\left(\hat{a}_I,\hat{b}_I\right)\right]\right\}\right] \tag{15}$$

where $(\hat{a}_{I^*}, \hat{b}_{I^*})$ and $(\hat{a}_I, \hat{b}_I)$ maximize $I^*(\hat{a}_{I^*}, \hat{b}_{I^*})$ and $I(\hat{a}_I, \hat{b}_I)$, respectively. The $\sum^*$ and $\sum$ are minus the inverse Hessians of $I^*(\hat{a}_{I^*}, \hat{b}_{I^*})$ and $I(\hat{a}_I, \hat{b}_I)$ at $(\hat{a}_{I^*}, \hat{b}_{I^*})$ and $(\hat{a}_I, \hat{b}_I)$, respectively.

Least square and weighted least squares estimates

Let $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ be order statistics of a random sample with n sizes having LKw (a, b) distribution. Then, the expected value of the empirical cumulative distribution function (ecdf):

$$E\left[F\left(X_{i:n}\right)\right]=\frac{i}{n+1}, \quad Var\left[F\left(X_{i:n}\right)\right]=\frac{i\left(n-i+1\right)}{\left(n+1\right)^2\left(n+2\right)}, \; i=1,2,...,n \tag{16}$$

The least square estimates of a and b, $\hat{a}_{\mathrm{LSE}}$ and $\hat{b}_{\mathrm{LSE}}$ can be obtained by minimizing following eq. (17):

$$Z\left(a,b\right)=\sum_{i=1}^{n}\left\{1-\left[1-\left(1-\mathrm{e}^{-x_{i:n}}\right)^a\right]^b-\frac{i}{n+1}\right\}^2 \tag{17}$$

In this case, $\hat{a}_{\mathrm{LSE}}$ and $\hat{b}_{\mathrm{LSE}}$ can be obtained via the simultaneously solution of the following system of equations:

$$\frac{\partial Z\left(a,b\right)}{\partial a}=\sum_{i=1}^{n}k\left(x_{i:n},a,b\right)\left\{1-\left[1-\left(1-\mathrm{e}^{-x_{i:n}}\right)^a\right]^b-\frac{i}{n+1}\right\}=0 \tag{18}$$

$$\frac{\partial Z\left(a,b\right)}{\partial b}=\sum_{i=1}^{n}\mathrm{g}\left(x_{i:n},a,b\right)\left\{1-\left[1-\left(1-\mathrm{e}^{-x_{i:n}}\right)^a\right]^b-\frac{i}{n+1}\right\}=0 \tag{19}$$

where

$$k\left(x_{i:n},a,b\right)=b\left[1-\left(1-\mathrm{e}^{-x_{i:n}}\right)^a\right]^{b-1}\left(1-\mathrm{e}^{-x_{i:n}}\right)^a\log\left(1-\mathrm{e}^{-x_{i:n}}\right)$$

$$\mathrm{g}\left(x_{i:n},a,b\right)=-\log\left[1-\left(1-\mathrm{e}^{-x_{i:n}}\right)^a\right]\left[1-\left(1-\mathrm{e}^{-x_{i:n}}\right)^a\right]^b$$

Equations (18) and (19) can be simultaneously solved using some iterative methods. The weighted least squared estimators (WLSE) shown with $\hat{a}_{\mathrm{WLSE}}$ and $\hat{b}_{\mathrm{WLSE}}$ can be obtained by minimizing following equation with respect to a and b parameters:

$$\varpi\left(a,b\right)=\sum_{i=1}^{n}\tau_i\left\{1-\left[1-\left(1-\mathrm{e}^{-x_{i:n}}\right)^a\right]^b-\frac{i}{n+1}\right\}^2 \tag{20}$$

where

$$\tau_i=\frac{1}{Var\left[F\left(X_{i:n}\right)\right]}=\frac{\left(n+1\right)^2\left(n+2\right)}{i\left(n-i+1\right)}$$

Percentile estimates

In this subsection, the percentile estimates (PE) of a and b for LKw(a, b) distribution, $\hat{a}_{\mathrm{PE}}$ and $\hat{b}_{\mathrm{PE}}$ are obtained. This estimation method was firstly suggested by Kao [22, 23]. There are many studies based on percentile estimation of unknown parameters for various statistical distributions. Some of these studies are Gupta and Kundu [24], Alkasabeh and Raqab [25], Erisoglu and Erisoglu [26]. The quantile function of LKw (a, b):

$$Q\left(a,b\right)=-\log\left\{1-\left[1-\left(1-p_i\right)^{1/b}\right]^{1/a}\right\} \tag{21}$$

Let $x_{i:n}$ is be value of i^{th} order statistics. The $\hat{a}_{\mathrm{PE}}$ and $\hat{b}_{\mathrm{PE}}$ can be obtained by minimizing the following equation with respect to a and b parameters:

$$\kappa(a,b)=\sum_{i=1}^{n}\left\{x_{i:n}+\log\left(1-\left[1-\left(1-\frac{i}{n+1}\right)^{1/b}\right]^{1/a}\right)\right\}^2 \tag{22}$$

Cramer-von Mises estimates

The Cramer-von Mises estimator is one of the goodness of-fit estimators. This method is based on the difference between the estimate of the cdf and ecdf. The bias of these estimators is smaller than the bias of other minimum distance estimators studied by Luceno [27], Ramos and Louzada [9] and Macdonald [28]. The Cramer-von Mises estimators (CVME), $\hat{a}_{\mathrm{CVME}}$ and $\hat{b}_{\mathrm{CVME}}$, can be derived by minimizing following equation:

$$C(a,b)=\frac{1}{12n}+\sum_{i=1}^{n}\left\{1-\left[1-\left(1-\mathrm{e}^{-x_{i:n}}\right)^{a}\right]^{b}-\frac{2i-1}{n+1}\right\}^2 \tag{23}$$

Simulation study

In this section, it is performed a extensive Monte-Carlo simulation study in order to compare MLE, BE, LSE, WLSE, PE, and CVME for LKw(a, b) distribution. The biases and mean square errors (MSE) of these estimators are simulated based on 10000 repetitions by considering different samples of sizes such as 25, 50, 100, 250, and 500 and different initial values as (a = 0.5, b = 0.9), (a = 3.3, b = 1.5), (a = 2.3, b = 1.2) and (a = 4, b = 2) for LKw(a, b) distribution. In the bayesian analysis, we consider (d_1 = 0.01, e_1 = 0.01) and (d_2 = 0.01, e_2 = 0.01) as the values of prior parameters. The results of simulation study are given in tabs. 1 and 2.

Table 1. The biases and MSE of $\hat{a}$ and $\hat{b}$ by using different estimation methods for a = 0.5, b = 0.9 and a = 3.3, b = 1.5

	n	MLE		BE		LSE		WLSE		PE		CVME	
		bias	MSE	bias	MSE	bias	MSE	bias	MSE	bias	MSE	bias	MSE
a = 0.5, b = 0.9													
$\hat{a}$	25	0.0539	0.0280	0.0434	0.0261	0.0410	0.0393	0.0581	0.0385	–0.0392	0.0848	0.0605	0.0435
	50	0.0267	0.0114	0.0218	0.0110	0.0208	0.0161	0.0325	0.0147	–0.0494	0.0515	–0.0494	0.0515
	100	0.0137	0.0051	0.0111	0.0050	0.0107	0.0073	0.0182	0.0064	–0.0414	0.0299	0.0154	0.0076
	250	0.0053	0.0019	0.0042	0.0019	0.0038	0.0027	0.0075	0.0022	–0.0299	0.0139	0.0056	0.0027
	500	0.0027	0.0009	0.0022	0.0009	0.0019	0.0013	0.0040	0.0011	–0.0228	0.0073	0.0028	0.0013
$\hat{b}$	25	0.1047	0.0993	0.0913	0.0959	0.1710	0.2120	0.2110	0.2156	–0.0554	0.1477	0.1334	0.1899
	50	0.0504	0.0361	0.0435	0.0352	0.0800	0.0614	0.1047	0.0592	–0.0605	0.0686	–0.0605	0.0686
	100	0.0245	0.0155	0.0211	0.0153	0.0376	0.0240	0.0530	0.0220	–0.0477	0.0346	0.0293	0.0230
	250	0.0097	0.0056	0.0083	0.0056	0.0140	0.0082	0.0215	0.0070	–0.0315	0.0149	0.0107	0.0080
	500	0.0049	0.0028	0.0042	0.0028	0.0070	0.0039	0.0111	0.0033	–0.0228	0.0077	0.0053	0.0039
a = 3.3, b = 01.5													
$\hat{a}$	25	0.2909	0.8819	0.2230	0.8184	0.2171	1.2228	0.3209	1.1924	–0.2200	0.9716	0.3203	1.3294
	50	0.1451	0.3700	0.1113	0.3548	0.1113	0.5145	0.1828	0.4734	–0.1813	0.5188	0.1617	0.5389
	100	0.0752	0.1694	0.0584	0.1656	0.0577	0.2387	0.1036	0.2089	–0.1286	0.2773	0.0827	0.2449
	250	0.0291	0.0631	0.0224	0.0625	0.0205	0.0883	0.0434	0.0739	–0.0820	0.1177	0.0304	0.0892
	500	0.0146	0.0309	0.0113	0.0308	0.0100	0.0438	0.0227	0.0359	–0.0580	0.0592	0.0150	0.0440
$\hat{b}$	25	0.2123	0.3789	0.1953	0.3737	0.3393	0.8982	0.4154	0.8987	–0.0850	0.2692	0.2813	0.8277
	50	0.1003	0.1287	0.0910	0.1266	0.1543	0.2283	0.2012	0.2181	–0.0795	0.1258	0.1275	0.2148
	100	0.0489	0.0536	0.0441	0.0531	0.0723	0.0859	0.1014	0.0780	–0.0601	0.0639	0.0596	0.0830
	250	0.0192	0.0192	0.0173	0.0191	0.0267	0.0286	0.0410	0.0243	–0.0383	0.0269	0.0218	0.0282
	500	0.0095	0.0093	0.0086	0.0093	0.0133	0.0138	0.0210	0.0113	–0.0271	0.0138	0.0108	0.0137

Table 2. The biases and MSE of $\hat{a}$ and $\hat{b}$ by using different estimation methods for a = 2.3, b = 1.2 and a = 4, b = 2

	n	MLE		BE		LSE		WLSE		PE		CVME	
		bias	MSE	bias	MSE	bias	MSE	bias	MSE	bias	MSE	bias	MSE
	a = 2.3 b = 1.2												
	25	0.2197	0.4875	0.1716	0.4532	0.1649	0.6781	0.2398	0.6624	–0.1639	0.6212	0.2435	0.7421
	50	0.1093	0.2025	0.0854	0.1943	0.0843	0.2825	0.1357	0.2598	–0.1428	0.3400	0.1225	0.2971
$\hat{a}$	100	0.0564	0.0921	0.0446	0.0900	0.0436	0.1303	0.0766	0.1140	–0.1042	0.1848	0.0625	0.1340
	250	0.0218	0.0342	0.0171	0.0339	0.0155	0.0480	0.0319	0.0401	–0.0682	0.0799	0.0230	0.0486
	500	0.0110	0.0168	0.0087	0.0168	0.0076	0.0238	0.0168	0.0195	–0.0488	0.0409	0.0114	0.0239
	25	0.1556	0.2097	0.1397	0.2046	0.2508	0.4733	0.3083	0.4773	–0.0687	0.1693	0.2026	0.4305
	50	0.0741	0.0735	0.0657	0.0720	0.1155	0.1280	0.1510	0.1228	–0.0648	0.0804	0.0933	0.1198
$\hat{b}$	100	0.0361	0.0310	0.0319	0.0306	0.0543	0.0490	0.0763	0.0446	–0.0492	0.0412	0.0437	0.0472
	250	0.0142	0.0112	0.0125	0.0111	0.0201	0.0165	0.0309	0.0141	–0.0315	0.0175	0.0160	0.0162
	500	0.0071	0.0055	0.0063	0.0055	0.0100	0.0079	0.0159	0.0066	–0.0222	0.0091	0.0079	0.0079
	a = 4 b = 2												
	25	0.3186	1.1454	0.2391	1.0634	0.2287	1.5822	0.3506	1.5239	–0.2415	1.1483	0.3426	1.7043
	50	0.1543	0.4656	0.1146	0.4467	0.1145	0.6602	0.1993	0.6019	–0.1981	0.5751	0.1699	0.6881
$\hat{a}$	100	0.0726	0.2115	0.0528	0.2071	0.0526	0.2981	0.1061	0.2595	–0.1486	0.3043	0.0802	0.3046
	250	0.0330	0.0854	0.0250	0.0846	0.0247	0.1189	0.0513	0.1005	–0.0853	0.1302	0.0357	0.1200
	500	0.0140	0.0398	0.0101	0.0396	0.0074	0.0560	0.0229	0.0459	–0.0606	0.0644	0.0128	0.0562
	25	0.3363	0.8813	0.3210	0.8834	0.5173	2.3023	0.6244	2.1665	–0.0935	0.5344	0.4450	2.1695
	50	0.1472	0.2700	0.1377	0.2680	0.2211	0.5011	0.2893	0.4755	–0.1048	0.2328	0.1878	0.4758
$\hat{b}$	100	0.0670	0.1077	0.0619	0.1071	0.0995	0.1769	0.1413	0.1598	–0.0852	0.1186	0.0837	0.1718
	250	0.0298	0.0399	0.0278	0.0397	0.0412	0.0608	0.0617	0.0518	–0.0497	0.0499	0.0351	0.0600
	500	0.0138	0.0185	0.0128	0.0184	0.0181	0.0275	0.0296	0.0225	–0.0355	0.0248	0.0150	0.0273

Empirical applications

In this section, it is performed two real data analysis in order to illustrate usefulness of LKw(a, b) in real life. A comparison the performances of MLE, BE, LSE, WLSE, Pe, and CVME for parameters of LKw(a, b) distribution is given in this section. For these purposes it is used Anderson-Darling (A^*), Cramer-Von Mises (W^*), Kolmogorov-Smirnov test statistics (K-S) and its (p-value).

Gauge lengths data

The first data set based on gauge lengths of 20 mm consists of 69 observations obtained by Bader and Priest [29]. These data previously used by Kundu and Raqab, [30], Ghitany *et al.* [31] and Nofal *et al.* [32]. These data are given by:
1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.14, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.57, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.88, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

The results of real data analysis for gauge lengths data are given in tab. 3. Also, cdf and pdf curves of these estimators for LKw(a, b) distribution are given in fig. 1.

Table 3. The parameter estimates and selection criteria statistics for gauge lengths data

Estimator	$\hat{a}$	$\hat{b}$	A^*	W^*	K-S	p-value
MLE	22.3085	5.4072	0.2687	1.2633	0.0534	0.9892
BE	22.5930	5.3970	0.3119	1.6777	0.0600	0.9647
LSE	23.5756	5.6703	0.2444	0.8684	0.0436	0.9994
WLSE	23.7161	5.8354	0.2672	0.8705	0.0488	0.9966
PE	22.0063	4.9715	0.2735	1.2863	0.0517	0.9927
CVME	23.7881	5.6424	0.2476	0.7391	0.0432	0.9995

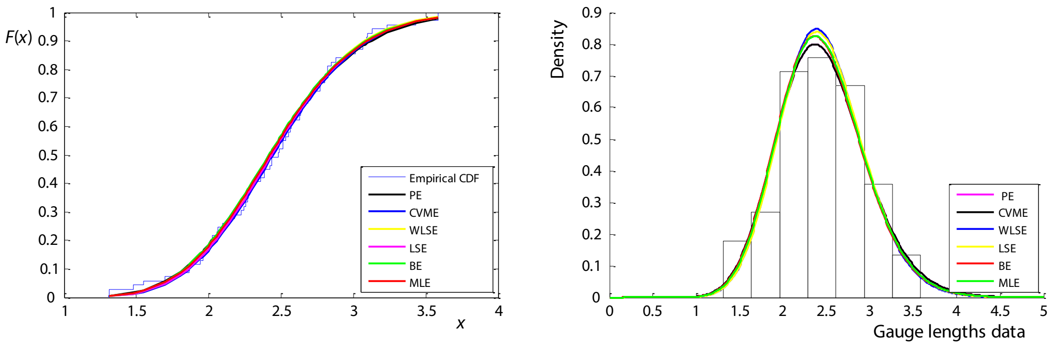

Figure 1. The curves for gauge lengths data; (a) cdf, (b) density

Carbon fibers (in Gba) data

The second data set consists of 50 observations on breaking stress of carbon fibers (in Gba) obtained by Nichols and Padgett [33] is as follows:
3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92.

The parameter estimates, and some selection statistics for carbon fibres data set are given in tab. 4.

Table 4. The parameter estimates and selection criteria statistics for carbon fibres data

Estimator	$\hat{a}$	$\hat{b}$	A^*	W^*	K-S	p-value
MLE	28.4793	3.1609	0.5142	1.9442	0.0845	0.8673
BE	28.0556	3.1329	0.5546	2.2357	0.0813	0.8959
LSE	35.3256	4.7092	0.5545	0.9684	0.0757	0.9369
WLSE	34.8359	4.6324	0.5531	0.9705	0.0790	0.9138
PE	26.8687	2.8253	0.5960	2.4165	0.0984	0.7182
CVME	35.7719	4.6719	0.5395	0.7860	0.0643	0.9860

The cdf and pdf curves according to six different estimators are presented for carbon fibres data in fig. 2.

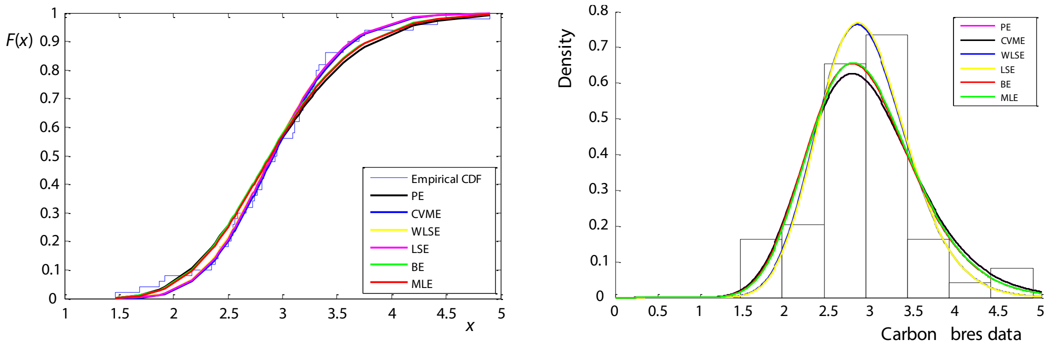

Figure 2. The curves for carbon fibres data; (a) cdf, (b) density

Concluding remarks

It has been considered ML, B, LS, WLS, P, and CVM estimation methods to estimate unknown parameters of LKw(a, b) distribution. Then, it is performed a Monte-Carlo simulation study to compare the performances of these estimators in terms of biases and MSE at different size of samples. According to results of simulation study, it is clearly seen that approximate bayes estimator is best the estimator among all estimators. Besides, as size of samples increases, biases and MSE of all estimators decrease. Also, it is seen that the biases and MSE of maximum likelihood estimators and approximate Bayes estimators approach each other in big size of samples. On the other hand, we illustrate usefulness of LKw(a, b) distribution for gauge lengths and carbon fibers data sets. Further, it is compared the fits of these estimators for LKw(a, b) distribution via Anderson-Darling, Cramer-Von Mises, Kolmogorov-Smirnov statistics and its p-values. It is seen that least square estimator is the best for gauge lengths data according to tab. 3. Approximate bayes estimator is the better than other estimators in modelling carbon fiber data according to tab. 4.

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