SOME NEW SOLUTIONS OF THE CONFORMABLE EXTENDED ZAKHAROV-KUZNETSOV EQUATION USING ATANGANA-BALEANU CONFORMABLE DERIVATIVE

by

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The generalized Riccati equation mapping method, coupled with Atangana's conformable derivative is implemented to solve non-linear extended Zakharov-Kuznetsov equation which results in producing hyperbolic, trigonometric and the rational solutions. The obtained results are new and are of great importance in engineering and applied sciences.

Key words: *conformable extended Zakharov-Kuznetsov equation, solitons, generalized Riccati equation mapping method, Atangana's conformable derivative*

Introduction

Fractional differential equations (FDE) are known to be generalized sense of integer order differential equations. Several remarkable research works in this field during recent years, have further indicated its scope for discovering more interesting and valuable facts in mathematical physics, heat transfer, thermal science models, biological sciences, system identification, theory of control, mathematical finance, signal processing and fractional dynamics, see [1] and references therein. The literature includes numerous mathematical techniques to attain exact solutions of non-linear PDE of integer and fractional order such as modified extended tanh function method [2], F-expansion method [3], fan sub-equation method [4], direct algebraic method [5], extended trial equation method [6], first integral method [7], auxiliary equation method [8], modified simple equation method, rxponential rational function method, exp-function method [9], and so on. Motivated by the ongoing research in this area, generalized Riccati equation mapping method [9] is applied for seeking the solutions of non-linear extended Zakharov-Kuznetsov (ZK) equation of fractional order [9]. For latest research in this area, see [10-13] and the references therein. It is observed that obtained results are new, credible and are of physical nature related to engineering and applied sciences.

Conformable (1+2)-extended ZK equation

Consider the conformable ZK equation [9]:

$$ {}_0^A D_t^{\alpha} u + \beta u \, {}_0^A D_z^{\alpha} u + \gamma \left({}_0^A D_y^{3\alpha} u + {}_0^A D_z^{3\alpha} u \right) + \delta \left({}_0^A D_{zyy}^{3\alpha} u + {}_0^A D_{yzz}^{3\alpha} u \right) = 0 \tag{1} $$

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By means of the following transformation:

$$u(x,t)=u(\xi),\quad \xi=\frac{\chi_1}{\alpha}\left[y+\frac{1}{\Gamma(\alpha)}\right]^{\alpha}+\frac{\chi_2}{\alpha}\left[z+\frac{1}{\Gamma(\alpha)}\right]^{\alpha}+\frac{\lambda}{\alpha}\left[t+\frac{1}{\Gamma(\alpha)}\right]^{\alpha} \tag{2}$$

Equation (1) leads to the non-linear conformable ODE:

$$\lambda u'(\xi)+\beta\chi_2 u(\xi)u'(\xi)+\gamma\chi_1^3 u'''(\xi)+\gamma\chi_2^3 u'''(\xi)+\delta\chi_1^2\chi_2 u'''(\xi)+\delta\chi_2^2\chi_1 u'''(\xi)=0 \tag{3}$$

Integrating eq. (3) once with respect to ξ yields:

$$\lambda u(\xi)+\frac{1}{2}\beta\chi_2 u(\xi)^2+\gamma\chi_1^3 u''(\xi)+\gamma\chi_2^3 u''(\xi)+\delta\chi_1^2\chi_2 u''(\xi)+\delta\chi_2^2\chi_1 u''(\xi)=0 \tag{4}$$

where $u'=\mathrm{d}u/\mathrm{d}\xi$. Positive integer $N=2$ can be found by balancing the dispersion term $u''(\xi)$ and non-linear term $u(\xi)^2$ present in eq. (3). The solution of eq. (3) takes the form:

$$u(\xi)=a_{-2}\phi(\xi)^{-2}+a_{-1}\phi(\xi)^{-1}+a_0+a_1\phi(\xi)+a_2\phi(\xi)^2 \tag{5}$$

Applying Riccati differential equation method:

$$a_0=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2},\quad a_1=0$$

$$a_{-1}=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2},\quad a_2=0$$

$$a_{-2}=-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2},\quad a_2$$

$$\lambda=-(\chi_1+\chi_2)\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]\left(p^2-4qr\right),\quad \chi_1=\chi_1,\quad \chi_2=\chi_2 \tag{6}$$

Consequently,

Family 1: When $p^2-4qr>0$ and $pq\neq 0$ or $qq\neq 0$, the hyperbolic function solutions:

$$u_{1,1}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-$$

$$-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\left\{-\frac{1}{2q}\left[p+\sqrt{p^2-4qr}\tanh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)\right]\right\}^{-1}-$$

$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\left\{-\frac{1}{2q}\left[p+\sqrt{p^2-4qr}\tanh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)\right]\right\}^{-2} \tag{7}$$

$$u_{1,2}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-$$

$$-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\left\{-\frac{1}{2q}\left[p+\sqrt{p^2-4qr}\coth\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)\right]\right\}^{-1}-$$

$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\left\{-\frac{1}{2q}\left[p+\sqrt{p^2-4qr}\coth\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)\right]\right\}^{-2} \tag{8}$$

$$u_{1,3}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q-}{\beta\chi_2}$$

$$-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\cdot$$

$$\cdot\left(-\frac{1}{2q}\left\{\frac{p+\sqrt{p^2-4qr}}{\left[\tanh\left(\sqrt{p^2-4qr}\,\xi\right)\pm i\operatorname{sech}\left(\sqrt{p^2-4qr}\,\xi\right)\right]}\right\}\right)^{-1}-$$

$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\cdot$$

$$\cdot\left(-\frac{1}{2q}\left\{\frac{p+\sqrt{p^2-4qr}}{\left[\tanh\left(\sqrt{p^2-4qr}\,\xi\right)\pm i\operatorname{sech}\left(\sqrt{p^2-4qr}\,\xi\right)\right]}\right\}\right)^{-2} \tag{9}$$

$$u_{1,4}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-$$

$$-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\cdot$$

$$\cdot\left(-\frac{1}{2q}\left\{\frac{p+\sqrt{p^2-4qr}}{\left[\coth\left(\sqrt{p^2-4qr}\,\xi\right)\pm\operatorname{csch}\left(\sqrt{p^2-4qr}\,\xi\right)\right]}\right\}\right)^{-1}-$$

$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\left(-\frac{1}{2q}\left\{\frac{p+\sqrt{p^2-4qr}}{\left[\coth\left(\sqrt{p^2-4qr}\,\xi\right)\pm\operatorname{csch}\left(\sqrt{p^2-4qr}\,\xi\right)\right]}\right\}\right)^{-2} \tag{10}$$

$$u_{1,5}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-$$

$$-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\left(-\frac{1}{4q}\left\{\frac{2p+\sqrt{p^2-4qr}}{\left[\tanh\left(\frac{\sqrt{p^2-4qr}}{4}\xi\right)\pm\coth\left(\frac{\sqrt{p^2-4qr}}{4}\xi\right)\right]}\right\}\right)^{-1}-$$

$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\left(-\frac{1}{4q}\left\{\frac{2p+\sqrt{p^2-4qr}}{\left[\tanh\left(\frac{\sqrt{p^2-4qr}}{4}\xi\right)\pm\coth\left(\frac{\sqrt{p^2-4qr}}{4}\xi\right)\right]}\right\}\right)^{-2} \tag{11}$$

$$u_{1,6}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\cdot$$

$$\cdot\left\{\frac{1}{2q}\left[-p+\frac{\sqrt{(A^2+B^2)(p^2-4qr)}-A\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}\,\xi\right)}{A\sinh\left(\sqrt{p^2-4qr}\,\xi\right)+B}\right]\right\}^{-1}-$$

$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\cdot$$

$$\cdot\left\{\frac{1}{2q}\left[-p+\frac{\sqrt{(A^2+B^2)(p^2-4qr)}-A\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}\,\xi\right)}{A\sinh\left(\sqrt{p^2-4qr}\,\xi\right)+B}\right]\right\}^{-2} \tag{12}$$

$$u_{1,7}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\cdot$$

$$\cdot\left\{\frac{1}{2q}\left[-p-\frac{\sqrt{(B^2-A^2)(p^2-4qr)}+A\sqrt{p^2-4qr}\sinh\left(\sqrt{p^2-4qr}\,\xi\right)}{A\cosh\left(\sqrt{p^2-4qr}\,\xi\right)+B}\right]\right\}^{-1}-$$

$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\cdot$$

$$\cdot\left\{\frac{1}{2q}\left[-p-\frac{\sqrt{(B^2-A^2)(p^2-4qr)}+A\sqrt{p^2-4qr}\sinh\left(\sqrt{p^2-4qr}\,\xi\right)}{A\cosh\left(\sqrt{p^2-4qr}\,\xi\right)+B}\right]\right\}^{-2} \tag{13}$$

where A and B are two non-zeto real constants satisfying $B^2 - A^2 > 0$.

$$u_{1,8}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-$$

$$-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\left[\frac{2r\cosh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)}{\sqrt{p^2-4qr}\sinh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)-p\cosh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)}\right]^{-1}-$$

$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\left[\frac{2r\cosh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)}{\sqrt{p^2-4qr}\sinh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)-p\cosh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)}\right]^{-2} \quad (14)$$

$$u_{1,9}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-$$

$$-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\left[\frac{-2r\sinh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)}{p\sinh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)-\sqrt{p^2-4qr}\cosh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)}\right]^{-1}-$$

$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\left[\frac{-2r\sinh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)}{p\sinh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)-\sqrt{p^2-4qr}\cosh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)}\right]^{-2} \quad (15)$$

$$u_{1,10}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\cdot$$

$$\cdot\left[\frac{2r\cosh\left(\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr}\sinh\left(\sqrt{p^2-4qr}\xi\right)-p\cosh\left(\sqrt{p^2-4qr}\xi\right)\pm i\sqrt{p^2-4qr}}\right]^{-1}-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\cdot$$

$$\cdot\left[\frac{2r\cosh\left(\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr}\sinh\left(\sqrt{p^2-4qr}\xi\right)-p\cosh\left(\sqrt{p^2-4qr}\xi\right)\pm i\sqrt{p^2-4qr}}\right]^{-2} \quad (16)$$

$$u_{1,11}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-$$
$$-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\cdot$$
$$\cdot\left[\frac{2r\sinh\left(\sqrt{p^2-4qr}\xi\right)}{-p\sinh\left(\sqrt{p^2-4qr}\xi\right)+\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}\xi\right)\pm\sqrt{p^2-4qr}}\right]^{-1}-$$
$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\cdot$$
$$\cdot\left[\frac{2r\sinh\left(\sqrt{p^2-4qr}\xi\right)}{-p\sinh\left(\sqrt{p^2-4qr}\xi\right)+\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}\xi\right)\pm\sqrt{p^2-4qr}}\right]^{-2} \tag{17}$$

Family 2: When $p^2-4qr<0$ and $pq\neq 0$ or $qr\neq 0$, the trigonometric solutions:

$$u_{1,12}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-$$
$$-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\left\{\frac{1}{2q}\left[-p+\sqrt{4qr-p^2}\tan\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)\right]\right\}^{-1}-$$
$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\left\{\frac{1}{2q}\left[-p+\sqrt{4qr-p^2}\tan\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)\right]\right\}^{-2} \tag{18}$$

$$u_{1,13}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-$$
$$-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\left\{-\frac{1}{2q}\left[p+\sqrt{4qr-p^2}\cot\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)\right]\right\}^{-1}-$$
$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\left\{-\frac{1}{2q}\left[p+\sqrt{4qr-p^2}\cot\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)\right]\right\}^{-2} \tag{19}$$

$$u_{1,14}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-$$
$$-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\cdot$$
$$\cdot\left(\frac{1}{2q}\left\{-p+\sqrt{4qr-p^2}\left[\tan\left(\sqrt{4qr-p^2}\,\xi\right)\pm\sec\left(\sqrt{4qr-p^2}\,\xi\right)\right]\right\}\right)^{-1}-$$
$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\cdot$$
$$\cdot\left(\frac{1}{2q}\left\{-p+\sqrt{4qr-p^2}\left[\tan\left(\sqrt{4qr-p^2}\,\xi\right)\pm\sec\left(\sqrt{4qr-p^2}\,\xi\right)\right]\right\}\right)^{-2} \tag{20}$$

$$u_{1,15}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-$$
$$-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\cdot$$
$$\cdot\left(-\frac{1}{2q}\left\{p+\sqrt{4qr-p^2}\left[\cot\left(\sqrt{4qr-p^2}\,\xi\right)\pm\csc\left(\sqrt{4qr-p^2}\,\xi\right)\right]\right\}\right)^{-1}-$$
$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\cdot$$
$$\cdot\left(-\frac{1}{2q}\left\{p+\sqrt{4qr-p^2}\left[\cot\left(\sqrt{4qr-p^2}\,\xi\right)\pm\csc\left(\sqrt{4qr-p^2}\,\xi\right)\right]\right\}\right)^{-2} \tag{21}$$

$$u_{1,16}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\cdot$$
$$\cdot\left(\frac{1}{4q}\left\{-2p+\sqrt{4qr-p^2}\left[\tan\left(\frac{\sqrt{4qr-p^2}}{4}\xi\right)-\cot\left(\frac{\sqrt{4qr-p^2}}{4}\xi\right)\right]\right\}\right)^{-1}-$$
$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\cdot$$
$$\cdot\left(\frac{1}{4q}\left[-2p+\sqrt{4qr-p^2}\left\{\tan\left(\frac{\sqrt{4qr-p^2}}{4}\xi\right)-\cot\left(\frac{\sqrt{4qr-p^2}}{4}\xi\right)\right\}\right]\right)^{-2} \tag{22}$$

$$u_{1,17}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\cdot$$
$$\cdot\left\{\frac{1}{2q}\left[-p+\frac{\pm\sqrt{\left(A^2-B^2\right)\left(4qr-p^2\right)}-A\sqrt{4qr-p^2}\cos\left(\sqrt{4qr-p^2}\,\xi\right)}{A\sin\left(\sqrt{4qr-p^2}\,\xi\right)+B}\right]\right\}^{-1}-$$
$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\cdot$$
$$\cdot\left\{\frac{1}{2q}\left[-p+\frac{\pm\sqrt{\left(A^2-B^2\right)\left(4qr-p^2\right)}-A\sqrt{4qr-p^2}\cos\left(\sqrt{4qr-p^2}\,\xi\right)}{A\sin\left(\sqrt{4qr-p^2}\,\xi\right)+B}\right]\right\}^{-2} \tag{23}$$

$$u_{1,18}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\cdot$$
$$\cdot\left\{\frac{1}{2q}\left[-p-\frac{\sqrt{\left(A^2-B^2\right)\left(4qr-p^2\right)}+A\sqrt{4qr-p^2}\cos\left(\sqrt{4qr-p^2}\,\xi\right)}{A\sin\left(\sqrt{4qr-p^2}\,\xi\right)+B}\right]\right\}^{-1}-$$
$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\cdot$$
$$\cdot\left\{\frac{1}{2q}\left[-p-\frac{\sqrt{\left(A^2-B^2\right)\left(4qr-p^2\right)}+A\sqrt{4qr-p^2}\cos\left(\sqrt{4qr-p^2}\,\xi\right)}{A\sin\left(\sqrt{4qr-p^2}\,\xi\right)+B}\right]\right\}^{-2} \tag{24}$$

$$u_{1,19}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-\frac{12(\chi_1+\chi_2)r\left(\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right)p}{\beta\chi_2}\cdot$$
$$\cdot\left[-\frac{2r\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}{\sqrt{4qr-p^2}\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)+p\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}\right]^{-1}-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\cdot$$
$$\cdot\left[-\frac{2r\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}{\sqrt{4qr-p^2}\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)+p\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}\right]^{-2} \tag{25}$$

$$u_{1,20}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-$$

$$-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\left\{\frac{2r\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}{-p\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)+\sqrt{4qr-p^2}\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}\right\}^{-1}-$$

$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\left[\frac{2r\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}{-p\sin\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)+\sqrt{4qr-p^2}\cos\left(\frac{\sqrt{4qr-p^2}}{2}\xi\right)}\right]^{-2} \tag{26}$$

$$u_{1,21}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\cdot$$

$$\cdot\left[-\frac{2r\cos\left(\sqrt{4qr-p^2}\,\xi\right)}{\sqrt{4qr-p^2}\sin\left(\sqrt{4qr-p^2}\,\xi\right)+p\cos\left(\sqrt{4qr-p^2}\,\xi\right)\pm\sqrt{4qr-p^2}}\right]^{-1}-$$

$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\cdot$$

$$\cdot\left[-\frac{2r\cos\left(\sqrt{4qr-p^2}\,\xi\right)}{\sqrt{4qr-p^2}\sin\left(\sqrt{4qr-p^2}\,\xi\right)+p\cos\left(\sqrt{4qr-p^2}\,\xi\right)\pm\sqrt{4qr-p^2}}\right]^{-2} \tag{27}$$

$$u_{1,22}(\xi)=-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2}-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\cdot$$

$$\cdot\left[\frac{2r\sin\left(\sqrt{4qr-p^2}\,\xi\right)}{-p\sin\left(\sqrt{4qr-p^2}\,\xi\right)+\sqrt{4qr-p^2}\cos\left(\sqrt{4qr-p^2}\,\xi\right)\pm\sqrt{4qr-p^2}}\right]^{-1}-$$

$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\cdot$$

$$\cdot\left[\frac{2r\sin\left(\sqrt{4qr-p^2}\,\xi\right)}{-p\sin\left(\sqrt{4qr-p^2}\,\xi\right)+\sqrt{4qr-p^2}\cos\left(\sqrt{4qr-p^2}\,\xi\right)\pm\sqrt{4qr-p^2}}\right]^{-2} \tag{28}$$

Family 3: When $r = 0$ and $pq \neq 0$ the solutions of eq. (1) are:

$$u_{1,23}(\xi) = -\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2} -$$

$$-\frac{12(\chi_1+\chi_2)r\left(\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right)p}{\beta\chi_2}\left(-\frac{pd}{q\left(d+\cosh(p\xi)-\sinh(p\xi)\right)}\right)^{-1} -$$

$$-\frac{12\left(\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right)(\chi_1+\chi_2)r^2}{\beta\chi_2}\left(-\frac{pd}{q\left(d+\cosh(p\xi)-\sinh(p\xi)\right)}\right)^{-2} \tag{29}$$

$$u_{1,24}(\xi) = -\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2} -$$

$$-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\left\{\frac{-p\left[\cosh(p\xi)+\sinh(p\xi)\right]}{q\left[d+\cosh(p\xi)-\sinh(p\xi)\right]}\right\}^{-1} -$$

$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\left\{\frac{-p\left[\cosh(p\xi)+\sinh(p\xi)\right]}{q\left[d+\cosh(p\xi)-\sinh(p\xi)\right]}\right\}^{-2} \tag{30}$$

Family 4: When $r = p = 0$ and $q \neq 0$ the rational solutions of eq. (1):

$$u_{1,25}(\xi) = -\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]q}{\beta\chi_2} -$$

$$-\frac{12(\chi_1+\chi_2)r\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right]p}{\beta\chi_2}\left(-\frac{1}{q\xi+c}\right)^{-1} -$$

$$-\frac{12\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](\chi_1+\chi_2)r^2}{\beta\chi_2}\left(-\frac{1}{q\xi+c}\right)^{-2} \tag{31}$$

where

$$\xi = \frac{\chi_1}{\alpha}\left[y+\frac{1}{\Gamma(\alpha)}\right]^{\alpha} + \frac{\chi_2}{\alpha}\left[z+\frac{1}{\Gamma(\alpha)}\right]^{\alpha} - \frac{(\chi_1+\chi_2)\left[\gamma\chi_2^2+\chi_1(-\gamma+\delta)\chi_2+\gamma\chi_1^2\right](p^2-4qr)}{\alpha}\left[t+\frac{1}{\Gamma(\alpha)}\right]^{\alpha}$$

The 3-D ($\alpha = 0.4$, 1) and 2-D ($\alpha = 0.4$, 0.8, 1) graphical representations of some of the solutions are given.

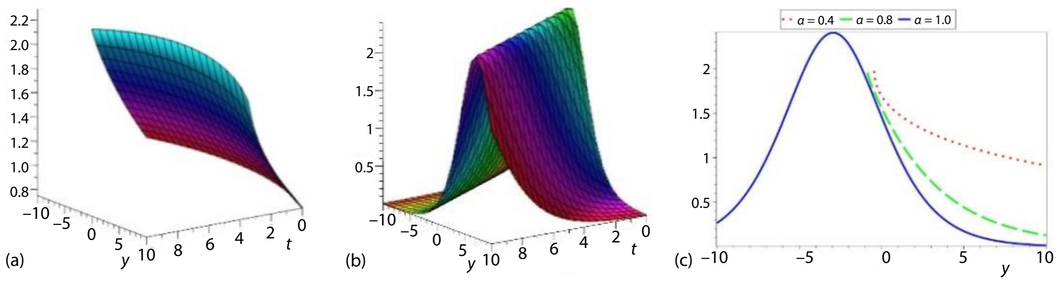

Figure 1. Soliton solution of $u_{1,1}(\xi)$

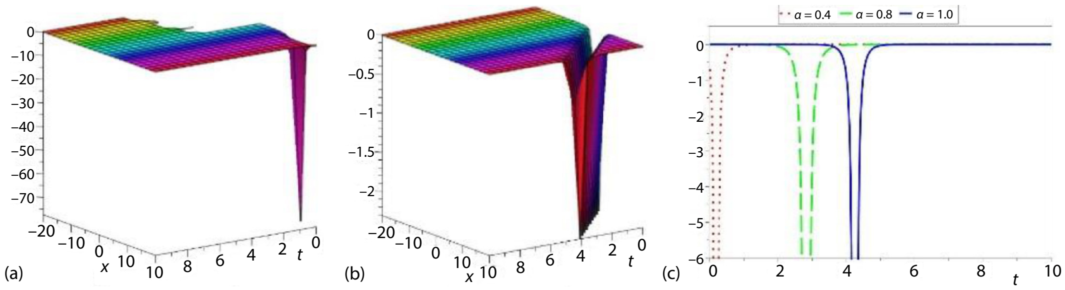

Figure 2. Singular soliton solution of $u_{1,2}(\xi)$

Conclusion

Some soliton solitons including, bell shaped, periodic, multiple periodic and singular soliton solutions of extended Zakharov-Kuznetsov equation are formed by using Generalized Riccati equation mapping method. All the solutions for this article have been checked by MAPLE software.

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