ANALYTICAL AND SEMI-ANALYTICAL WAVE SOLUTIONS FOR LONGITUDINAL WAVE EQUATION VIA MODIFIED AUXILIARY EQUATION METHOD AND ADOMIAN DECOMPOSITION METHOD

by

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This paper studies the analytical and semi-analytical wave solutions for the longitudinal wave equation. Moreover, it examines the performance of the modified auxiliary equation method and Adomian decomposition method on this model. This model describes the dispersion in the circular rod that dispersion caused by the transverse Poisson's effect in electro-magneto-elastic. Many explicit wave solutions are found by using the analytical technique. These solutions allow studying the physical properties of this model. The comparison between the analytical and semi-analytical solutions is discussed to show the value of the absolute error between them.

Key words: *longitudinal wave equation, modified auxiliary equation method, analytical and semi-analytical wave solutions, adomian decomposition method*

Introduction

Partial differential equations have been focused the attention of many researchers in different fields because of its ability for modelling many non-linear phenomena. The PDE represent many of non-linear physical phenomena by representing them with liner or non-linear PDE with integer or fractional orders. These phenomena have been investigated to study its physical properties by using the exact solutions of these models. According to this goal, many researchers have been trying to derive analytical techniques for getting explicit wave solutions [1-10]. Many kinds of solutions are obtained, such as trigonometric, exponential, hyperbolic, periodic, rational, and elliptic solutions [11-15].

Auxiliary equation method is one of the most modern techniques derived in this field. It derived by Khater [16]. Even though many research papers used this method [17-20], the solutions obtained via this method are computational solutions, not an exact solution. Given that, a modified auxiliary equation method (modified Khater method) was derived to obtain exact traveling wave solutions [21-23].

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According to the high technology level and increasing usage of tools such as sensors, actuators, *etc*. many researchers have been formulating mathematical modules that describe the dispersion in the circular rod. That dispersion caused by the transverse Poisson's effect in electro-magneto-elastic (EME). The longitudinal wave equation [24] is given in the following form:

$$u_{tt} - a^2 u_{xx} - \left(\frac{a}{2}u^2 + b u_{tt}\right)_{xx} = 0 \tag{1}$$

where a and b represent a linear longitudinal wave velocity and dispersion parameter, respectively. Both of these parameters depend on the mathe terial property and the geometry of the rod. For more explanation of this model, we consider the material of the electro-magneto-elastic rod is $BaTiO_3$-$CoFe_2O_4$ with a different values of $BaTiO_3$ in rod radius equal 0.05 m. The fraction volume of the mixture effects on the material properties of the composite. For more details, you see [25-30]. Many analytical traveling wave solutions are applied to this model for obtaining the exact and solitary wave solutions. Applying the following traveling wave transformation on eq. (1):

$$u(x,t) = v(\xi), \;\; \xi = x + ct$$

moreover, by twice integrations for the obtained ODE with zero constant of integration, we get:

$$2\left(c^2 - a^2\right)v - a v^2 - 2bc^2 v'' = 0 \tag{2}$$

Balancing the terms in eq. (2) between the highest derivative term and non-linear term, yields $n = 2$.

Application

In this part, we apply the modified auxiliary equation method and Adomian decomposition method [31-35] to the longitudinal wave equation.

Modified auxiliary equation method

According to the general solutions suggested by the method, we get the general solution of eq. (2) in the next form:

$$v(\xi) = a_0 + k^{f(\xi)}a_1 + k^{2f(\xi)}a_2 + k^{-f(\xi)}b_1 + k^{-2f(\xi)}b_2 \tag{3}$$

where a_0, a_1, a_2, b_1, b_2, and k are arbitrary constants while $f(\xi)$ satisfies the following auxiliary equation:

$$\left\{ f'(\xi) = \frac{1}{\mathrm{Ln}(k)}\left[\alpha k^{-f(\xi)} + \beta + \sigma k^{f(\xi)}\right]\right\}$$

where α, β are arbitrary constants. Substituting eq. (3) and its derivatives into eq. (2). Collecting all terms of the same power of $k^{f(\xi)}$. Solving the obtained algebric system by any computer software program, leads to:

– *Family 1*

$$a_0 = \frac{-2bc\left(\beta^2 + 2\alpha\sigma\right)}{\sqrt{1 + b\left(\beta^2 - 4\alpha\sigma\right)}}, \quad a_1 = \frac{-12bc\beta\sigma}{\sqrt{1 + b\beta^2 - 4b\alpha\sigma}}$$

$$a_2 = \frac{-12bc\sigma^2}{\sqrt{1 + b\beta^2 - 4b\alpha\sigma}}, \quad b_1 = b_2 = 0, \quad a = c\sqrt{1 + b\beta^2 - 4b\alpha\sigma}$$

– *Family 2*

$$a_0=\frac{-2bc\left(\beta^2+2\alpha\sigma\right)}{\sqrt{1+b\left(\beta^2-4\alpha\sigma\right)}},\quad a_1=a_2=0,\quad b_1=\frac{-12bc\alpha\beta}{\sqrt{1+b\beta^2-4b\alpha\sigma}}$$

$$b_2=\frac{-12bc\alpha^2}{\sqrt{1+b\beta^2-4b\alpha\sigma}},\quad a=c\sqrt{1+b\beta^2-4b\alpha\sigma}$$

According to the value of parameters in *Family 1*, we get the solitary wave solutions of eq. (1):
when [$\beta^2-4\alpha\sigma<0,\ \sigma\neq 0$] we get:

$$u(x,t)=\frac{bc\left(\beta^2-4\alpha\sigma\right)\left(-2+3\operatorname{Sec}\left[\frac{1}{2}\sqrt{-\beta^2+4\alpha\sigma}\left(x+\frac{at}{\sqrt{1+b\beta^2-4b\alpha\sigma}}\right)\right]^2\right)}{\sqrt{1+b\left(\beta^2-4\alpha\sigma\right)}} \tag{4}$$

$$u(x,t)=\frac{bc\left(\beta^2-4\alpha\sigma\right)\left(-2+3\operatorname{Csc}\left[\frac{1}{2}\sqrt{-\beta^2+4\alpha\sigma}\left(x+\frac{at}{\sqrt{1+b\beta^2-4b\alpha\sigma}}\right)\right]^2\right)}{\sqrt{1+b\left(\beta^2-4\alpha\sigma\right)}} \tag{5}$$

when [$\beta^2-4\alpha\sigma>0,\ \sigma\neq 0$] we get:

$$u(x,t)=\frac{bc\left(\beta^2-4\alpha\sigma\right)\left(-2+3\operatorname{Sech}\left[\frac{1}{2}\sqrt{\beta^2-4\alpha\sigma}\left(x+\frac{at}{\sqrt{1+b\beta^2-4b\alpha\sigma}}\right)\right]^2\right)}{\sqrt{1+b\left(\beta^2-4\alpha\sigma\right)}} \tag{6}$$

$$u(x,t)=\frac{bc\left(\beta^2-4\alpha\sigma\right)\left(2+3\operatorname{Csch}\left[\frac{1}{2}\sqrt{\beta^2-4\alpha\sigma}\left(x+\frac{at}{\sqrt{1+b\beta^2-4b\alpha\sigma}}\right)\right]^2\right)}{\sqrt{1+b\left(\beta^2-4\alpha\sigma\right)}} \tag{7}$$

when [$\beta^2+4\alpha^2<0,\ \alpha=-\sigma$] we get:

$$u(x,t)=\frac{bc\left(4\alpha^2+\beta^2\right)\left(1+3\operatorname{Tan}\left[\frac{1}{2}\sqrt{-4\alpha^2-\beta^2}\left(x+\frac{at}{\sqrt{1+4b\alpha^2+b\beta^2}}\right)\right]^2\right)}{\sqrt{1+b\left(4\alpha^2+\beta^2\right)}} \tag{8}$$

$$u(x,t)=\frac{bc\left(4\alpha^2+\beta^2\right)\left(1+3\operatorname{Cot}\left[\frac{1}{2}\sqrt{-4\alpha^2-\beta^2}\left(x+\frac{at}{\sqrt{1+4b\alpha^2+b\beta^2}}\right)\right]^2\right)}{\sqrt{1+b\left(4\alpha^2+\beta^2\right)}} \tag{9}$$

when [$\beta^2+4\alpha^2>0$, $\alpha=-\sigma$] we get:

$$u(x,t)=\frac{bc\left(4\alpha^2+\beta^2\right)\left(-1+3\operatorname{Tanh}\left[\frac{1}{2}\sqrt{4\alpha^2+\beta^2}\left(x+\frac{at}{\sqrt{1+4b\alpha^2+b\beta^2}}\right)\right]^2\right)}{\sqrt{1+b\left(4\alpha^2+\beta^2\right)}} \tag{10}$$

$$u(x,t)=\frac{bc\left(4\alpha^2+\beta^2\right)\left(2+3\operatorname{Csch}\left[\frac{1}{2}\sqrt{4\alpha^2+\beta^2}\left(x+\frac{at}{\sqrt{1+4b\alpha^2+b\beta^2}}\right)\right]^2\right)}{\sqrt{1+b\left(4\alpha^2+\beta^2\right)}}. \tag{11}$$

when [$\beta^2-4\alpha^2<0$, $\alpha=\sigma$] we get:

$$u(x,t)=\frac{bc\left(4\alpha^2-\beta^2\right)\left(1+3\operatorname{Tan}\left[\frac{1}{2}\sqrt{4\alpha^2-\beta^2}\left(x+\frac{at}{\sqrt{1-4b\alpha^2+b\beta^2}}\right)\right]^2\right)}{\sqrt{1+b\left(-4\alpha^2+\beta^2\right)}} \tag{12}$$

$$u(x,t)=\frac{bc\left(4\alpha^2-\beta^2\right)\left\{1+3\operatorname{Cot}\left[\frac{1}{2}\sqrt{4\alpha^2-\beta^2}\left(x+\frac{at}{\sqrt{1-4b\alpha^2+b\beta^2}}\right)\right]^2\right\}}{\sqrt{1+b\left(-4\alpha^2+\beta^2\right)}} \tag{13}$$

when [$\beta^2-4\alpha^2>0$, $\alpha=\sigma$] we get:

$$u(x,t)=\frac{bc\left(4\alpha^2-\beta^2\right)\left\{-1+3\operatorname{Tanh}\left[\frac{1}{2}\sqrt{-4\alpha^2+\beta^2}\left(x+\frac{at}{\sqrt{1-4b\alpha^2+b\beta^2}}\right)\right]^2\right\}}{\sqrt{1+b\left(-4\alpha^2+\beta^2\right)}} \tag{14}$$

$$u(x,t)=\frac{bc\left(4\alpha^2-\beta^2\right)\left\{2+3\operatorname{Csch}\left[\frac{1}{2}\sqrt{-4\alpha^2+\beta^2}\left(x+\frac{at}{\sqrt{1-4b\alpha^2+b\beta^2}}\right)\right]^2\right\}}{\sqrt{1+b\left(-4\alpha^2+\beta^2\right)}} \tag{15}$$

when [$\alpha\sigma > 0$, $\beta = 0$] we get:

$$u(x,t) = \frac{4bc\alpha\sigma\left\{1+3\mathrm{Tan}\left[\sqrt{\alpha\sigma}\left(x+\frac{at}{\sqrt{1-4b\alpha\sigma}}\right)\right]^2\right\}}{\sqrt{1-4b\alpha\sigma}} \tag{16}$$

$$u(x,t) = \frac{4bc\alpha\sigma\left\{1+3\mathrm{Cot}\left[\sqrt{\alpha\sigma}\left(x+\frac{at}{\sqrt{1-4b\alpha\sigma}}\right)\right]^2\right\}}{\sqrt{1-4b\alpha\sigma}} \tag{17}$$

when [$\alpha\sigma < 0$, $\beta = 0$] we get:

$$u(x,t) = \frac{4bc\alpha\sigma\left\{-1+3\mathrm{Tanh}\left[\sqrt{-\alpha\sigma}\left(x+\frac{at}{\sqrt{1-4b\alpha\sigma}}\right)\right]^2\right\}}{\sqrt{1-4b\alpha\sigma}} \tag{18}$$

$$u(x,t) = \frac{4bc\alpha\sigma\left\{2+3\mathrm{Csch}\left[\sqrt{-\alpha\sigma}\left(x+\frac{at}{\sqrt{1-4b\alpha\sigma}}\right)\right]^2\right\}}{\sqrt{1-4b\alpha\sigma}} \tag{19}$$

when [$\beta = 0$, $\alpha = -\sigma$] we get:

$$u(x,t) = -\frac{4bc\alpha^2\left\{2+3\mathrm{Csch}\left[\alpha\left(x+\frac{at}{\sqrt{1+4b\alpha^2}}\right)\right]^2\right\}}{\sqrt{1+4b\alpha^2}} \tag{20}$$

when [$\beta = \sigma = \kappa$, $\alpha = 0$] we get:

$$u(x,t) = \frac{bc\kappa^2\left\{2+3\mathrm{Csch}\left[\frac{1}{2}\kappa\left(x+\frac{at}{\sqrt{1+b\kappa^2}}\right)\right]^2\right\}}{\sqrt{1+b\kappa^2}} \tag{21}$$

when [$\alpha = 0$] we get:

$$u(x,t) = \frac{2bc\beta^2\left[4+\mathrm{e}^{\beta\left(x+\frac{at}{\sqrt{1+b\beta^2}}\right)}\sigma\left(8+\mathrm{e}^{\beta\left(x+\frac{at}{\sqrt{1+b\beta^2}}\right)}\sigma\right)\right]}{\sqrt{1+b\beta^2}\left[-2+\mathrm{e}^{\beta\left(x+\frac{at}{\sqrt{1+b\beta^2}}\right)}\sigma\right]^2} \tag{22}$$

when [$\beta = \alpha = 0$] we get:

$$u(x,t) = -\frac{12bc}{(at+x)^2} \tag{23}$$

when [$\beta = 0$, $\sigma = \alpha$] we get:

$$u(x,t) = -\frac{4bc\alpha^2\left\{1+3\mathrm{Tan}\left[C+\alpha\left(x+\frac{at}{\sqrt{1-4b\alpha^2}}\right)\right]^2\right\}}{\sqrt{1-4b\alpha^2}} \tag{24}$$

when [$\beta^2 - 4\alpha\sigma = 0$] we get:

$$u(x,t) = 2bc\left[-\beta^2 - 2\alpha\sigma + \frac{12\alpha\sigma\left(x+\frac{2}{\beta}+at\right)}{at+x} - \frac{24\alpha^2\sigma^2(2+x\beta+at\beta)^2}{(at+x)^2\beta^4}\right] \tag{25}$$

According to the value of parameters in *Family 2*, we get the solitary wave solutions of eq. (1):

when [$\beta^2 - 4\alpha\sigma < 0$, $\sigma \neq 0$] we get:

$$u(x,t) = \frac{1}{\sqrt{1+b\left(\beta^2-4\alpha\sigma\right)}}\cdot$$

$$\cdot\left[2bc\left(-\beta^2 + \frac{12\alpha\beta\sigma}{\beta-\sqrt{-\beta^2+4\alpha\sigma}\mathrm{Tan}\left[\frac{1}{2}\sqrt{-\beta^2+4\alpha\sigma}\left(x+\frac{at}{\sqrt{1+b\beta^2-4b\alpha\sigma}}\right)\right]}+\right.\right.$$

$$\left.\left.+2\alpha\sigma\left(-1-\frac{12\alpha\sigma}{\left(\beta-\sqrt{-\beta^2+4\alpha\sigma}\mathrm{Tan}\left[\frac{1}{2}\sqrt{-\beta^2+4\alpha\sigma}\left(x+\frac{at}{\sqrt{1+b\beta^2-4b\alpha\sigma}}\right)\right]\right)^2}\right)\right)\right] \tag{26}$$

$$u(x,t) = \frac{1}{\sqrt{1+b\left(\beta^2-4\alpha\sigma\right)}}\cdot$$

$$\cdot\left[2bc\left(-\beta^2 + \frac{12\alpha\beta\sigma}{\beta-\sqrt{-\beta^2+4\alpha\sigma}\mathrm{Cot}\left[\frac{1}{2}\sqrt{-\beta^2+4\alpha\sigma}\left(x+\frac{at}{\sqrt{1+b\beta^2-4b\alpha\sigma}}\right)\right]}+\right.\right.$$

$$+2\alpha\sigma\left(-1-\frac{12\alpha\sigma}{\left(\beta-\sqrt{-\beta^2+4\alpha\sigma}\operatorname{Cot}\left[\frac{1}{2}\sqrt{-\beta^2+4\alpha\sigma}\left(x+\frac{at}{\sqrt{1+b\beta^2-4b\alpha\sigma}}\right)\right]\right)^2}\right)\Bigg]\Bigg] \tag{27}$$

when [$\beta^2-4\alpha\sigma>0$, $\sigma\neq 0$] we get:

$$u(x,t)=\frac{1}{\sqrt{1+b\left(\beta^2-4\alpha\sigma\right)}}\cdot$$

$$\cdot\left[2bc\left(-\beta^2+\frac{12\alpha\beta\sigma}{\beta+\sqrt{\beta^2-4\alpha\sigma}\operatorname{Tanh}\left[\frac{1}{2}\sqrt{\beta^2-4\alpha\sigma}\left(x+\frac{at}{\sqrt{1+b\beta^2-4b\alpha\sigma}}\right)\right]}+\right.\right.$$

$$\left.\left.+2\alpha\sigma\left(-1-\frac{12\alpha\sigma}{\left(\beta+\sqrt{\beta^2-4\alpha\sigma}\operatorname{Tanh}\left[\frac{1}{2}\sqrt{\beta^2-4\alpha\sigma}\left(x+\frac{at}{\sqrt{1+b\beta^2-4b\alpha\sigma}}\right)\right]\right)^2}\right)\right)\right] \tag{28}$$

$$u(x,t)=\frac{1}{\sqrt{1+b\left(\beta^2-4\alpha\sigma\right)}}\cdot$$

$$\cdot\left[2bc\left(-\beta^2+\frac{12\alpha\beta\sigma}{\beta+\sqrt{\beta^2-4\alpha\sigma}\operatorname{Coth}\left[\frac{1}{2}\sqrt{\beta^2-4\alpha\sigma}\left(x+\frac{at}{\sqrt{1+b\beta^2-4b\alpha\sigma}}\right)\right]}+\right.\right.$$

$$\left.\left.+2\alpha\sigma\left(-1-\frac{12\alpha\sigma}{\left(\beta+\sqrt{\beta^2-4\alpha\sigma}\operatorname{Coth}\left[\frac{1}{2}\sqrt{\beta^2-4\alpha\sigma}\left(x+\frac{at}{\sqrt{1+b\beta^2-4b\alpha\sigma}}\right)\right]\right)^2}\right)\right)\right] \tag{29}$$

when $[\beta^2 + 4\alpha^2 < 0,\ \alpha = -\sigma]$ we get:

$$u(x,t)=\frac{1}{\sqrt{1+b\left(4\alpha^2+\beta^2\right)}}\cdot\left[2bc\left(-\beta^2+2\alpha^2\cdot\right.\right.$$

$$\left.\left.\cdot\left(1-\frac{6\left(2\alpha^2+\beta^2-\beta\sqrt{-4\alpha^2-\beta^2}\,\mathrm{Tan}\left[\frac{1}{2}\sqrt{-4\alpha^2-\beta^2}\left(x+\frac{at}{\sqrt{1+4b\alpha^2+b\beta^2}}\right)\right]\right)}{\left(\beta-\sqrt{-4\alpha^2-\beta^2}\,\mathrm{Tan}\left[\frac{1}{2}\sqrt{-4\alpha^2-\beta^2}\left(x+\frac{at}{\sqrt{1+4b\alpha^2+b\beta^2}}\right)\right]\right)^2}\right)\right)\right] \tag{30}$$

$$u(x,t)=\frac{1}{\sqrt{1+b\left(4\alpha^2+\beta^2\right)}}\left[2bc\left(-\beta^2+2\alpha^2\cdot\right.\right.$$

$$\left.\left.\cdot\left(1-\frac{6\left(2\alpha^2+\beta^2-\beta\sqrt{-4\alpha^2-\beta^2}\,\mathrm{Cot}\left[\frac{1}{2}\sqrt{-4\alpha^2-\beta^2}\left(x+\frac{at}{\sqrt{1+4b\alpha^2+b\beta^2}}\right)\right]\right)}{\left(\beta-\sqrt{-4\alpha^2-\beta^2}\,\mathrm{Cot}\left[\frac{1}{2}\sqrt{-4\alpha^2-\beta^2}\left(x+\frac{at}{\sqrt{1+4b\alpha^2+b\beta^2}}\right)\right]\right)^2}\right)\right)\right] \tag{31}$$

when $[\beta^2 + 4\alpha^2 > 0,\ \alpha = -\sigma]$ we get:

$$u(x,t)=\frac{1}{\sqrt{1+b\left(4\alpha^2+\beta^2\right)}}\left[2bc\left(-\beta^2+2\alpha^2\cdot\right.\right.$$

$$\left.\left.\cdot\left(1-\frac{6\left(2\alpha^2+\beta^2+\beta\sqrt{4\alpha^2+\beta^2}\,\mathrm{Tanh}\left[\frac{1}{2}\sqrt{4\alpha^2+\beta^2}\left(x+\frac{at}{\sqrt{1+4b\alpha^2+b\beta^2}}\right)\right]\right)}{\left(\beta+\sqrt{4\alpha^2+\beta^2}\,\mathrm{Tanh}\left[\frac{1}{2}\sqrt{4\alpha^2+\beta^2}\left(x+\frac{at}{\sqrt{1+4b\alpha^2+b\beta^2}}\right)\right]\right)^2}\right)\right)\right] \tag{32}$$

$$u(x,t)=\frac{1}{\sqrt{1+b\left(4\alpha^2+\beta^2\right)}}\left[2bc\left(-\beta^2+2\alpha^2\cdot\right.\right.$$

$$\left.\left.\cdot\left(1-\frac{6\left(2\alpha^2+\beta^2+\beta\sqrt{4\alpha^2+\beta^2}\,\mathrm{Coth}\left[\frac{1}{2}\sqrt{4\alpha^2+\beta^2}\left(x+\frac{at}{\sqrt{1+4b\alpha^2+b\beta^2}}\right)\right]\right)}{\left(\beta+\sqrt{4\alpha^2+\beta^2}\,\mathrm{Coth}\left[\frac{1}{2}\sqrt{4\alpha^2+\beta^2}\left(x+\frac{at}{\sqrt{1+4b\alpha^2+b\beta^2}}\right)\right]\right)^2}\right)\right)\right] \tag{33}$$

when [$\beta^2 + 4\alpha^2 < 0$, $\alpha = -\sigma$] we get:

$$u(x,t)=\frac{1}{\sqrt{1+b\left(-4\alpha^2+\beta^2\right)}}\Bigg[2bc\Bigg(-\beta^2+2\alpha^2\cdot$$

$$\cdot\left(-1+\frac{6\left(-2\alpha^2+\beta^2-\beta\sqrt{4\alpha^2-\beta^2}\,\mathrm{Tan}\left[\frac{1}{2}\sqrt{4\alpha^2-\beta^2}\left(x+\frac{at}{\sqrt{1-4b\alpha^2+b\beta^2}}\right)\right]\right)}{\left(\beta-\sqrt{4\alpha^2-\beta^2}\,\mathrm{Tan}\left[\frac{1}{2}\sqrt{4\alpha^2-\beta^2}\left(x+\frac{at}{\sqrt{1-4b\alpha^2+b\beta^2}}\right)\right]\right)^2}\right)\Bigg)\Bigg] \tag{34}$$

$$u(x,t)=\frac{1}{\sqrt{1+b\left(-4\alpha^2+\beta^2\right)}}\Bigg[2bc\Bigg(-\beta^2+2\alpha^2\cdot$$

$$\cdot\left(-1+\frac{6\left(-2\alpha^2+\beta^2-\beta\sqrt{4\alpha^2-\beta^2}\,\mathrm{Cot}\left[\frac{1}{2}\sqrt{4\alpha^2-\beta^2}\left(x+\frac{at}{\sqrt{1-4b\alpha^2+b\beta^2}}\right)\right]\right)}{\left(\beta-\sqrt{4\alpha^2-\beta^2}\,\mathrm{Cot}\left[\frac{1}{2}\sqrt{4\alpha^2-\beta^2}\left(x+\frac{at}{\sqrt{1-4b\alpha^2+b\beta^2}}\right)\right]\right)^2}\right)\Bigg)\Bigg] \tag{35}$$

when [$\beta^2 - 4\alpha^2 > 0$, $\alpha = \sigma$] we get:

$$u(x,t)=\frac{1}{\sqrt{1+b\left(-4\alpha^2+\beta^2\right)}}\Bigg[2bc\Bigg(-\beta^2+2\alpha^2\cdot$$

$$\cdot\left(-1+\frac{6\left(-2\alpha^2+\beta^2+\beta\sqrt{-4\alpha^2+\beta^2}\,\mathrm{Tanh}\left[\frac{1}{2}\sqrt{-4\alpha^2+\beta^2}\left(x+\frac{at}{\sqrt{1-4b\alpha^2+b\beta^2}}\right)\right]\right)}{\left(\beta+\sqrt{-4\alpha^2+\beta^2}\,\mathrm{Tanh}\left[\frac{1}{2}\sqrt{-4\alpha^2+\beta^2}\left(x+\frac{at}{\sqrt{1-4b\alpha^2+b\beta^2}}\right)\right]\right)^2}\right)\Bigg)\Bigg] \tag{36}$$

$$u(x,t)=\frac{1}{\sqrt{1+b\left(-4\alpha^2+\beta^2\right)}}\Bigg[2bc\Bigg(-\beta^2+2\alpha^2\cdot$$

$$\cdot\left(-1+\frac{6\left(-2\alpha^2+\beta^2+\beta\sqrt{-4\alpha^2+\beta^2}\,\mathrm{Coth}\left[\frac{1}{2}\sqrt{-4\alpha^2+\beta^2}\left(x+\frac{at}{\sqrt{1-4b\alpha^2+b\beta^2}}\right)\right]\right)}{\left(\beta+\sqrt{-4\alpha^2+\beta^2}\,\mathrm{Coth}\left[\frac{1}{2}\sqrt{-4\alpha^2+\beta^2}\left(x+\frac{at}{\sqrt{1-4b\alpha^2+b\beta^2}}\right)\right]\right)^2}\right)\Bigg)\Bigg] \tag{37}$$

when [$\alpha\sigma > 0$, $\beta = 0$] we get:

$$u(x,t)=\frac{4bc\alpha\sigma\left\{1+3\mathrm{Cot}\left[\sqrt{\alpha\sigma}\left(x+\frac{at}{\sqrt{1-4b\alpha\sigma}}\right)\right]^2\right\}}{\sqrt{1-4b\alpha\sigma}} \tag{38}$$

$$u(x,t)=\frac{4bc\alpha\sigma\left\{1+3\mathrm{Tan}\left[\sqrt{\alpha\sigma}\left(x+\frac{at}{\sqrt{1-4b\alpha\sigma}}\right)\right]^2\right\}}{\sqrt{1-4b\alpha\sigma}} \tag{39}$$

when [$\alpha\sigma < 0$, $\beta = 0$] we get:

$$u(x,t)=-\frac{4bc\alpha\sigma\left\{1+3\mathrm{Cot}\left[\sqrt{\alpha}\sqrt{\sigma}\left(x+\frac{at}{\sqrt{1-4b\alpha\sigma}}\right)\right]^2\right\}}{\sqrt{1-4b\alpha\sigma}} \tag{40}$$

$$u(x,t)=\frac{4bc\alpha\sigma\left\{1+3\mathrm{Tan}\left[\sqrt{\alpha}\sqrt{\sigma}\left(x+\frac{at}{\sqrt{1-4b\alpha\sigma}}\right)\right]^2\right\}}{\sqrt{1-4b\alpha\sigma}} \tag{41}$$

when [$\beta = 0$, $\alpha = -\sigma$] we get:

$$u(x,t)=\frac{4bc\alpha^2\left\{-2+3\mathrm{Sech}\left[\alpha\left(x+\frac{at}{\sqrt{1+4b\alpha^2}}\right)\right]^2\right\}}{\sqrt{1+4b\alpha^2}} \tag{42}$$

when [$\beta = \kappa$, $\alpha = 2\kappa$, $\sigma = 0$] we get:

$$u(x,t)=\frac{2bc\left[-1-\frac{24}{\left(-2+\mathrm{e}^{\kappa\left(x+\frac{at}{\sqrt{1+b\kappa^2}}\right)}\right)^2}-\frac{12}{-2+\mathrm{e}^{\kappa\left(x+\frac{at}{\sqrt{1+b\kappa^2}}\right)}}\right]\kappa^2}{\sqrt{1+b\kappa^2}} \tag{43}$$

when [$\beta = \sigma = 0$] we get:

$$u(x,t)=-\frac{12bc}{(at+x)^2} \tag{44}$$

when [$\beta = 0$, $\sigma = \alpha$] we get:

$$u(x,t) = -\frac{4bc\alpha^2\left\{1+3\mathrm{Cot}\left[C+\alpha\left(x+\frac{at}{\sqrt{1-4b\alpha^2}}\right)\right]^2\right\}}{\sqrt{1-4b\alpha^2}} \tag{45}$$

when [$\sigma = 0$] we get:

$$u(x,t) = \frac{2bc\beta^2\left(-1-\frac{6\mathrm{e}^{\beta\left(x+\frac{at}{\sqrt{1+b\beta^2}}\right)}\alpha\beta}{\left[\alpha-\mathrm{e}^{\beta\left(x+\frac{at}{\sqrt{1+b\beta^2}}\right)}\beta\right]^2}\right)}{\sqrt{1+b\beta^2}} \tag{46}$$

when [$\beta^2 - 4\alpha\sigma = 0$] we get:

$$u(x,t) = bc\left[-2\left(\beta^2+2\alpha\sigma\right)-\frac{3(at+x)^2\beta^4}{(2+x\beta+at\beta)^2}+\frac{6(at+x)\beta^3}{2+x\beta+at\beta}\right] \tag{47}$$

Adomian decomposition method

Applying the Adomian decomposition method on eq. (2) enables rewriting it to be in the following form:

$$Lv(\xi)+Rv(\xi)+Nv(\xi)=0 \tag{48}$$

where L, R, N represent a differential operator, a linear operator and non-linear term, respectively.

Using the inverse operator L^{-1} on eq. (48), we get:

$$\sum_{i=0}^{\infty} v_i(\xi) = v(0)+v'(0)\xi+\frac{2\left(c^2-a^2\right)}{2bc^2}L^{-1}\left[\sum_{i=0}^{\infty} v_i(\xi)\right]-\frac{a}{2bc}L^{-1}\left(\sum_{i=0}^{\infty} A_i\right) \tag{49}$$

Under the following condition:

$$\left[\beta=4,\ \alpha=2,\ \sigma=1,\ b=1,\ a=-2,\ a_0=\frac{80}{9},\ a_1=\frac{32}{3},\ a_2=\frac{8}{3},\ b_1=0,\ b_2=0,\ c=-\frac{2}{3}\right]$$

on eq. (6), we get:

$$v_0=-\frac{16}{9} \tag{50}$$

$$v_1 = \frac{128\xi^2}{27} \tag{51}$$

$$v_2 = -\frac{256\xi^4}{243} \tag{52}$$

$$v_3 = \frac{512\xi^4}{243} - \frac{1024\xi^6}{1215} \tag{53}$$

According to eqs. (50)-(53), we get an approximate solution of eq. (2) in the next formula:

$$v(\xi) = -\frac{16}{9} + \frac{128\xi^2}{27} + \frac{256\xi^4}{243} - \frac{1024\xi^6}{1215} + \dots \tag{54}$$

In tab. 1, we discuss the exact and approximate solutions of the longitudinal wave equation show the value of the absolute error between them.

Table 1. Shows for increasing the value ξ, the absolute error increases gradually; that means the Adomian decomposition method gives more accurate solutions for the values near to zero

Value of ξ	Exact solution	Approximate solution	Absolute error
0.0001	1.7777776711111133	1.7777777303703703	$5.925925705695 \cdot 10^{-8}$
0.0002	1.777777351111135	1.7777775881481463	$2.370370122406484 \cdot 10^{-7}$
0.0003	1.7777768177778916	1.7777773511111026	$5.333332109280775 \cdot 10^{-7}$
0.0004	1.7777760711114752	1.7777770192592324	$9.481477571959829 \cdot 10^{-7}$
0.0005	1.7777751111119997	1.7777765925925266	0.00000148148052692143

Figure

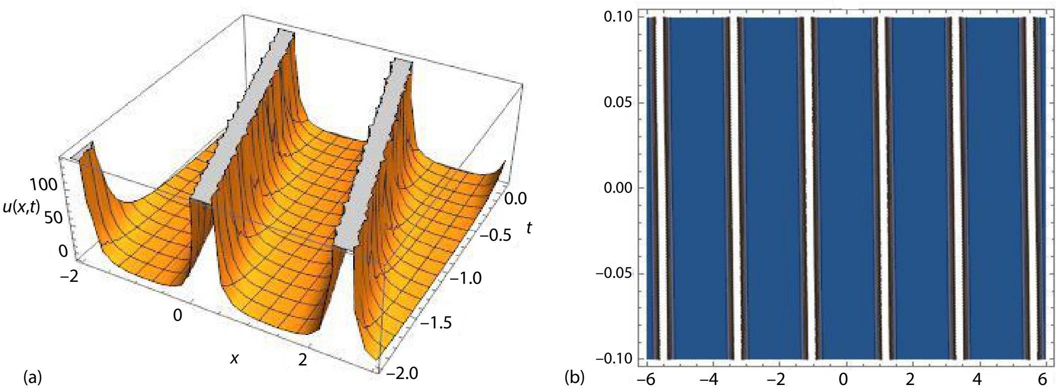

Figure 1. Cuspon wave in 3-D and contour plot of eq. (4) when [$\beta = 2$, $\alpha = 3$, $\sigma = 1$, $a = -2$, $b = -1$, $c = -2/3$, $a_0 = -40/9$, $a_1 = -16/3$, $a_2 = -8/3$, $b_1 = b_2 = 0$]

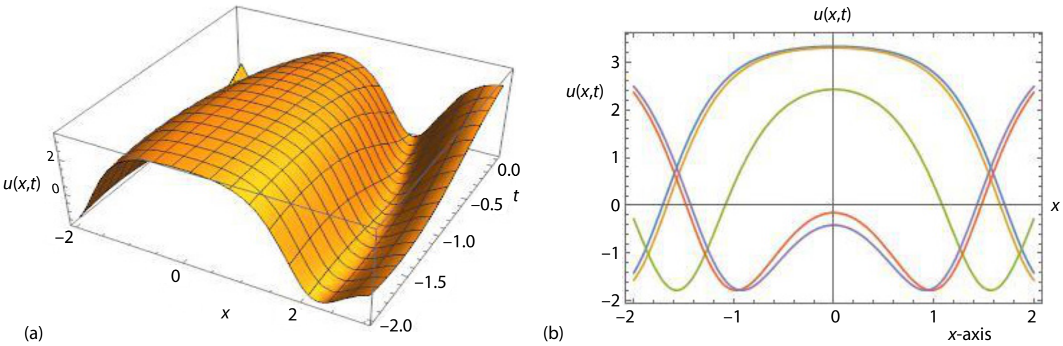

Figure 2. Periodic soliton wave in 3-D and 2-D plot of eq. (6) when [$\beta = 4$, $\alpha = 2$, $\sigma = 1$, $a = -2$, $b = 1$, $c = -2/3$, $a_0 = 80/9$, $a_1 = -32/3$, $a_2 = 8/3$, $b_1 = b_2 = 0$]

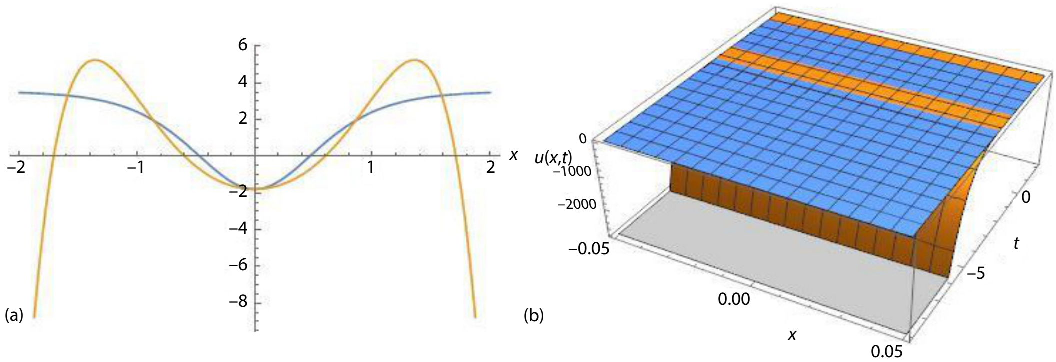

Figure 3. Convergance between exact and approximate solutions in 2- and 3-D plots for both types of solutions eqs. (6) and (54)

Conclusion

In this paper, we succeed in obtaining analytical and semi-analytical wave solutions of the longitudinal wave equation. We obtained novel and different solitary wave solutions of this model. We also obtained the approximate solutions and discuss both solutions to show the absolute value of the error tab. 1. The results show the effectiveness of the Adomian decomposition method for interval near zero. Some solitary and approximate solutions are sketched to investigate more of the physical properties of this model figs. (1)-(3). The performance of both methods shows useful and powerful in studying many of non-linear partial differential equations.

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