ON BACKLUND TRANSFORMATIONS OF SURFACES BY EXTENDED HARRY-DYM FLOW

by

Muhammed Talat SARIAYDIN *

Department of Mathematics, Faculty of Science, Selcuk University, Konya, Turkey

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The present paper deals with the introduction of Backlund transformations by extended Harry-Dym flow and with the aid of the extended version of the Riccati mapping method is obtained new solutions. Then, we give the Backlund transformation of the Schrodinger flow and obtain its the Bonnet surface. In finally, results obtained with the mathematical model are evaluated by applying to mathematica.

Key words: *fluid-flow, Riccati mapping method, PDE, time-evolution equation*

Introduction

Backlund transformation, which was introduced in 1880 by A. V. Backlund, can be used to create a series of solutions to a PDE from a known trivial solution. Moreover, a non-linear PDE into another PDE can be transformed by Backlund transformation. In this respect, a powerful method for generating solutions to non-linear PDE is the application of the Backlund transformation. On the other hand, a transform which leaves a PDE invariant is called an auto-Backlund transformation and a different second solution the same PDE can be transformed by this transformation, [1]. The Backlund transformations, which are n times to a particular solution of sine-Gordon equation, are written by a family of solutions of sine-Gordon equation. With the aid of the Bianchi's permutability formula through purely algebraic means can be achieved by these solutions, in [2]. Due to the aforementioned features, numerous studies have been carried out on Backlund transformations from past to present. For example, Palmer studied Backlund transformations for surfaces in [3], Schief gave analog of Darboux's Backlund transformation for isothermic surfaces in [4] and Goulart obtained Backlund and Ribaucour transformations for some special surfaces in [5]. Korpinar gave numerical solutions of heat-like equation in [6] and Ijaz studied Heat Transfer Analysis in MHD flow in [7]. Gokmen studied the general formulation for inextensible flows of curves in n-dimensional Euclidean space in [8]. Kwon gave new equations of Evolution of inelastic plane curves, and Inextensible flows of curves and developable surfaces in [1, 9]. Additionally, there are many works related to curves [6, 10-23].

In applied differential geometry, the flow of a space curve and surface, and the time evolution of a space curve or surface constructed by flow can be called inextensible, if its arc-length does not change. Since flow of curves has a very important place in the field of industry, there are many studies on evolving curves in the direction of their curvature vector field, [7, 8]. Therefore, we deal with the introduction of Backlund transformations by extended Harry-Dym flow in this study. Also this work has been organized: firstly, we tersely summarized the basic concepts of flows and Backlund transformations; then, we give relationships between

* Author's e-mail: talatsariaydin@gmail.com

Backlund transformations and a Bonnet surface; it is also presented some special theorems for this transformation; at the end, results obtained with the mathematical model are evaluated by applying to MATHEMATICA.

Preliminaries

Let us recall some known concepts Backlund transformations for some curve flows in R^3 Assume that γ is a smooth curve parameterized by arc-length s, torsion τ of $\gamma(s)$ is constant and its Frenet frame is $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ in R^3. We suppose that a solution of the differential equation:

$$\frac{\mathrm{d}\xi}{\mathrm{d}s} = C\sin\xi - \kappa \tag{1}$$

is $\xi = \xi[s; \kappa(s); C]$ where C is any constant. Thus, a curve which is parameterized by arclength s and constant torsion τ:

$$\tilde{\gamma}(s) = \gamma(s) + \frac{2C}{C^2 + \tau^2}(\cos\xi\mathbf{t} + \sin\xi\mathbf{n}) \tag{2}$$

Moreover, we will choose the geometric curve flow as follows throughout the article:

$$\gamma_t = \mathfrak{b}\mathbf{t} + f\mathbf{n} + \mathbf{r}\mathbf{b} \tag{3}$$

where $\mathfrak{b}$, f, and $\mathbf{r}$ depend on κ and τ of the space curve γ.

In a given geometry, if we take $\gamma(s, t)$ as a special curve, then $\tilde{\gamma}(s, t)$ can be written as another curve, which related to the following Backlund transformation:

$$\tilde{\gamma}(s,t) = \gamma(s,t) + \alpha(s,t)\mathbf{t} + \beta(s,t)\mathbf{n} + \chi(s,t)\mathbf{b} \tag{4}$$

In this paper, we choose that both curve flows for γ and $\tilde{\gamma}$ are governed by the same integrable system, that means the curvatures of the curves $\tilde{\gamma}$ determined by the flow (4) satisfy the integrable systems as for the curves [24].

Space curve flows and Backlund transformations in R^3

Let $\gamma: I \to R^3$ be a curve with arc-length parameter s, $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Serret-Frenet frame and κ and τ be curvature and torsion of the space curve γ. Also, we assume that the velocities $\mathfrak{b}$, f, and $\mathbf{r}$ depend on κ and τ of the spatial curve γ. In this respect, the integrable flows for space curves in R^3:

$$\gamma_t = \mathfrak{b}\mathbf{n} + f\mathbf{b} + \mathbf{r}\mathbf{t} \tag{5}$$

Moreover, we know that the Serret-Frenet equation of this curve:

$$\mathbf{t}_s = \kappa\mathbf{n}, \quad \mathbf{n}_s = -\kappa\mathbf{t} + \tau\mathbf{b}, \quad \mathbf{b}_s = -\tau\mathbf{n} \tag{6}$$

Take into consideration the eq. (5), we immediately have the time evolutions:

$$\begin{aligned}
\mathbf{t}_t &= \left(\frac{\partial\mathbf{r}}{\partial s} - \tau f + \kappa\right)\mathbf{n} + \left(\frac{\partial f}{\partial s} + \tau\mathfrak{b}\right)\mathbf{b} \\
\mathbf{n}_t &= -\left(\frac{\partial\mathfrak{b}}{\partial s} - \tau f + \kappa\mathbf{r}\right)\mathbf{t} + \left[\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial s} + \tau\mathfrak{b}\right) + \frac{\tau}{\kappa}\left(\frac{\partial\mathfrak{b}}{\partial s} - \tau f + \kappa\mathbf{r}\right)\right]\mathbf{b} \\
\mathbf{b}_t &= -\left(\frac{\partial f}{\partial s} + \tau\mathfrak{b}\right)\mathbf{t} - \left[\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial s} + \tau\mathfrak{b}\right) + \frac{\tau}{\kappa}\left(\frac{\partial\mathfrak{b}}{\partial s} - \tau f + \kappa\mathbf{r}\right)\right]\mathbf{n} \\
g_t &= g\left(\frac{\partial\mathbf{r}}{\partial s} - \kappa\mathfrak{b}\right)
\end{aligned} \tag{7}$$

where $g = |\gamma_p|$denotes the metric of the curve γ. Thus, the equations for κ and τ with a simple calculation are obtained immediately [24]:

$$\begin{aligned}\frac{\partial\tau}{\partial t}&=\frac{\partial}{\partial s}\left[\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial s}+\tau\mathfrak{b}\right)+\frac{\tau}{\kappa}\left(\frac{\partial\mathfrak{b}}{\partial s}-\tau f\right)+\tau\int\kappa\mathfrak{b}\mathrm{d}s\right]+\kappa\tau\mathfrak{b}+\kappa\frac{\partial f}{\partial s}\\ \frac{\partial\kappa}{\partial t}&=\frac{\partial^2\mathfrak{b}}{\partial s^2}+(\kappa^2-\tau^2)\mathfrak{b}+\frac{\partial\kappa}{\partial s}\int\kappa\mathfrak{b}\mathrm{d}s-2\tau\frac{\partial f}{\partial s}-\kappa\frac{\partial\tau}{\partial s}f\end{aligned}\tag{8}$$

Bonnet surfaces as geometric space curve flows

Let us start by assuming γ is a surface and the standard unit normal vector field on γ is $\bar{\omega}_\gamma$. The fundamental forms of the surface γ:

$$\begin{aligned}\mathbf{I}&=\mathbf{E}\mathrm{d}s^2+2\mathbf{F}\mathrm{d}s\mathrm{d}t+\mathbf{G}\mathrm{d}t^2\\ \mathbf{II}&=\mathbf{e}\mathrm{d}s^2+2\mathbf{f}\mathrm{d}s\mathrm{d}t+\mathbf{g}\mathrm{d}t^2\end{aligned}\tag{9}$$

where **I** and **II** are the first fundamental form, **fff**, and the second fundamental form, **sff**, respectively. The **E**, **F**, **G** are the coefficients of the **fff** of the surface and **e**, **f**, **g** are the coefficients of the **sff**.

Definition 4.1. Let us start by assuming γ is a surface. Then, paratmerized form of A-net on this surface satisfying the conditons the conditions **E** = **G**, **F** = 0, **f** = c = constant ≠ 0 is called an A-net [10].

Theorem 4.2. A surface is a Bonnet surface if and only if it has an A-net [10].

Lemma 4.3. Let γ be geometric space curve flows:

$$\begin{aligned}\mathbf{E}&=g(\gamma_s,\gamma_s)=1\\ \mathbf{F}&=g(\gamma_s,\gamma_t)=\mathbf{r}\\ \mathbf{G}&=g(\gamma_t,\gamma_t)=\mathfrak{b}^2+f^2+\mathbf{r}^2\end{aligned}\tag{10}$$

Lemma 4.4. Let γ be be geometric space curve flows. If γ is regular surface:

$$\mathfrak{b},f\neq 0\tag{11}$$

Moreover, eq. (5) imply:

$$\bar{\omega}_\gamma(s,t)=\mathfrak{b}\mathbf{b}-f\mathbf{n}\tag{12}$$

Theorem 4.5.

$$\begin{aligned}\gamma_{tt}=&\left\{\mathfrak{b}_t-f\left[\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial s}+\tau\mathfrak{b}\right)+\frac{\tau}{\kappa}\left(\frac{\partial\mathfrak{b}}{\partial s}-\tau f+\kappa\mathfrak{r}\right)\right]+\mathfrak{r}\left(\frac{\partial\mathfrak{b}}{\partial s}-\tau f+\kappa\mathfrak{r}\right)\right\}\mathbf{n}+\\ &+\left\{f_t+\mathfrak{b}\left[\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial s}+\tau\mathfrak{b}\right)+\frac{\tau}{\kappa}\left(\frac{\partial\mathfrak{b}}{\partial s}-\tau f+\kappa\mathfrak{r}\right)\right]+\mathfrak{r}\left(\frac{\partial f}{\partial s}+\tau\mathfrak{b}\right)\right\}\mathbf{b}+\\ &+\left[\mathfrak{r}_t-\mathfrak{b}\left(\frac{\partial\mathfrak{b}}{\partial s}-\tau f+\kappa\mathfrak{r}\right)-f\left(\frac{\partial f}{\partial s}+\tau\mathfrak{b}\right)\right]\mathbf{t}\end{aligned}\tag{13}$$

Lemma 4.6. Let γ be geometric space curve flows:

$$\varpi_\gamma\left(s,t\right)_t=\left\{\mathfrak{b}_t-f\left[\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial s}+\tau\mathfrak{b}\right)+\frac{\tau}{\kappa}\left(\frac{\partial\mathfrak{b}}{\partial s}-\tau f+\kappa\mathfrak{r}\right)\right]\right\}\mathbf{b}+$$
$$+\left\{-f_t-\mathfrak{b}\left[\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial s}+\tau\mathfrak{b}\right)+\frac{\tau}{\kappa}\left(\frac{\partial\mathfrak{b}}{\partial s}-\tau f+\kappa\mathfrak{r}\right)\right]\right\}\mathbf{n}+$$
$$+\left[f\left(\frac{\partial\mathfrak{b}}{\partial s}-\tau f+\kappa\mathfrak{r}\right)-\mathfrak{b}\left(\frac{\partial f}{\partial s}+\tau\mathfrak{b}\right)\right]\mathbf{t} \tag{14}$$

Theorem 4.7. Let γ be geometric space curve flows. The γ is a Bonnet surface if and only if:

$$\mathfrak{r}=0$$
$$\mathfrak{b}^2+f^2=1 \tag{15}$$
$$f\left(\frac{\partial\mathfrak{b}}{\partial s}-\tau f+\kappa\mathfrak{r}\right)-\mathfrak{b}\left(\frac{\partial f}{\partial s}+\tau\mathfrak{b}\right)=c$$

The extension of Harry-Dym flow

Let us recall the Extended version of Harry-Dym flow given by Qu [24]. The extended version of Harry-Dym flow:

$$\gamma_t=\tau^{-1/2}\mathbf{b} \tag{16}$$

with the choicess $\mathfrak{b}=\mathbf{r}=0$ and $f=\tau^{-1/2}$ in the space curve flow (5). We note that the curve flow with constant curvature κ is considered here. In the case of $\kappa=1$:

$$\frac{\partial\mathbf{r}}{\partial s}=\kappa\mathfrak{b} \tag{17}$$

that the torsion of satisfies the extended Harry-Dym equation [25]. Hence, the time evolution of frame vectors:

$$\mathbf{t}_t=-\tau^{-1/2}\mathbf{n}-\frac{1}{2}\tau^{-3/2}\tau_s\mathbf{b}$$
$$\mathbf{n}_t=\tau^{1/2}\mathbf{t}+\left[\left(\tau^{-1/2}\right)_{ss}-\tau^{3/2}\right]\mathbf{b} \tag{18}$$
$$\mathbf{b}_t=\frac{1}{2}\tau^{-3/2}\tau_s\mathbf{t}-\left[\left(\tau^{-1/2}\right)_{ss}-\tau^{3/2}\right]\mathbf{n}$$

We will obtain Backlund transformation of the Schrodinger flow. Let us recall following equation:

$$\tilde{\gamma}(s,t)=\gamma(s,t)+\alpha(s,t)\mathbf{t}+\beta(s,t)\mathbf{n}+\chi(s,t)\mathbf{b} \tag{19}$$

As a consequence of the eq. (19), we immediately have the equations:

$$\tilde{\gamma}_s=(1+\alpha_s-\beta\kappa)\mathbf{t}+(\beta_s+\alpha\kappa-\chi\tau)\mathbf{n}+(\chi_s+\beta\tau)\mathbf{b}$$
$$\tilde{\gamma}_t=\left[\alpha_t+\beta\tau^{1/2}+\chi(\tau^{-1/2})_s\right]\mathbf{t}+$$
$$+\left\{\beta_t-\alpha\tau^{1/2}-\chi\left[(\tau^{-1/2})_{ss}-\tau^{3/2}\right]\right\}\mathbf{n}+$$
$$+\left\{\tau^{-1/2}+\alpha(\tau^{-1/2})_s+\beta\left[(\tau^{-1/2})_{ss}-\tau^{3/2}\right]+\chi_t\right\}\mathbf{b} \tag{20}$$

Thus, we immediately have the normal vector of Backlund transformation of the Schrodinger flow:

$$\begin{aligned}\varpi_{\tilde{\gamma}} = &\Big\{(\beta_s + \alpha\kappa - \chi\tau)\Big[\tau^{-1/2} + \alpha(\tau^{-1/2})_s + \beta\Big((\tau^{-1/2})_{ss} - \tau^{3/2}\Big) + \chi_t\Big] - \\ &- (\chi_s + \beta\tau)\Big[\beta_t - \alpha\tau^{1/2} - \chi\Big((\tau^{-1/2})_{ss} - \tau^{3/2}\Big)\Big]\Big\}\mathbf{t} + \\ &+ \Big\{(\chi_s + \beta\tau)\Big[\alpha_t + \beta\tau^{1/2} + \chi(\tau^{-1/2})_s\Big] - \\ &- (1 + \alpha_s - \beta\kappa)\Big[\tau^{-1/2} + \alpha(\tau^{-1/2})_s + \beta\Big((\tau^{-1/2})_{ss} - \tau^{3/2}\Big) + \chi_t\Big]\Big\}\mathbf{n}\cdot \\ &\cdot\Big\{(1 + \alpha_s - \beta\kappa)\Big[\beta_t - \alpha\tau^{1/2} - \chi\Big((\tau^{-1/2})_{ss} - \tau^{3/2}\Big)\Big] - \\ &- (\beta_s + \alpha\kappa - \chi\tau)\Big[\alpha_t + \beta\tau^{1/2} + \chi(\tau^{-1/2})_s\Big]\Big\}\mathbf{b}\end{aligned} \tag{21}$$

From previous equations:

$$\begin{aligned}\tilde{\gamma}_{ss} = &[(1 + \alpha_s - \beta\kappa)_s - (\beta_s + \alpha\kappa - \chi\tau)\kappa]\mathbf{t} + \\ &+ [(\beta_s + \alpha\kappa - \chi\tau)_s + (1 + \alpha_s - \beta\kappa)\kappa - (\chi_s + \beta\tau)\tau]\mathbf{n} + \\ &+ [(\chi_s + \beta\tau)_s + (\beta_s + \alpha\kappa - \chi\tau)\tau]\mathbf{b}\end{aligned} \tag{22}$$

and

$$\begin{aligned}\tilde{\gamma}_{ts} = &\Big(\Big[\alpha_t + \beta\tau^{1/2} + \chi(\tau^{-1/2})_s\Big]_s - \Big\{\beta_t - \alpha\tau^{1/2} - \chi\Big[(\tau^{-1/2})_{ss} - \tau^{3/2}\Big]\Big\}\kappa\Big)\mathbf{t} + \\ &+ (\Big[\alpha_t + \beta\tau^{1/2} + \chi(\tau^{-1/2})_s\Big]\kappa + \Big\{\beta_t - \alpha\tau^{1/2} - \chi\Big[(\tau^{-1/2})_{ss} - \tau^{3/2}\Big]\Big\}_s - \\ &- \Big\{\tau^{-1/2} + \alpha(\tau^{-1/2})_s + \beta\Big[(\tau^{-1/2})_{ss} - \tau^{3/2}\Big] + \chi_t\Big\}\tau)\mathbf{n} + \\ &+ (\Big\{\beta_t - \alpha\tau^{1/2} - \chi\Big[(\tau^{-1/2})_{ss} - \tau^{3/2}\Big]\Big\}\tau + \\ &+ \Big\{\tau^{-1/2} + \alpha(\tau^{-1/2})_s + \beta\Big[(\tau^{-1/2})_{ss} - \tau^{3/2}\Big] + \chi_t\Big\}_s)\mathbf{b}\end{aligned} \tag{23}$$

Theorem 5.1. Let γ be Backlund transformation of the Schrodinger flow. The γ is a Bonnet surface if and only if:

$$\begin{aligned}&\Big[\alpha_t + \beta\tau^{1/2} + \chi\Big(\tau^{-1/2}\Big)_s\Big]^2 + \Big\{\beta_t - \alpha\tau^{1/2} - \chi\Big[\Big(\tau^{-1/2}\Big)_{ss} - \tau^{3/2}\Big]\Big\}^2 + \Big\{\tau^{-1/2} + \alpha\Big(\tau^{-1/2}\Big)_s + \\ &+ \beta\Big[\Big(\tau^{-1/2}\Big)_{ss} - \tau^{3/2}\Big] + \chi_t\Big\}^2 = (1 + \alpha_s - \beta\kappa)^2 + (\beta_s + \alpha\kappa - \chi\tau)^2 + (\chi_s + \beta\tau)^2 \\ &(1 + \alpha_s - \beta\kappa)\Big[\alpha_t + \beta\tau^{1/2} + \chi\Big(\tau^{-1/2}\Big)_s\Big] + (\beta_s + \alpha\kappa - \chi\tau)\Big\{\beta_t - \alpha\tau^{1/2} - \chi\Big[\Big(\tau^{-1/2}\Big)_{ss} - \tau^{3/2}\Big]\Big\} + \\ &+ (\chi_s + \beta\tau)\Big\{\tau^{-1/2} + \alpha\Big(\tau^{-1/2}\Big)_s + \beta\Big[\Big(\tau^{-1/2}\Big)_{ss} - \tau^{3/2}\Big] + \chi_t\Big\} = 0\end{aligned} \tag{24}$$

and

$$\begin{aligned}c = &\Big\{(\beta_s + \alpha\kappa - \chi\tau)\Big[\tau^{-1/2} + \alpha(\tau^{-1/2})_s + \beta\Big((\tau^{-1/2})_{ss} - \tau^{3/2}\Big) + \chi_t\Big] - \\ &- (\chi_s + \beta\tau)\Big[\beta_t - \alpha\tau^{1/2} - \chi\Big((\tau^{-1/2})_{ss} - \tau^{3/2}\Big)\Big]\Big\}\cdot \\ &\cdot\Big\{\Big[\alpha_t + \beta\tau^{1/2} + \chi(\tau^{-1/2})_s\Big]_s - \Big[\beta_t - \alpha\tau^{1/2} - \chi\Big((\tau^{-1/2})_{ss} - \tau^{3/2}\Big)\Big]\kappa\Big\} + \Big\{(\chi_s + \beta\tau)\cdot\end{aligned} \tag{25}$$

$$\begin{gathered}\cdot\left[\alpha_t+\beta\tau^{1/2}+\chi(\tau^{-1/2})_s\right]-(1+\alpha_s-\beta\kappa)\left[\tau^{-1/2}+\alpha(\tau^{-1/2})_s+\beta\left((\tau^{-1/2})_{ss}-\tau^{3/2}\right)+\chi_t\right]\Big\}\cdot\\ \cdot\left[\alpha_t+\beta\tau^{1/2}+\chi(\tau^{-1/2})_s\right]\kappa+\left[\beta_t-\alpha\tau^{1/2}-\chi\left((\tau^{-1/2})_{ss}-\tau^{3/2}\right)\right]_s-\\ -\left\{\tau^{-1/2}+\alpha(\tau^{-1/2})_s+\beta\left[(\tau^{-1/2})_{ss}-\tau^{3/2}\right]+\chi_t\right\}\tau+\{(1+\alpha_s-\beta\kappa)\cdot\\ \cdot\left[\beta_t-\alpha\tau^{1/2}-\chi\left((\tau^{-1/2})_{ss}-\tau^{3/2}\right)\right]-(\beta_s+\alpha\kappa-\chi\tau)\left[\alpha_t+\beta\tau^{1/2}+\chi(\tau^{-1/2})_s\right]\Big\}\cdot\\ \cdot\left\{\left[\beta_t-\alpha\tau^{1/2}-\chi\left((\tau^{-1/2})_{ss}-\tau^{3/2}\right)\right]\tau+\left[\tau^{-1/2}+\alpha(\tau^{-1/2})_s+\beta\left((\tau^{-1/2})_{ss}-\tau^{3/2}\right)+\chi_t\right]_s\right\}\end{gathered}\tag{25}$$

Proof. First fundamental form of Backlund transformation of the Schrodinger flow:

$$\begin{gathered}\mathbf{E}=(1+\alpha_s-\beta\kappa)^2+(\beta_s+\alpha\kappa-\chi\tau)^2+(\chi_s+\beta\tau)^2\\ \mathbf{F}=(1+\alpha_s-\beta\kappa)\left[\alpha_t+\beta\tau^{1/2}+\chi(\tau^{-1/2})_s\right]+\\ +(\beta_s+\alpha\kappa-\chi\tau)\left\{\beta_t-\alpha\tau^{1/2}-\chi\left[(\tau^{-1/2})_{ss}-\tau^{3/2}\right]\right\}+\\ +(\chi_s+\beta\tau)\left\{\tau^{-1/2}+\alpha(\tau^{-1/2})_s+\beta\left[(\tau^{-1/2})_{ss}-\tau^{3/2}\right]+\chi_t\right\}\\ \mathbf{G}=\left[\alpha_t+\beta\tau^{1/2}+\chi(\tau^{-1/2})_s\right]^2+\left\{\beta_t-\alpha\tau^{1/2}-\chi\left[(\tau^{-1/2})_{ss}-\tau^{3/2}\right]\right\}^2+\\ +\left\{\tau^{-1/2}+\alpha(\tau^{-1/2})_s+\beta\left[(\tau^{-1/2})_{ss}-\tau^{3/2}\right]+\chi_t\right\}^2\end{gathered}\tag{26}$$

On the other hand:

$$\begin{gathered}\mathbf{f}=((\beta_s+\alpha\kappa-\chi\tau)\left\{\tau^{-1/2}+\alpha(\tau^{-1/2})_s+\beta\left[(\tau^{-1/2})_{ss}-\tau^{3/2}\right]+\chi_t\right\}-\\ -(\chi_s+\beta\tau)\left\{\beta_t-\alpha\tau^{1/2}-\chi\left[(\tau^{-1/2})_{ss}-\tau^{3/2}\right]\right\})\cdot\\ \cdot(\left[\alpha_t+\beta\tau^{1/2}+\chi(\tau^{-1/2})_s\right]_s-\left\{\beta_t-\alpha\tau^{1/2}-\chi\left[(\tau^{-1/2})_{ss}-\tau^{3/2}\right]\right\}\kappa)+\\ +((\chi_s+\beta\tau)\left[\alpha_t+\beta\tau^{1/2}+\chi(\tau^{-1/2})_s\right]-(1+\alpha_s-\beta\kappa)\cdot\\ \cdot\left\{\tau^{-1/2}+\alpha(\tau^{-1/2})_s+\beta\left[(\tau^{-1/2})_{ss}-\tau^{3/2}\right]+\chi_t\right\})\cdot\\ \cdot\left[\alpha_t+\beta\tau^{1/2}+\chi(\tau^{-1/2})_s\right]\kappa+\left\{\beta_t-\alpha\tau^{1/2}-\chi\left[(\tau^{-1/2})_{ss}-\tau^{3/2}\right]\right\}_s-\\ -\left\{\tau^{-1/2}+\alpha(\tau^{-1/2})_s+\beta\left[(\tau^{-1/2})_{ss}-\tau^{3/2}\right]+\chi_t\right\}\tau+\{(1+\alpha_s-\beta\kappa)\cdot\\ \cdot\left\{\beta_t-\alpha\tau^{1/2}-\chi\left[(\tau^{-1/2})_{ss}-\tau^{3/2}\right]\right\}-(\beta_s+\alpha\kappa-\chi\tau)\cdot\\ \cdot\left[\alpha_t+\beta\tau^{1/2}+\chi(\tau^{-1/2})_s\right]\}(\left\{\beta_t-\alpha\tau^{1/2}-\chi\left[(\tau^{-1/2})_{ss}-\tau^{3/2}\right]\right\}\tau+\\ +\left\{\tau^{-1/2}+\alpha(\tau^{-1/2})_s+\beta\left[(\tau^{-1/2})_{ss}-\tau^{3/2}\right]+\chi_t\right\}_s)\end{gathered}$$

Application Mathematica

Let us recall the curvature κ and torsion τ given by Qu [24]:

$$\frac{\partial\tau}{\partial t}=\frac{\partial}{\partial s}\left[\frac{1}{\kappa}\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial s}+\tau\mathfrak{b}\right)+\frac{\tau}{\kappa}\left(\frac{\partial\mathfrak{b}}{\partial s}-\tau f\right)+\tau\int\kappa\mathfrak{b}\mathrm{d}s\right]+\kappa\tau\mathfrak{b}+\kappa\frac{\partial f}{\partial s}$$
$$\frac{\partial\kappa}{\partial t}=\frac{\partial^2\mathfrak{b}}{\partial s^2}+(\kappa^2-\tau^2)\mathfrak{b}+\frac{\partial\kappa}{\partial s}\int\kappa\mathfrak{b}\mathrm{d}s-2\tau\frac{\partial f}{\partial s}-\kappa\frac{\partial\tau}{\partial s}f \tag{27}$$

From previous equations, Hasimoto transformation:

$$\phi=\kappa\eta,\quad \eta=\exp[i\tau(t,s)\mathrm{d}s] \tag{28}$$

Let $\mathfrak{b}=-\kappa, f=\kappa\tau$. Then $\mathbf{r}=-1/2\kappa^2$, and ϕ satisfies the mKdV system:

$$\phi_t+\phi_{sss}+\frac{3}{2}|\phi|^2\phi_s=0 \tag{29}$$

Consider the traveling wave variable:

$$\phi(s,t)=q(\varsigma),\quad \varsigma=s-Ht \tag{30}$$

Then, using eq. (30), eq. (29) is changed into an ordinary differential equation for $q(\varsigma)$:

$$-Hq'(\varsigma)+\frac{3}{2}|q(\varsigma)|^2q'(\varsigma)+q'''(\varsigma)=0 \tag{31}$$

We can give the extended generalized Riccati mapping method to obtain the solution of eq. (29). By balancing $|q(\varsigma)|^2q'(\varsigma)$ with $q'''(\varsigma)$ in eq. (29), we yield $N=1$.

Therefore, the solution of eq. (31):

$$q(\varsigma)=a_1\left[\frac{G'(\varsigma)}{G(\varsigma)}\right]+a_0,\quad a_1\neq 0 \tag{32}$$

Equation (32) can be re-written:

$$q(\varsigma)=a_1\left[hG^{-1}(\varsigma)+f+gG(\varsigma)\right]+a_0 \tag{33}$$

where f, g, h are arbitrary constants, $g\neq 0$ and $G'(\varsigma)=h+fG(\varsigma)+gG^2(\varsigma)$ is auxiliary equation.

By substituting eq. (33) into eq. (32), we find a set of algebraic equations for a_0, a_1, f, g, h, and Q from coefficients of $G^k(\varsigma)$ and $G^{-k}(\varsigma)$ ($k=0, 1, 2,...$). Solving the system of algebraic equations by using software MATHEMATICA:

$$a_0=if,\quad a_1=-2i,\ H=\frac{1}{2}(-f^2-8gh) \tag{34}$$

One of solutions of eq. (33) is ($f=3$, $g=2$, $h=1$), fig. 1:

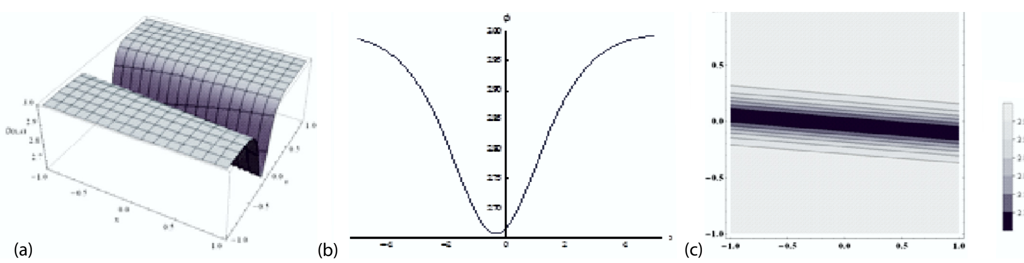

Figure 1. Shape of solution for imaginary part of eq. (35); (a) in 3-D, (b) in 2-D ($t=0$), and (c) its contour

$$q=-2i\left\{\frac{\Delta\sec h^2\left(\frac{\sqrt{\Delta}}{2}\varphi\right)}{2\left[f+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\varphi\right)\right]}\right\}+if \tag{35}$$

where $\Delta=f^2-4gh$.

Conclusion

A non-linear PDE into another PDE can be transformed by Backlund transformation. In this respect, a powerful method for generating solutions to non-linear PDE is the application of the Backlund transformation. In this study, we give relationships between Backlund transformations and a Bonnet surface. It is also presented some special theorems for this transformation. At the end, results obtained with the mathematical model are evaluated by application Mathematica.

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