SOME IMPORTANT DETAILS ON TECHNICAL SYSTEM
Matrix Representation, Odds Ratio, Embedding Subsystems

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The system signature calculated independently from the distribution of the lifetime of the units in the analysis of technical systems enables us to have important information about the reliability of the system. System signature is a probability vector obtained from possible sequences of units according to the moments of deterioration. The reliability of the system can be easily obtained if the distribution of the lifetime of the units is known and if the distributions can be listed as open among themselves. However, when the distributions cannot be listed as open among themselves, it is a very important finding that the system reliability is calculated with the help of system signature by generating the order statistics. It is also possible to calculate the odds ratios of the units when the lifetime of the units is certain. In this study, odds ratio could be calculated by calculating the probability of fail the system of each unit by considering the possible situations of the system, as in the system signature, regardless of the distribution of lifetime of the units. In addition, the technical system is represented by a matrix by establishing a relation between the system fail probabilities of the units obtained in the study and the system signature. However, examples of some technical systems given in the study.

Key words: *system signature, odds ratio, subsystems, system reliability*

Introduction

Reliability is defined as the probability that a unit will perform its intended functions satisfactory for a specified period of time under specified operating conditions. Based on this definition, reliability is measured as a probability. Probability theory has been used to analyze the reliability of components as well as the reliability of systems consisting of these components [1]. Reliability has always been a key role in the design of engineering systems [2]. Reliability study is concerned with random occurrences of undesirable events or failures during the life of a physical systems and engineering systems [3]. Estimation of system reliability has been discussed by [4-10].

Components are the building blocks of a system. Thus, the performance of the system depends on the performance of every component. A serial system works if all of the component works. A parallel system works if and only if at least one-component works. An component system that fails if and only if at least k of the n components fail is called a k-out of-n:F system. The reliability of a system defined by the ϕ structure function is given by:

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$$R = E\phi = Pr\{\phi = 1\} \tag{1}$$

Let T be the random variable has a distribution function F representing the lifetime of a system. Thus, the reliability function of the system:

$$R(t) = Pr\{T > t\} = 1 - F(t) \tag{2}$$

The expected value or the mean the lifetime T is also called the mean time to failure (MTTF) of the system. The MTTF is given:

$$MTTF = ET = \int_0^\infty [1 - F(t)]\mathrm{d}t \tag{3}$$

The signature s of a coherent system of order n is the n-dimensional probability vector whose i^{th} element is $s_i = P(T = X_{i:n})$, where T – the system lifetime and $X_{1:n}, X_{2:n},\ldots,X_{n:n}$ are the order statistics of the not independent and identically distribution component lifetimes.

Fundamental matrix of system

Considering the ways in which units are connected to each other in a technical system where the system signature is handled, the system failure status of order statistics can be easily calculated by taking into account the possible situations. For example, connection of a technical system consisting of three units defined.

$$\phi = \min\{\max\{X_1, X_2\}, X_3\} \tag{4}$$

In this case, the system signature is calculated as $Im(2/6, 4/6, 0)$. Where the i^{th} component of the system signature indicates the probability that i^{th} failed component must failed the system. In the first two order statistics in the example, because the system is absolutely fail, the probability of system corruption of third order statistics is naturally zero.

Now let us show the i^{th} corruption occur $\{\xi_i = 0\}$ and in this case, the probability of system corruption is shown.

$$Pr\{\xi_i = 0\} = s_i \tag{5}$$

Likewise, $\alpha_{ij} = Pr\ \{\xi_i = 0,\ Y_i = X_j\}$ the probabilities can also be calculated. The totals of the α_{ij} probabilities according to the first index will give possibility $\alpha(j)$ of system corruption of the j^{th} unit. The matrix $M = [\alpha_{ij}]$ obtained from $\min\{\max\{X_1, X_2\}, X_3\}$ system is calculated:

$$M = \begin{bmatrix} 0 & 0 & 2/6 \\ 1/6 & 1/6 & 2/6 \\ 0 & 0 & 0 \end{bmatrix} \tag{6}$$

The sum of the line elements of this matrix will give the s_i components of the system signature. Accordingly, the following equations can be written:

$$MI = Im \tag{7}$$

$$I^t M = \alpha = [\alpha(1), \cdots, \alpha(n)] \tag{8}$$

$$MIe_j = s_j \tag{9}$$

Here, n_{x1} dimensional elements 1 and e_j, j^{th} element 1 and others are 0 vectors.

Odds ratio of system

In this section, odds ratios will be calculated according to the connection methods of some technical systems. Here, the odds ratio of a technical system will be calculated according to the principle of dividing the probability of system operation of the probability of system corruption. When calculating this ratio, the system will be evaluated completely and parallel and serial connections will be evaluated separately and odds ratios will be calculated. These ratios will then be merged. For this, we first examine the odds ratio of a technical system consisting of two units connected in series. Let us show the operation event of a serial connected technical system consisting of two units with $\{S = 1\}$ and corrupt state $\{S = 0\}$. The possibilities of these situations:

$$Pr\{S=1\} = p_1 p_2 \tag{10}$$

$$Pr\{S=0\} = q_1 q_2 + p_1 q_2 + q_1 p_2 \tag{11}$$

In this case, the odds ratio of the system will be as follows:

$$OR_{\text{seri sistem}} = \frac{q_1 q_2 + p_1 q_2 + q_1 p_2}{p_1 p_2} = \left(\frac{q_1}{p_1}\right)\left(\frac{q_2}{p_2}\right) + \frac{q_1}{p_1} + \frac{q_2}{p_2} \tag{12}$$

Odds ratios of units are also $OR_{j\text{-unci unite}} = q_j/p_j$, the odds ratio of the two units connected in series can be calculated in terms of odds ratios of the units depending on the following functional relationship:

$$f_s(x, y) = xy + x + y \tag{13}$$

Similarly, the odds ratio of the two units connected in parallel:

$$OR_{\text{paralel sistem}} = \left\{\left(\frac{p_1}{q_1}\right)\left(\frac{p_2}{q_2}\right) + \frac{p_1}{q_1} + \frac{p_2}{q_2}\right\}^{-1} \tag{14}$$

Accordingly, odds ratios of the two units connected in parallel can be calculated according to the following functional relationship:

$$f_p(x, y) = \left\{\left(\frac{1}{x}\right)\left(\frac{1}{y}\right) + \frac{1}{x} + \frac{1}{y}\right\}^{-1} \tag{15}$$

In the previous functional relations, $f(x, y) = f(y, x)$ and $(x, y, z) = f[f(x, y), z]$ are present. As a result, a technical system with the help of these relations can be examined easily by reducing to dual subsystems.

Example 1

In this example, let us examine $\max\{\min\{\max\{X_3, X_4\}, \min\{X_1, X_2\}\}$. The technical system is composed of four subsystems. The odds ratio of this system can be calculated with the help of the following functional equation.

$$OR_{\text{sistem}} = f_p\left\{f_s\left[f_p\left(X_3, X_4\right), X_5\right], f_s\left(X_1, X_2\right)\right\}$$

In the tab. 1, the odds ratio of the system is calculated for different corruption possibilities by equalizing the failure possibilities of the components that make up the system.

Table 1. Odds ratios of the system according to the corruption rate of the components

Possibility of corruption of components	Odds ratio of components	Odds ratio of the system
$q = 0$	0	0
$q = 0.1$	0.11	0.035
$q = 0.2$	0.25	1.028
$q = 0.3$	0.43	2.202
$q = 0.4$	0.66	4.42
$q = 0.5$	1	9.64
$q = 0.6$	1.5	23.37
$q = 0.7$	2.33	71.41
$q = 0.8$	4	346.5
$q = 0.9$	9	5259
$q = 1$	∞	∞

Example 2

In this example, let us examine the 3-out-of-5:*F* system. Let us show all triple combinations of the components of the system with A_j, $j = 1, 2,...,10$. The 3-out-of-5:*F* system can be expressed:

$$\phi = \min\left\{\max A_1, \cdots, \max A_{10}\right\}$$

Accordingly, the odds ratio of the system can be calculated with the following functional expression.

$$OR_{\text{sistem}} = f_s\left[f_p\left(A_1\right), \cdots, f_p\left(A_1\right)\right]$$

The tab. 2 is prepared equal to the corruption probabilities of the units.

Table 2. Odds ratios of the system according to the corruption ratio of the components in system *3-out-of-5:F*

Possibility of corruption of components	Odds ratio of components	Odds ratio of the system
$q = 0$	0	0
$q = 0.1$	0.11	0.0095
$q = 0.2$	0.25	0.082
$q = 0.3$	0.43	0.312
$q = 0.4$	0.66	0.908
$q = 0.5$	1	0.941
$q = 0.6$	1.5	2.366
$q = 0.7$	2.33	7.101
$q = 0.8$	4	34.726
$q = 0.9$	9	682.09
$q = 1$	∞	∞

Example 3

In this example, let us examine consecutive 3-out-of-5:*F* systems. Possible consecutive sequential states of the system are $A_j = \{X_j, X_{j+1}, X_{j+2}\}$, $j = 1, 2, 3$, the system is expressed with $\phi = \min\{\max A_1, \max A_2, \max A_3\}$. Accordingly, the odds ratio of the system can be calculated with the following functional expression.

$$OR_{\text{sistem}} = f_s\left[f_p\left(A_1\right), f_p\left(A_2\right), f_p\left(A_3\right)\right]$$

System reliability by combining embedding subsystems

In this section, the reliability of the technical system will be calculated by combining the parts divided into the smallest subsystems. When a technical system is divided into two-component parts, these smallest parts are connected in series or parallel. The odds ratio of the system was calculated by using this property in the examples in the previous section, tab. 3.

Table 3. Odds ratios of the system according to the corruption ratio of the components in system consecutive *3-out-of-5:F*

Possibility of corruption of components	Odds ratio of components	Odds ratio of the system
$q = 0$	0	0
$q = 0.1$	0.11	0.0028
$q = 0.2$	0.25	0.024
$q = 0.3$	0.43	0.082
$q = 0.4$	0.66	0.214
$q = 0.5$	1	0.484
$q = 0.6$	1.5	1.047
$q = 0.7$	2.33	2.511
$q = 0.8$	4	7.589
$q = 0.9$	9	49.199
$q = 1$	∞	∞

First, let us consider the stochastic process described below in order to identify the parallel or serial connected parts of the system. The components of the technical system are X_1, X_2, and state space $\{1, 0\}$, $Z_{(1)p} = X_1$ and the variable Z_n is defined for parallel connected components:

$$Z_{(n+1)p} = \max\left\{Z_{(n)p}, X_{n+1}\right\}, n \geq 1$$

Similarly for serial connected systems, $Z_{(1)s} = X_1$:

$$Z_{(n+1)s} = \min\left\{Z_{(n)s}, X_{n+1}\right\}, n \geq 1$$

One-step transition possibilities for parallel-connected components:

$$Pr\left\{Z_{(n+1)p} = 1 | Z_{(n)p} = 1\right\} = 1$$

$$Pr\left\{Z_{(n+1)p}=1|Z_{(n)p}=0\right\}=p_{n+1}$$

One-step transition possibilities for serial-connected components:

$$Pr\left\{Z_{(n+1)s}=1|Z_{(n)s}=1\right\}=p_{n+1}$$

$$Pr\left\{Z_{(n+1)s}=0|Z_{(n)s}=0\right\}=1$$

A step transition matrix depending on the n-th step of the process $\{Z_{(n)p}:n\geq 1\}$, with the help of the transition probabilities:

$$P_p\left[X_{(n+1)}\right]=\begin{bmatrix}1 & 0\\ p_{n+1} & q_{n+1}\end{bmatrix}$$

Similarly, a step transition matrix depending on the n^{th} step of the process $\{Z_{(n)p}:n\geq 1\}$, with the help of the transition probabilities:

$$P_s\left[X_{(n+1)}\right]=\begin{bmatrix}p_{n+1} & q_{n+1}\\ 0 & 1\end{bmatrix}$$

Let distributions of $Z_{(n)p}$ and $Z_{(n)s}$ variables be $\pi_{(n)p}$ and $\pi_{(n)s}$, respectively. Accordingly there is $\pi_{(n+1)p}=\pi_{(n)p}P_{\text{parallel}}(n)$ and $\pi_{(n+1)s}=\pi_{(n)s}P_{\text{serial}}(n)$ equality between the distribution of the process and the transition matrix, respectively. Naturally, when a technical system is divided into lower parts, $Z_{(n)p}$ and $Z_{(n)s}$ must be more than one. In this case, we will show the processes of the subparts $Z_{j(i)s}$, and with the variable representing the i^{th} step of the j^{th} sub-part connected in series. In this case the reliability of the technical system is the possibility $Pr\{Z_{r(v)}=1\}$. Where $Z_{r(v)}$ is the last variable obtained after all subsystems are connected.

Example 4

In this example, let us examine the system $\ell=\min\{\max\{X_1,X_2\},X_3,\max\{X_4,X_5,X_6\}\}$. The ℓ system consists of three basic parts. These are $\ell_1=\max\{X_1,X_2\}$, $\ell_1=X_3$, $\ell_3=\max\{X_4, X_5, X_6\}$, respectively. Let the processes representing these subsystems be $Z_{1(j)\text{p}}$ for ℓ_1, $Z_{2(j)\text{p}}$ for ℓ_2, and $Z_{3(j)\text{p}}$ for ℓ_3 respectively. Now let us get the distribution of these processes, respectively.

$$Z_{1(1)p}=X_1,\ \pi_{1(1)p}=(p_1,q_1)$$

$$Z_{1(2)p}=\max\left\{Z_{1(1)p},X_2\right\},\pi_{1(2)p}=\pi_{1(1)p}P_p\left(X_{1(2)}\right)=\left(p_1+q_1p_2,q_1q_2\right)$$

$$Z_{2(1)}=X_3,\ \ \pi_{2(1)}=\left(p_3,q_3\right)$$

$$Z_{3(1)p}=X_4,\ \ \pi_{3(1)}=\left(p_4,q_4\right)$$

$$Z_{3(2)p}=\max\left\{Z_{3(1)p},X_5\right\},\ \ \pi_{3(2)p}=\pi_{3(1)}P_p\left(X_{3(5)}\right)=\left(p_4+q_4p_5,q_4q_5\right)$$

$$Z_{3(3)p}=\max\left\{Z_{3(2)p},X_6\right\},\ \ \pi_{3(3)p}=\pi_{3(2)p}P_{\text{paralel}}\left[X_{3(6)}\right]=\left(p_4+q_4p_5+q_4q_5p_6,q_4q_5q_6\right)$$

Obtained these distributions will be combined with $Z_{4(j)s}$ process:

$$Z_{4(1)s} = Z_{1(2)p}, \ \pi_{4(1)} = \left(p_1 + q_1 p_2, q_1 q_2\right)$$

$$Z_{4(2)s} = \min\left\{Z_{4(1)s}, Z_{2(1)}\right\}, \ \pi_{4(2)s} = \pi_{4(1)} P_s\left(Z_{2(1)}\right) = \left(p_1 p_3 + q_1 p_2 p_3, p_1 q_3 + q_1 p_2 q_3 + q_1 q_2\right)$$

$$Z_{4(3)s} = \min\left\{Z_{4(2)s}, Z_{3(3)}\right\}, \ \pi_{4(3)s} = Z_{4(2)s} P_s\left(Z_{3(3)}\right) =$$

$$= \left[\left(p_1 p_3 + q_1 p_2 p_3\right)\left(p_4 + q_4 p_5 + q_4 q_5 p_6\right),\right.$$

$$\left.\left(p_1 p_3 + q_1 p_2 p_3\right)\left(q_4 q_5 q_6\right) + p_1 q_3 + q_1 p_2 q_3 + q_1 q_2\right]$$

As a result, the reliability of the system is $Pr\{Z_{4(3)s} = 1\} = (p_1p_3 + q_1p_2p_3)(p_4 + q_4p_5 + q_4q_5p_6)$. The $X_{j(i)}$ used in the example refers to the variable X_i in the subcomponent.

Conclusion

The connection shapes of technical systems can be quite complicated. In this case, as the number of components that make up the system increases, it is very difficult to calculate the reliability and mean time to failure of the system. In this respect, the method presented in the study is very useful. Calculating the odds ratio for the system based on the odds ratios calculated from these parts by subdividing a technical system into the lower parts is sufficient for the reliability of the system and mean time to failure.

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