ON EXACT AND APPROXIMATE SOLUTIONS OF (2+1)-D KONOPELCHENKO-DUBROVSKY EQUATION VIA MODIFIED SIMPLEST EQUATION AND CUBIC B-SPLINE SCHEMES

by

***Suleman H. ALFALQI*[a], *Jameel F. ALZAIDI*[a], *Dianchen LU*[b], *and Mostafa M. A. KHATER*[b*]**

[a] Department of Mathematics, Faculty of Science and Arts in Mahayil Asir King Khalid University, Abha, Saudi Arabia
[b] Department of Mathematics, Faculty of Science, Jiangsu University, Zhenjiang, China

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This paper studies the analytical and numerical solutions for (2+1)-D Konopelchenko-Dubrovsky equation. It also examines the performance of the modified simplest equation method and the cubic B-spline scheme on this model. Many explicit wave solutions are found by using the analytical technique. These solutions allow studying the physical properties of this model. The comparison between the analytical and numerical solutions are discussed to show which one of cubic B-spline scheme families is more accurate in finding the numerical solutions of this model.

Key words: *analytical and numerical wave solution, B-spline schemes, solitary wave solution, modified simplest equation method*

Introduction

Non-linear PDE are the most suitable technique to express many significant phenomena. It is also able to study the mechanism and physical properties of these phenomena. To this end, exact and numerical solutions are particularly significant. Indeed, a lot of physicists and mathematicians have been investigating the exact and numerical solutions. Up to now, many numerical and solitary schemes have been formulating, such as generalized extended tanh-function method, Khater method, modified auxiliary equation method (modified Khater method), Wronskian technique, linear superposition principle, and Hirota direct method [1-15].

Recently, for studying the exact traveling wave solutions, the modified simplest equation method has been proposed to investigate many kinds of solutions such as rational, exponential, hyperbolic, trigonometric, kink, rogue, lump, bilinear, and solitary [16-20]. Moreover, the complexion solutions had been investigated such that it defines as an interaction of exponential and trigonometric waves [21-25] while the B-spline schemes have been being used to study the numerical solutions of many various forms of non-linear PDE [26-30].

In this paper, we study (2+1)-D Konopelchenko-Dubrovsky equation derived by Konopelchenko BG and Dubrovsky VG [31]. The classical form of his equation is given:

* Corresponding author, e-mail: mostafa.khater2024@yahoo.com

$$\phi_t - \phi_{xxx} - 3\frac{\mathrm{d}}{\mathrm{d}y}\left(\int \phi_y \,\mathrm{d}x\right) + \frac{3}{2}a^2\phi^2\phi_x + 3a\phi_x \int \phi_y \,\mathrm{d}x - 6b\phi\phi_x = 0 \tag{1}$$

where a, b are arbitrary constants and $\phi = \phi(x, y, t)$ is an analytical function in x, y, t. Equation (1) when, ($a = 0$) becomes the Kadomtsev-Petviashvili (KP) equation while, when ($b = 0$) becomes the modified Kadomtsev-Petviashvili (mKP) equation. With the following dependent variable transformation $a = 0$, $\phi = 2/b\ln(\varphi)_{xx}$, eq. (1) transforms:

$$\left(D_x D_t - D_x^4 - 3D_y^2\right)\varphi\varphi = 0 \tag{2}$$

While, the (2+1)-D Konopelchenko-Dubrovsky system takes the following form [32-35]:

$$\begin{cases} S_t - S_{xxx} - 6bSS_x + \dfrac{3}{2}a^2S^2S_x - 3R_y + 3aS_xR = 0 \\ S_y = R_x \end{cases} \tag{3}$$

where a and b are arbitrary constant and $S(x, y, t)$, $R(x, y)$ represent a wave function. Using the traveling wave transformation $S(x, y, t) = S(\xi)$, $R(x, y, t) = R(\xi)$, where ($\xi = x + y + ct$) on the system (3), obtains:

$$\begin{cases} cS' - S''' - 6bSS' + \dfrac{3}{2}a^2S^2S' - 3R' + 3aS'R = 0 \\ S' = R' \end{cases} \tag{3a}$$

Integration of the second equation in the previous system gives:

$$S = R \tag{3b}$$

Substituting eq. (3b) into the first equation in the system of eq. (3a) and then integrate the obtained equation with zero constant of integration:

$$(c-3)S - \left(3b + \frac{3}{2}a\right)S^2 + \frac{a^2}{2}S^3 - S'' = 0 \tag{4}$$

Application

In this part, we apply the modified simplest equation and the B-spline methods [36-40] to (2+1)-D Konopelchenko-Dubrovsky equation.

Modified simplest equation method

According to the general solutions that suggested by the method and balance rule between S'' and S^3, we get $N = 1$ and the general solution of eq. (4):

$$S(\xi) = \sum_{i=-N}^{N} a_i f(\xi)^i = \frac{a_{-1}}{f(\xi)} + a_0 + a_1 f(\xi) \tag{5}$$

where a is arbitrary constant and $f(\xi)$ satisfies the following auxiliary equation

$$[f'(\xi) = \alpha + \lambda f(\xi) + \mu f(\xi)^2]$$

where α, λ, μ are arbitrary constants. Substituting eq. (5) and its derivatives into eq. (4). and collecting all terms of the same power of $f(\xi)^i$ where $\{i = -3, -2, -1, 0, 1, 2, 3\}$. Solving the obtained algebric equations by MAPLE or MATHEMATICA softwares:

– *Family 1*

$$a_0 = \frac{2\lambda}{a},\ a_1 = \frac{2\mu}{a},\ a_{-1} = \frac{2\alpha}{a},\ c = 3 + \lambda^2 - 4\alpha\mu,\ b = \frac{1}{2}a(-1+\lambda)$$

where μ or $\alpha \neq 0$, $a \neq 0, \lambda \in \Re - \{1\}$, and $3 + \lambda^2 \neq 4\alpha\mu$

– *Family 2*

$$a_0 = -\frac{\lambda}{a},\ a_1 = 0, a_{-1} = -\frac{2\alpha}{a},\ c = \frac{1}{2}\left(6 - \lambda^2 + 4\alpha\mu\right),\ b = -\frac{a}{2}$$

where $\alpha \neq 0$, $a \neq 0$, and $6 - \lambda^2 \neq 4\alpha\mu$

– *Family 3*

$$a_0 = \frac{\lambda}{a},\ a_1 = \frac{2\mu}{a},\ a_{-1} = 0,\ c = \frac{1}{2}\left(6 - \lambda^2 + 4\alpha\mu\right),\ b = -\frac{a}{2}$$

where $\mu \neq 0$, $a \neq 0$, and $6 - \lambda^2 \neq 4\alpha\mu$

According to the value of parameters in *Family 1*, we get the solitary wave solutions of eq. (1) in the following formulas:

Case 1

When, $\lambda = 0$, we get:

When $\alpha\mu > 0$

$$S_1 = \frac{4\sqrt{\alpha\mu}}{a}\mathrm{Csc}\left\{2\sqrt{\alpha\mu}\left[x + y + \vartheta + t\left(3 - 4\alpha\mu\right)\right]\right\} \tag{6}$$

When $\alpha\mu < 0$

$$S_2 = -\frac{4\sqrt{-\alpha\mu}}{a}\mathrm{Csch}\left(2\left\{\sqrt{-\alpha\mu}\left[x + y + t\left(3 - 4\alpha\mu\right)\right] \mp \frac{\mathrm{Log}[\vartheta]}{2}\right\}\right) \tag{7}$$

Case 2

When, $\alpha = 0$, we get:

When $\lambda > 0$

$$S_3 = -\frac{2\lambda}{a\left\{-1 + \mathrm{e}^{\lambda\left[x+y+\vartheta+t\left(3+\lambda^2\right)\right]}\mu\right\}} \tag{8}$$

When $\lambda > 0$

$$S_4 = \frac{2}{a}\left(\lambda + \frac{1}{\mathrm{e}^{\lambda\left[x+y+\vartheta+t\left(3+\lambda^2\right)\right]} + \dfrac{1}{\mu}} - \mu\right) \tag{9}$$

Case 3

When $\lambda \neq 0$, $\alpha \neq 0$, $\mu \neq 0$

When $\alpha\mu > \lambda^2$ and $\mu > 0$

$$S_5=\frac{\left(\lambda^2-4\alpha\mu\right)\mathrm{Sec}\left\{\frac{1}{2}\sqrt{-\lambda^2+4\alpha\mu}\left[x+y+\vartheta+t\left(3+\lambda^2-4\alpha\mu\right)\right]\right\}^2}{a\left(\lambda-\sqrt{-\lambda^2+4\alpha\mu}\mathrm{Tan}\left\{\frac{1}{2}\sqrt{-\lambda^2+4\alpha\mu}\left[x+y+\vartheta+t\left(3+\lambda^2-4\alpha\mu\right)\right]\right\}\right)} \tag{10}$$

$$S_6=\frac{\left(\lambda^2-4\alpha\mu\right)\mathrm{Csc}\left\{\frac{1}{2}\sqrt{-\lambda^2+4\alpha\mu}\left[x+y+\vartheta+t\left(3+\lambda^2-4\alpha\mu\right)\right]\right\}^2}{a\left(\lambda-\sqrt{-\lambda^2+4\alpha\mu}\mathrm{Cot}\left\{\frac{1}{2}\sqrt{-\lambda^2+4\alpha\mu}\left[x+y+\vartheta+t\left(3+\lambda^2-4\alpha\mu\right)\right]\right\}\right)} \tag{11}$$

When $4\alpha\mu>\lambda^2$ and $\mu<0$

$$S_7=\frac{1}{a}\left(3\lambda+\sqrt{-\lambda^2+4\alpha\mu}\mathrm{Tan}\left\{\frac{1}{2}\sqrt{-\lambda^2+4\alpha\mu}\left[x+y+\vartheta+t\left(3+\lambda^2-4\alpha\mu\right)\right]\right\}+\right.$$
$$\left.+\frac{4\alpha\mu}{\lambda+\sqrt{-\lambda^2+4\alpha\mu}\mathrm{Tan}\left\{\frac{1}{2}\sqrt{-\lambda^2+4\alpha\mu}\left[x+y+\vartheta+t\left(3+\lambda^2-4\alpha\mu\right)\right]\right\}}\right) \tag{12}$$

$$S_8=\frac{1}{a}\left(3\lambda+\sqrt{-\lambda^2+4\alpha\mu}\mathrm{Cot}\left\{\frac{1}{2}\sqrt{-\lambda^2+4\alpha\mu}\left[x+y+\vartheta+t\left(3+\lambda^2-4\alpha\mu\right)\right]\right\}+\right.$$
$$\left.+\frac{4\alpha\mu}{\lambda+\sqrt{-\lambda^2+4\alpha\mu}\mathrm{Cot}\left\{\frac{1}{2}\sqrt{-\lambda^2+4\alpha\mu}\left[x+y+\vartheta+t\left(3+\lambda^2-4\alpha\mu\right)\right]\right\}}\right) \tag{13}$$

According to the value of parameters in *Family 2*, we get the solitary wave solutions of eq. (1) in the following formulas:

Case 1

When, $\lambda=0$, we get:

When $\alpha\mu>0$

$$S_9=-\frac{2\sqrt{\alpha\mu}}{a}\mathrm{Cot}\left\{\sqrt{\alpha\mu}\left[x+y+\vartheta+t\left(3+2\alpha\mu\right)\right]\right\} \tag{14}$$

$$S_{10}=-\frac{2\sqrt{\alpha\mu}}{a}\mathrm{Tan}\left\{\sqrt{\alpha\mu}\left[x+y+\vartheta+t\left(3+2\alpha\mu\right)\right]\right\} \tag{15}$$

When $\alpha\mu<0$

$$S_{11}=\frac{2\sqrt{-\alpha\mu}}{a}\mathrm{Coth}\left\{\sqrt{-\alpha\mu}\left[x+y+t\left(3+2\alpha\mu\right)\right]\mp\frac{\mathrm{Log}\left[\vartheta\right]}{2}\right\} \tag{16}$$

$$S_{12}=\frac{2\sqrt{-\alpha\mu}}{a}\mathrm{Tanh}\left\{\sqrt{-\alpha\mu}\left[x+y+t\left(3+2\alpha\mu\right)\right]\mp\frac{\mathrm{Log}\left[\vartheta\right]}{2}\right\} \tag{17}$$

Case 2

When $\lambda \neq 0$, $\alpha \neq 0$, $\mu \neq 0$

When $4\alpha\mu > \lambda^2$ and $\mu > 0$

$$S_{13} = \frac{1}{a}\left(-\lambda + \frac{4\alpha\mu}{\lambda - \sqrt{-\lambda^2 + 4\alpha\mu}\,\mathrm{Tan}\left\{\frac{1}{2}\sqrt{-\lambda^2 + 4\alpha\mu}\left[x + y + \vartheta + t\left(3 - \frac{\lambda^2}{2} + 2\alpha\mu\right)\right]\right\}}\right) \tag{18}$$

$$S_{14} = \frac{1}{a}\left(-\lambda + \frac{4\alpha\mu}{\lambda - \sqrt{-\lambda^2 + 4\alpha\mu}\,\mathrm{Cot}\left[\frac{1}{2}\sqrt{-\lambda^2 + 4\alpha\mu}\left\{x + y + \vartheta + t\left(3 - \frac{\lambda^2}{2} + 2\alpha\mu\right)\right\}\right]}\right) \tag{19}$$

When $4\alpha\mu > \lambda^2$ and $\mu < 0$

$$S_{15} = \frac{-1}{a}\left(\lambda + \frac{4\alpha\mu}{\lambda + \sqrt{-\lambda^2 + 4\alpha\mu}\,\mathrm{Tan}\left\{\frac{1}{2}\sqrt{-\lambda^2 + 4\alpha\mu}\left[x + y + \vartheta + t\left(3 - \frac{\lambda^2}{2} + 2\alpha\mu\right)\right]\right\}}\right) \tag{20}$$

$$S_{16} = \frac{-1}{a}\left[\lambda + \frac{4\alpha\mu}{\lambda + \sqrt{-\lambda^2 + 4\alpha\mu}\,\mathrm{Cot}\left\{\frac{1}{2}\sqrt{-\lambda^2 + 4\alpha\mu}\left[x + y + \vartheta + t\left(3 - \frac{\lambda^2}{2} + 2\alpha\mu\right)\right]\right\}}\right] \tag{21}$$

According to the value of parameters in *Family 3*, we get the solitary wave solutions of eq. (1) in the following formulas:

Case 1

When, $\lambda = 0$, we get:

When $\alpha\mu > 0$

$$S_{17} = \frac{2\sqrt{\alpha\mu}}{a}\mathrm{Tan}\left\{\sqrt{\alpha\mu}\left[x + y + \vartheta + t\left(3 + 2\alpha\mu\right)\right]\right\} \tag{22}$$

$$S_{18} = \frac{2\sqrt{\alpha\mu}}{a}\mathrm{Cot}\left\{\sqrt{\alpha\mu}\left[x + y + \vartheta + t\left(3 + 2\alpha\mu\right)\right]\right\} \tag{23}$$

When $\alpha\mu < 0$

$$S_{19} = \frac{2\sqrt{-\alpha\mu}}{a}\mathrm{Tanh}\left\{\sqrt{-\alpha\mu}\left[x + y + t\left(3 + 2\alpha\mu\right)\right] \mp \frac{\mathrm{Log}[\vartheta]}{2}\right\} \tag{24}$$

$$S_{20} = \frac{2\sqrt{-\alpha\mu}}{a}\mathrm{Coth}\left\{\sqrt{-\alpha\mu}\left[\left(x+y+t\left(3+2\alpha\mu\right)\right)\mp\frac{\mathrm{Log}\left[\vartheta\right]}{2}\right]\right\} \tag{25}$$

Case 2

When, $\alpha = 0$, we get:

When $\lambda > 0$

$$S_{21} = \frac{\lambda}{a}\left\{-1-\frac{2}{-1+\mathrm{e}^{\frac{\lambda\left[x+y+\vartheta-\frac{1}{2}t\left(-6+\lambda^2\right)\right]}{\mu}}}\right\} \tag{26}$$

When $\lambda < 0$

$$S_{22} = \frac{1}{a}\left(\lambda+2\mu\left\{-1+\frac{1}{1+\mathrm{e}^{\frac{\lambda\left[x+y+\vartheta-\frac{1}{2}t\left(-6+\lambda^2\right)\right]}{\mu}}}\right\}\right) \tag{27}$$

Case 3

When $\lambda \neq 0$, $\alpha \neq 0$, $\mu \neq 0$

When $4\alpha\mu > \lambda^2$ and $\mu > 0$

$$S_{23} = \frac{\sqrt{-\lambda^2+4\alpha\mu}}{a}\mathrm{Tan}\left\{\frac{1}{2}\sqrt{-\lambda^2+4\alpha\mu}\left[x+y+\vartheta+t\left(3-\frac{\lambda^2}{2}+2\alpha\mu\right)\right]\right\} \tag{28}$$

$$S_{24} = \frac{\sqrt{-\lambda^2+4\alpha\mu}}{a}\mathrm{Cot}\left\{\frac{1}{2}\sqrt{-\lambda^2+4\alpha\mu}\left[x+y+\vartheta+t\left(3-\frac{\lambda^2}{2}+2\alpha\mu\right)\right]\right\} \tag{29}$$

When $4\alpha\mu > \lambda^2$ and $\mu < 0$

$$S_{25} = \frac{1}{a}\left(2\lambda+\sqrt{-\lambda^2+4\alpha\mu}\mathrm{Tan}\left\{\frac{1}{2}\sqrt{-\lambda^2+4\alpha\mu}\left[x+y+\vartheta+t\left(3-\frac{\lambda^2}{2}+2\alpha\mu\right)\right]\right\}\right) \tag{30}$$

$$S_{26} = \frac{1}{a}\left(2\lambda+\sqrt{-\lambda^2+4\alpha\mu}\mathrm{Cot}\left\{\frac{1}{2}\sqrt{-\lambda^2+4\alpha\mu}\left[x+y+\vartheta+t\left(3-\frac{\lambda^2}{2}+2\alpha\mu\right)\right]\right\}\right) \tag{31}$$

Cubic B-spline scheme

In this section, we apply the cubic B-spline numerical scheme to (2+1)-D Konopelchenko-Dubrovsky equation study the numerical solution of this model. The numerical solution that obtained by this scheme can be written as a linear combination of cubic B-splines basis functions. Consider the following grid $a = \xi_0 < \xi_1 < ... < \xi_{n-1} < \xi_n = b$ as the uniform partition of solution $\xi_{i+1} - \xi_i = (b - a)/n = h$ where $i = 0,...n - 1$. The numerical solution of eq. (4) has the following general form:

$$S\left(\xi\right) = \sum_{i=-1}^{n+1} c_i\, B_i\left(\xi\right) \tag{32}$$

where c_i, $(i = -1,...,n + 1)$ are arbitrary constants and $B_i(\xi)$, $(i = -1,...,n + 1)$ are cubic B-spline function which satisfy the following condition:

$$B_i(\xi) = \frac{1}{h^3}\begin{cases} (\xi - \xi_{i-2})^3, & \xi \in [\xi_{i-2}, \xi_{i-1}] \\ (\xi - \xi_{i-2})^3 - 4(\xi - \xi_{i-1})^3, & \xi \in [\xi_{i-1}, \xi_i] \\ (\xi_{i+2} - \xi)^3 - 4(\xi_{i+1} - \xi)^3, & \xi \in [\xi_i, \xi_{i+1}] \\ (\xi_{i+2} - \xi)^3, & \xi \in [\xi_{i+1}, \xi_{i+2}] \\ 0, & \text{Otherwise} \end{cases} \tag{33}$$

where the coefficent of of has the following shown value in the tab. 1:

Table 1. Values of $B_i(\xi)$, $B_i'(\xi)$, and $B_i''(\xi)$

ξ	ξ_{i-2}	ξ_{i-1}	ξ_i	ξ_{i+1}	ξ_{i+2}
$B_i(\xi)$	0	1	4	1	0
$B_i'(\xi)$	0	$3/h$	0	$-3/h$	0
$B_i''(\xi)$	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

According to these values of $B_i(\xi)$, $B_i'(\xi)$, and $B_i''(\xi)$:

$$S(\xi) = c_{i-1} + 4c_i + c_{i+1} \tag{34}$$

$$S'(\xi) = \frac{3}{h}c_{i-1} - \frac{3}{h}c_{i+1} \tag{35}$$

$$S''(\xi) = \frac{6}{h^2}c_{i-1} - \frac{12}{h^2}4c_i + \frac{6}{h^2}c_{i+1} \tag{36}$$

Substituting eqs. (34)-(36) into eq. (4) with the following initial condition that obtained from eq. (37):

$$S = -\frac{2}{3}\text{Tanh}(2\xi) \tag{37}$$

where

$$\alpha = -4, \quad \mu = 1, \quad a = -6, \quad \vartheta = 1, \quad \lambda = 0, \quad c = -5, \quad b = 3$$

$$S(0) = 0$$

$$S(1) = -\frac{2\text{Tanh}[2]}{3}$$

$$S'(0) = -\frac{4}{3}$$

$$S'(1) = -\frac{4}{3}\text{Sech}[2]^2$$

We get a system of equations. Solving this system of equations we obtained results presented at tab. 2 and on figs. 1-5:

Table 2. Values of exact and approximate solutions

Value of ξ	Approximate	Exact	Absolute error
$\xi = 0$	$-1.734723475 \cdot 10^{-18}$	0	$1.734723475976 \cdot 10^{-18}$
$\xi = 0.1$	–0.0721748973954985	–0.13158354681660267	0.05940864942110417
$\xi = 0.2$	–0.13997442420104717	–0.2532993081701499	0.11332488396910273
$\xi = 0.3$	–0.2009696888883744	–0.35803304466535685	0.15706335577698244
$\xi = 0.4$	–0.2549185485599881	–0.442691801785659	0.18777263161857782
$\xi = 0.5$	–0.3033871007264205	–0.5077294373038432	0.20434233657742273
$\xi = 0.6$	–03494022279038176	–0.5557697380081035	0.2063675101042859
$\xi = 0.7$	–0.453963142954458	–05902344321348416	0.19280683075323546
$\xi = 0.8$	–0.453963142954458	–0.6144457029376476	0.16048255998318955
$\xi = 0.9$	–0.529365211731331	–0.6312040085641788	0.1018387968328478
$\xi = 1.0$	–0.6426850533838779	–0.6426850533838779	0

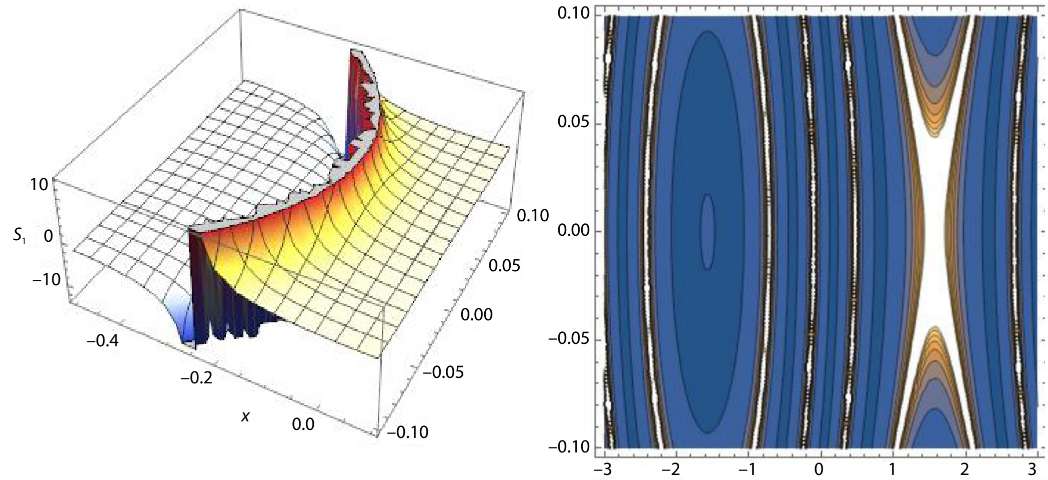

Figure 1. The 3-D and contour plots of eq. (6), when [$\alpha = 4$, $\mu = 2$, $\nu = 1$, $a = -6$, $y = 3$]

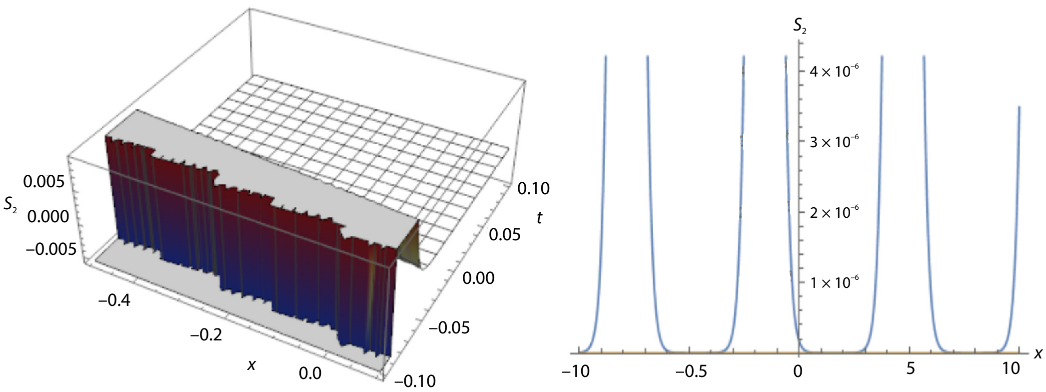

Figure 2. The 3- and 2-D and plots of eq. (7), when [$\alpha = -4$, $\mu = 2$, $\nu = 1$, $a = -6$, $y = 3$]

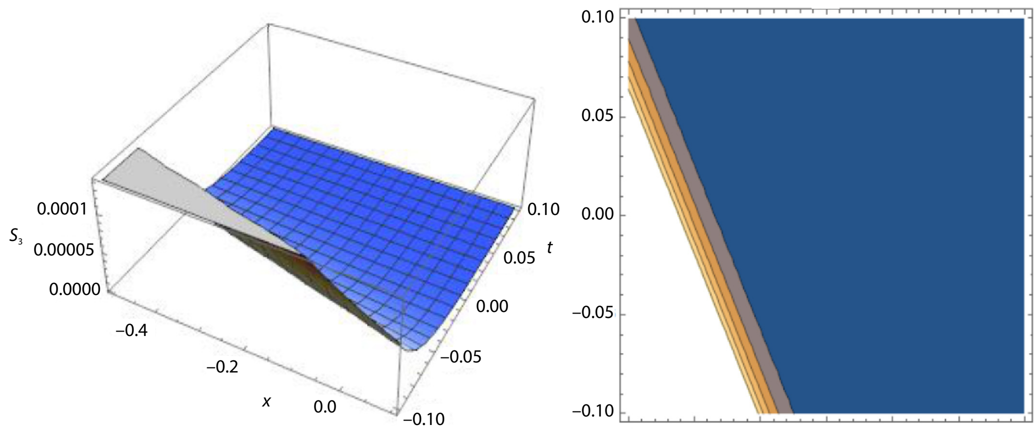

Figure 3. The 3-D and contour plots of eq. (8), when [$\alpha = 0$, $\mu = 2$, $\nu = 1$, $\lambda = 3$, $a = -6$, $y = 3$]

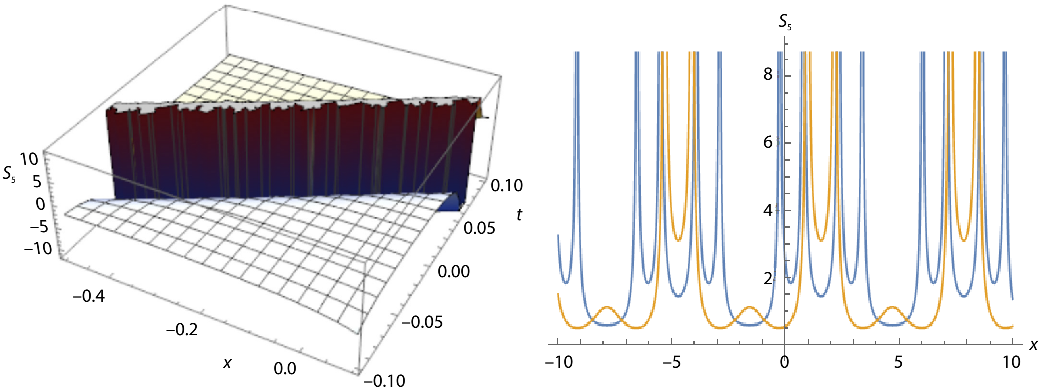

Figure 4. The 3- and 2-D plots of eq. (10), when [$\alpha = 3$, $\mu = 1$, $\nu = 1$, $\lambda = 2$, $a = -6$ $y = 3$]

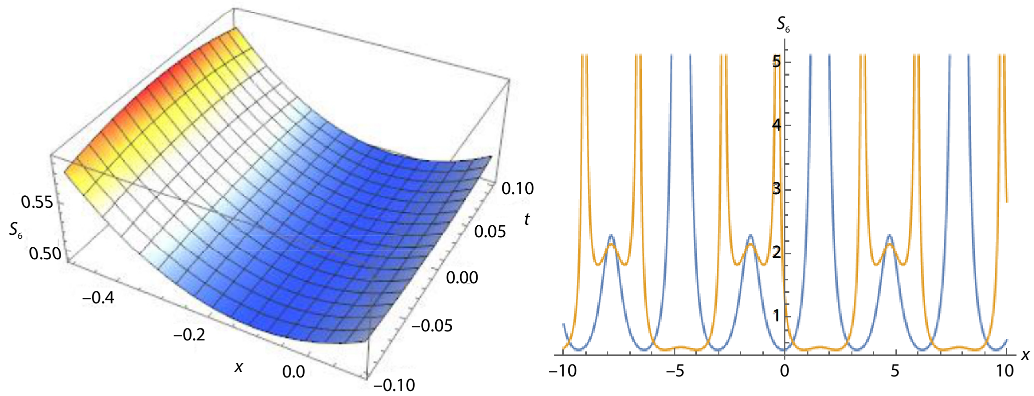

Figure 5. The 3- and 2-D plots of eq. (11), when [$\alpha = 3$, $\mu = 1$, $\nu = 1$, $\lambda = 2$, $a = -6$, $y = 3$]

Conclusion

In this paper, we used the modified simplest equation method and the cubic B-spline scheme to (2+1)-D Konopelchenko-Dubrovsky equation. We succeed in obtaining analytical and numerical solutions of the model. We obtained different forms of solutions such as shock waves, singular, solitary waves, periodic singular waves, plane waves, and others. We obtained novel and distinct, solitary wave solutions of this model. Some of our obtained solutions can be reduced to the known solutions in some instances. We also obtained the approximate solutions and discuss both solutions to show the absolute value of the error tab. 2. The results show the effectiveness of the Adomian decomposition method for interval near zero. Some solitary and approximate solutions are sketched to investigate more of the physical properties of this model figs. 1-5. The performance of both methods shows useful and powerful in studying many of non-linear partial differential equations.

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