# SYSTEM SIGNATURE AND RELIABILITY OF COHERENT SYSTEM UNDER CIRCULAR STRESS

by

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*The concept of the system signature is a useful tool not only in the analysis of engineering systems but also in comparing systems of different size. The system signature is a vector that is defined based on the failure of the components in the system and depends only on the design of the system. This article deals with the computation of signature vectors of coherent systems when systems are exposed to a circular stress. We also present how to compare coherent systems under circular stress.*

Key words: *system signature, coherent system, circular stress*

## Introduction

The system signature which is independent of the distribution of components' lifetimes when the system has independent and identically distributed component lifetimes. Let $T$ denote the lifetimeof a coherent system with the independent and identically distributed component lifetimes $T_1,...,T_n$. Then, the signature of a coherent system is defined by the probability vector $s = (s_1, s_2,...,s_n)$ with $s_i = P(T = T_{i:n})$, $i = 1, 2,...,n$ where $T_{i:n}$ is the $i^{\text{th}}$ smallest lifetime among $T_1,...,T_n$.. The concept of system signature is used not only computing the reliability of a coherent system but also comparing coherent systems of different size.

The reliability of a coherent system has been defined via the system signature by Samaniego [1]:

$$P(T > t) = \sum_{i=1}^{n} s_i P(T_{i:n} > t) = \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} \left[F(t)\right]^j \left[1 - F(t)\right]^{n-j} \tag{1}$$

where F is the common absolutely continuous distribution function of the component lifetimes $T_1,...,T_n$. For the evaluation of reliability of a coherent systems, the concept of system signature has been extensively studied in the literature [2-4].

The comparison of the same size of the coherent systems with stochastic, hazard rate and likelihood ratio sequences:

$$\text{If } s_1 \le_{st} s_2 \le_{st} \ldots \le_{st} s_k \text{, then } T_1 \le_{st} T_2 \le_{st} \ldots \le_{st} T_k$$

$$\text{If } s_1 \le_{hr} s_2 \le_{hr} \ldots \le_{hr} s_k \text{, then } T_1 \le_{hr} T_2 \le_{hr} \ldots \le_{hr} T_k$$

$$\text{If } s_1 \le_{lr} s_2 \le_{lr} \ldots \le_{lr} s_k \text{, then } T_1 \le_{lr} T_2 \le_{lr} \ldots \le_{lr} T_k$$

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where $s_1, s_2,...,s_k$ are the signatures of the $k$ mixed systems of order $n$ and $T_1, T_2,...,T_k$, are the $k$ mixed systems' respective lifetimes. Also, where *st*, *hr*, and *lr* are abbreviation of stochastic, hazard rate and likelihood ordering, respectively.

We will define the following definition for the tail probability vector to examine the system signature in the stochastically and hazard rate ordering [5].

*Definition 1.* The tail probability vector of $s = (s_1, s_2,...,s_n) \in [0.1]^n$ can be written as $V = (v_1,v_2,...,v_n)$ where $v_j = \sum_{i=j}^{n} s_i$, $j = 1, 2,...,n$. Let $V_1 = (v_{11}, v_{12},...,v_{1n})$ and $V_2 = (v_{21}, v_{22},...,v_n)$ be the tail proba bility vector of two systems. If $v_{1i} \le v_{2i}$ or $\sum_{i=j}^{n} s_{1i} \le \sum_{i=j}^{n} s_{2i}$ for all $i$, then $V_1 \le V_2$ where $i = 1, 2,...n$.

Let $s_1, s_2 \in [0.1]^n$, be the signatures of the two mixed systems of order $n$ and $v_1, v_2 \in [0.1]^n$ be the tail probability vector of $s_1$ and $s_2$, respectively. From the previous definition:

$$s_1 \le_{st} s_2 \Leftrightarrow v_1 \le v_2$$

and

$$s_1 \le_{hr} s_2 \Leftrightarrow \text{the} \frac{v_2}{v_1} \text{rate vector is not decreasing}$$

It may be easy to compare the system signature of the same size of the coherent systems with stochastically and hazard rate sequences. But systems that are compared in daily life can be different sizes. Samaniego [6] introduced how to compare coherent systems of different sizes with system signature.

*Theorem 1*. Let $s = (s_1, s_2,...,s_n)$ be the signature of a mixed system in *i.i.d* components with continuous distribution $F$. Then the mixed system with $(n + 1)$ components with *i.i.d* lifetimes $\sim F$ and corresponding to the signature vector:

$$s^* = \left( \frac{n}{n+1} s_1, \frac{1}{n+1} s_1 + \frac{n-1}{n+1} s_2, \frac{2}{n+1} s_2 + \frac{n-2}{n+1} s_3, \ldots, \frac{n}{n+1} s_n \right)$$

has the same lifetime distribution as the $n$ components system with signature $s$.

There has been increasing interest in the study of the concept of system signature in recent decades, see, *e. g.* [7-9]. They all assumed that the system has independent and identically distributed component lifetimes. But in practice, this acceptance may fail. For example, components operating under circular stress can show varying life spans. For this reason, the main purpose of this paper is to compute system signature for the case where components in the coherent system have different lifetimes.

## The system signature and system reliability of a coherent system

For a coherent system consisting of independent and identically distributed $n$ components, let $s = (s_1, s_2,..., s_n)$ be the vector called system signature. It is defined:

$$s_i = \frac{\#\text{of orderings for which the ith failure causes system failure}}{n!} \tag{2}$$

where $i = 1, 2,...,n$ and $\sum_{i=j}^{n} s_i = 1$. For the signature of a coherent system:

$$s_i = a_{n-i+1}(n) - a_{n-i}(n) \tag{3}$$

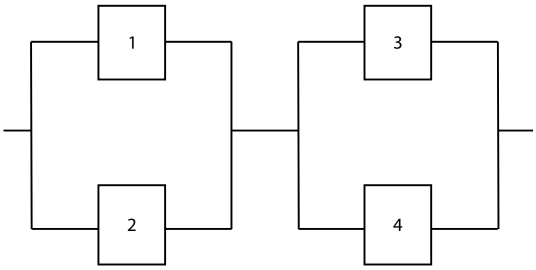

**Figure 1. Four-component serial-parallel system**

where

$$a_i(n) = \frac{r_i(n)}{\binom{n}{i}} \tag{4}$$

where $r_i(n)$ denotes the number of path sets including $i$ working components, *e. g.*, [7].

*Example 1.* Consider the four-component serial-parallel system whose structure is illustrated in fig. 1. Its structure function can be expressed:

$$\phi(x_1, x_2, x_3, x_4) = \min\left[\max(x_1, x_2), \max(x_3, x_4)\right]$$

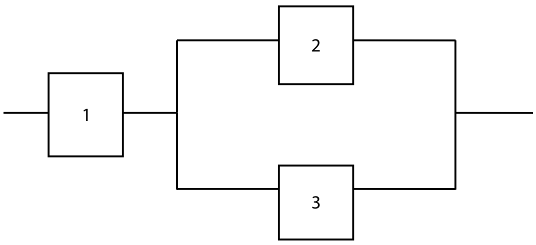

**Figure 2. Three-component serial-parallel system**

The cut sets of this structure are {1, 2}, {3, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}. Thus $r_1(4) = 0$, $r_2(4) = 2$, $r_3(4) = 4$, and $r_4(4) = 1$. From eqs. (3) and (4), the signature of the system is found to be $s_1 = (0, 1/3, 2/3, 0)$.

*Example 2*. Consider the three-component serial-parallel system whose structure is illustrated in fig. 2. Its structure function can be expressed:

$$\phi(x_1, x_2, x_3, x_4) = \min\left[x_1, \max(x_2, x_3)\right]$$

We can define the lifetime of the aforementioned system as $T = \min[T_1, \max(T_2, T_3)]$. Now with the help of the min-max operator, the order statistics which equals to $T$:

| Orderings | $T$ |
|---|---|
| $T_1 < T_2 < T_3$ | $T_{1:3}$ |
| $T_1 < T_3 < T_2$ | $T_{1:3}$ |
| $T_2 < T_1 < T_3$ | $T_{2:3}$ |
| $T_2 < T_3 < T_1$ | $T_{2:3}$ |
| $T3 < T_2 < T_1$ | $T_{2:3}$ |
| $T_3 < T_1 < T_2$ | $T_{2:3}$ |

From eq. (2) we have $P(T = T_{1:3}) = 2/6$, $P(T = T_{2:3}) = 4/6$, and $P(T = T_{3:3}) = 0$. Hence the signature of the system is found to be $s_2 = (1/3, 2/3, 0)$.

To find the reliability of the systems given in *Example 1* and *Example 2*, we can use eq. (1). Now, in both examples let us assume that the components in the system have exponential distribution with means $1/\lambda$. If eq. (1) is used for the first example:

$$P(T_1 > t) = \sum_{i=1}^{4} s_i \sum_{j=0}^{i-1} \binom{4}{j} \left(1 - \mathrm{e}^{-\lambda t}\right)^j \left(\mathrm{e}^{-\lambda t}\right)^{4-j} \tag{5}$$

where $T_1$ is the lifetime of the four-component serial-parallel system which is given in *Example 1*. Using the signature of the system $s_1 = (0.1/3, 2/3, 0)$ in eq. (5):

$$P\left(T_1 > t\right) = \mathrm{e}^{-2\lambda t}\left(\mathrm{e}^{-2\lambda t} - 4\mathrm{e}^{-\lambda t} + 4\right)$$

If eq. (1) is used for the 2nd *Example*:

$$P\left(T_2 > t\right) = \sum_{i=1}^{3} s_i \sum_{j=0}^{i-1} \binom{3}{j}\left(1-\mathrm{e}^{-\lambda t}\right)^j \left(\mathrm{e}^{-\lambda t}\right)^{3-j} \tag{6}$$

where $T_2$ is the lifetime of the three-component serial-parallel system which is given in *Example 2*. Using the signature of the system $s_2$ = (1/3, 2/3, 0) in eq. (6):

$$P\left(T_2 > t\right) = \mathrm{e}^{-2\lambda t}\left(2 - \mathrm{e}^{-\lambda t}\right)$$

For the comparison of different size of coherent systems, let us consider two coherent systems which are given in *Example 1* and *Example 2*. By using *Theorem 1*, we can rewrite system signature $s_2$:

$$\boldsymbol{s}_2^* = \left(\left(\frac{3}{4}\right)\left(\frac{1}{3}\right), \left(\frac{1}{4}\right)\left(\frac{1}{3}\right) + \left(\frac{2}{4}\right)\left(\frac{2}{3}\right), \left(\frac{2}{4}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{4}\right)(0), \left(\frac{3}{4}\right)(0)\right) = \left(\frac{1}{4}, \frac{5}{12}, \frac{1}{3}, 0\right)$$

For system signature $s_1$ = (0.1/3, 2/3, 0) and system signature $s_2^*$ = (1/4, 5/12, 1/3, 0) we have the tail probability vectors (1, 1, 2/3, 0) and (1, 3/4, 1/3, 0), respectively. As a result we can say that $s_2 \leq_{st} \mathrm{s}_1$.

## System signature under circular stress

Let us assume that $n$ components in the coherent system have exponential distribution under circular stress. Then, their cumulative functions can be written:

$$F_i\left(t\right) = 1 - \exp\left\{-\lambda_i\left[\alpha_i + \cos\left(u + \beta_i\right)\right]t\right\} \tag{7}$$

where $i = 1, 2, \ldots, n$, $t \geq 0$, $\lambda_i > 0$, $\alpha_i \geq 1$, $0 \leq u \leq \pi$, and $0 \leq \beta_i < \pi$. Let us consider a coherent system under circular stress with non-identical components. Then, for the signature of the system under circular stress, we can use eq. (2).

*Example 3.* Consider the three-component serial-parallel system under circular stress whose structure is illustrated in fig. 2. Let us assume that three components in the coherent system have exponential distribution under circular stress. For $n = 3$ in eq. (7):

$$F_1\left(t\right) = 1 - \exp\left\{-\lambda_1\left[\alpha_1 + \cos\left(u + \beta_1\right)\right]t\right\}$$

$$F_2\left(t\right) = 1 - \exp\left\{-\lambda_2\left[\alpha_2 + \cos\left(u + \beta_2\right)\right]t\right\}$$

and

$$F_3\left(t\right) = 1 - \exp\left\{-\lambda_3\left[\alpha_3 + \cos\left(u + \beta_3\right)\right]t\right\}$$

Let $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, $\alpha_1 = \alpha_2 = \alpha_3 = 1$, and $\beta_1 = 0$, $\beta_2 = \pi/4$, $\beta_3 = \pi/2$. Then, for $\lambda = 2$ and different values of $u$:

| $u$ | Orderings | $T$ |
|---|---|---|
| $0 \le u < \pi/8$ | $T_1 < T_2 < T_3$ | $T_{1:3}$ |
| $\pi/8 \le u < \pi/4$ | $T_1 < T_3 < T_2$ | $T_{1:3}$ |
| $\pi/4 \le u < 3\pi/8$ | $T_3 < T_1 < T_2$ | $T_{2:3}$ |
| $3\pi/8 \le u < \pi/2$ | $T_3 < T_2 < T_1$ | $T_{2:3}$ |
| $\pi/2 \le u < 5\pi/8$ | $T_3 < T_2 < T_1$ | $T_{2:3}$ |
| $5\pi/8 \le u < 3\pi/4$ | $T_2 < T_3 < T_1$ | $T_{2:3}$ |
| $3\pi/4 \le u < 7\pi/8$ | $T_2 < T_1 < T_3$ | $T_{2:3}$ |
| $7\pi/8 \le u < \pi$ | $T_1 < T_2 < T_3$ | $T_{2:3}$ |

From eq. (2) we have $P(T = T_{1:3}) = 3/8$, $P(T = T_{2:3}) = 5/8$, and $P(T = T_{3:3}) = 0$. Hence the signature of the system is found to be $s_3 = (3/8, 5/8, 0)$.

*Example 4.* Consider the four-component serial-parallel system under circular stress whose structure is illustrated in fig. 1. Let us assume that four components in the coherent system have exponential distribution under circular stress. For $n = 4$ in eq. (7):

$$F_1(t) = 1 - \exp\left\{-\lambda_1\left[\alpha_1 + \cos(u + \beta_1)\right]t\right\}$$

$$F_2(t) = 1 - \exp\left\{-\lambda_2\left[\alpha_2 + \cos(u + \beta_2)\right]t\right\}$$

$$F_3(t) = 1 - \exp\left\{-\lambda_3\left[\alpha_3 + \cos(u + \beta_3)\right]t\right\}$$

and

$$F_4(t) = 1 - \exp\left\{-\lambda_4\left[\alpha_4 + \cos(u + \beta_4)\right]t\right\}$$

Let $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, and $\beta_1 = 0$, $\beta_2 = \pi/6$, $\beta_3 = \pi/3$, $\beta_4 = \pi/2$. Then, for $\lambda = 2$ and different values of $u$:

| $u$ | Orderings | $T$ |
|---|---|---|
| $0 \le u < \pi/12$ | $T_1 < T_2 < T_3 < T_4$ | $T_{2:4}$ |
| $\pi/12 \le u < \pi/6$ | $T_1 < T_2 < T_4 < T_3$ | $T_{2:4}$ |
| $\pi/6 \le u < \pi/8$ | $T_1 < T_4 < T_2 < T_3$ | $T_{3:4}$ |
| $\pi/4 \le u < \pi/3$ | $T_4 < T_1 < T_3 < T_2$ | $T_{3:4}$ |
| $\pi/3 \le u < 5\pi/12$ | $T_4 < T_3 < T_1 < \mathrm{T}_2$ | $T_{2:4}$ |
| $5\pi/12 \le u < \pi/2$ | $T_4 < T_3 < T_2 < T_1$ | $T_{2:4}$ |
| $\pi/2 \le u < 7\pi/12$ | $T_4 < T_3 < T_2 < T_1$ | $T_{2:4}$ |
| $7\pi/12 \le u < 2\pi/3$ | $T_3 < T_4 < T_2 < T_1$ | $T_{2:4}$ |
| $2\pi/3 \le u < 3\pi/4$ | $T_3 < T_2 < T_4 < T_1$ | $T_{3:4}$ |
| $3\pi/4 \le u < 5\pi/6$ | $T_2 < T_3 < T_1 < T_4$ | $T_{3:4}$ |
| $5\pi/6 \le u < 11\pi/12$ | $T_2 < T_1 < T_3 < T_4$ | $T_{2:4}$ |
| $11\pi/12 \le u < \pi$ | $T_1 < T_2 < T_3 < T_4$ | $T_{2:4}$ |

From eq. (2) we have $P(T=T_{1:4})=0$, $P(T=T_{2:4})=8/12$, $P(T=T_{3:4})=4/12$, and $P(T=T_{4:4})=0$. and Hence the signature of the system is found to be $s_4 = (0, 8/12, 4/12, 0)$.

For the comparison of different size of coherent systems under circular stress, let us consider two coherent systems which are given in *Example 3* and *Example 4*. By using *Theorem 1*, we can rewrite system signature $s_3$:

$$\boldsymbol{s}_3^* = \left[\left(\frac{3}{4}\right)\left(\frac{3}{8}\right),\left(\frac{1}{4}\right)\left(\frac{3}{8}\right)+\left(\frac{2}{4}\right)\left(\frac{5}{8}\right),\left(\frac{2}{4}\right)\left(\frac{5}{8}\right)+\left(\frac{1}{4}\right)(0),\left(\frac{3}{4}\right)(0)\right]=\left(\frac{9}{32},\frac{13}{32},\frac{10}{32},0\right)$$

For system signature $s_4 = (0, 8/12, 4/12, 0)$ and system signature $s_3^* = (9/32, 13/32, 10/32, 0)$ we have the tail probability vectors (1, 1, 4/12, 0) and (1, 23/32, 10/32, 0), respectively. As a result we can say that $s_3 \leq_{st} s_4$.

## Conclusion

In the literature, many systems are well defined, see, *e. g.* [10]. However, lots of technical systems under stress are trouble. For system working under stress, the system may have circular stress due to its working structure. In this study, a system that works under circular stress is considered. The order statistics of the components that work under circular stress are replaced at certain intervals. By using this feature of the components, the system signature can be calculated. The method has been extensively studied by giving examples. Because the stress applied to the system and system signature can be examined together, the study is very important for the related literature. In addition, this study gives a different examination technique in many other related systems.

## References

[1] Samaniego F. J., On Closure the IFR Class Under Formation of Coherent Systems. *IEEE Trans. Reliab.*, *34* (1985), 1, pp. 69-72
[2] Navarro J., *et al.*, On the Application and Extension of System Signatures in Engineering Reliability, *Naval Research Logistic*, *55* (2005), 4, pp. 313-326
[3] Samaniego F. J., et al., Dynamic Signatures and Their Use in Comparing the Reliability of New and Used Systems. *Naval Research Logistic*, *56* (2009), 6, pp. 577-591
[4] Mahmoudi M., Asadi M., The Dynamic Signature of Coherent Systems. *IEEE Trans. Reliab.*, *60* (2011), pp. 817-822
[5] Kochar S., *et al.*, The Signature of a Coherent System and Its Application Comparison among Systems, *Naval Research Logistic*, *46* (1999), 5, pp. 507-523
[6] Samaniego F. J., On the Comparison of Engineered Systems of Different Sizes, *Proceedings,* 12th Annual Army Conference on Applied Statistics, Aberdeen Proving Ground, Army Research Laboratory, Aberdeen, UK, 2006
[7] Boland P. J., Signatures of Indirect Majority Systems. *Journal of Applied Probability*, *38* (2001), 2, pp. 597-603
[8] Eryılmaz S., Computing Reliability Indices of Repairable Systems via Signature, *Journal of Computational and Applied Mathematics*, *260* (2014), Apr., pp. 229-235
[9] Da G., *et al.*, On Computing Signatures of Coherent Systems, *Journal of Multivariate Analysis*, *103* (2012), 1, pp. 142-150
[10] Gurcan M., Gokdere G., Computing Reliability for Closed-Cycle Cooling System in Thermos-Electric Power Plants by Modelling to Circular Consecutive-2-out-of-n: F System, *Thermal Science, 22* (2018), Suppl. 1, pp. S177-S184

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