THE MATH MODELLING RESEARCH OF CONSTANT TEMPERATURE BATH

by

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This paper studies how the temperature in the bathroom keeps changing when people bathe in the bathtub. The shape, capacity, behavior and body posture of people in the bathroom are considered. In fluid mechanics point of view, we consider the impact of the flow of heat transfer, by the energy differential equations and boundary-layer momentum to establish a set of PDE, and use Laplace operator rewrite it. We use finite difference method, with Taylor series expansion. We use the function value of grid nodes of difference quotient instead of control equation of derivative, and discretize it, and solve the heat conduction equation.

Key words: *heat transfer, finite difference method, conservation of energy, laplace operator*

Introduction

In the real world, when we use the bathtub it is often based on water conservation as a primary condition. To keep water in the bathtub as possible to maintain the initial temperature or a desired temperature, we choose water in human skin acceptable temperature range for the central plains and some bath water temperature regulation, but only if water consumption a minimum. Taking into account the volume of the bath is often relatively large, when the need to adjust the temperature and the temperature difference between the original big, to save time, we can exclude all or part of the water through the drainage system, and then open the faucet starts to raise the temperature.

Analysis of the problem

In considering the unsteady energy change rate model, we have to introduce temperature difference (TD) to describe the temperature of bath water temperature rate of change over time.

The main idea of the model is the bath water temperature equals the rate of change of heat absorption and heat release rate of change.

Stable phase's energy conservation model is the basis of the previous model, so that differential equations can be equal to zero.

The improved model with the consideration of other elements think the impact energy equation and the boundary conditions of energy to pour PDE.

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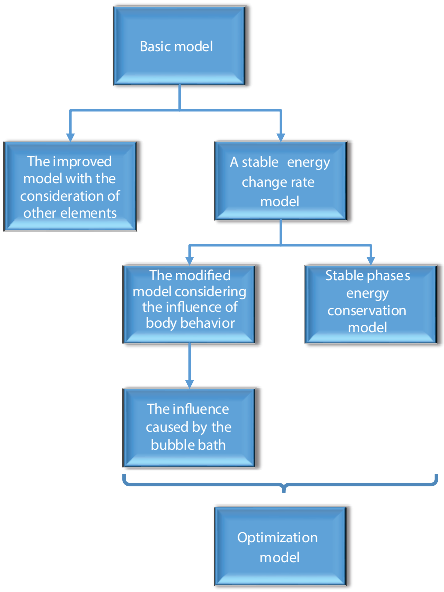

Figure 1. Flow chart

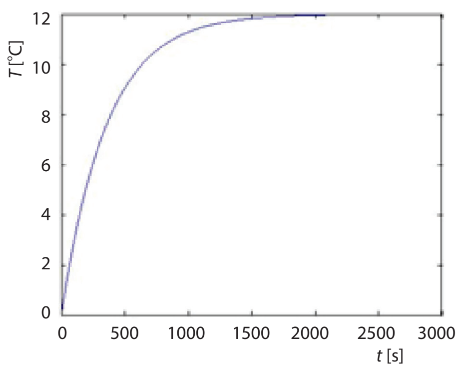

Figure 2. The temperature increase with time

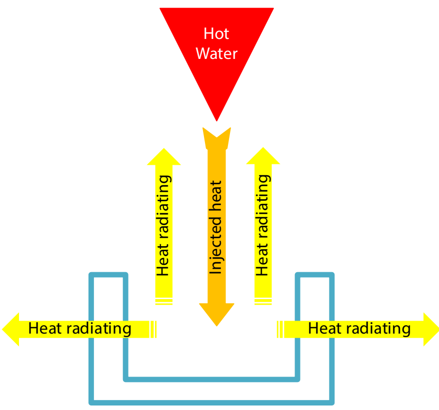

Figure 3. The flow of energy

The modified model considering the influence of body behavior consider the changes of convection cooling and evaporative heat index when people in the bathtub. According to the bathroom, the calculated evaporation heat index.

Figure 1 is this paper's flow chart.

The development of models and solution of the model

A stable energy change rate model

We look up the data. For ceramic tile whose volume is the V, A – the heat transfer area, ρ – the density, C – the specific heat for the thermal conductivity, t_0 – the initial temperature, t_∞ – the cooled by fluid whose temperature, and h – the surface heat transfer coefficient for the fixed value. We can assume that the heat conductor inside each point temperature can be used at any time. We can introduce temperature difference: $\theta = t - t_\infty$, $\theta_0 = t_0 - t_\infty$.

The formula of ceramic tile's temperature is [1]:

$$\theta = \theta_0 \exp\left(-\frac{hA}{\rho C}\tau\right) \tag{1}$$

Figure 2 shows that the temperature increase after a certain amount of time to reach a stable value, about 1500 seconds after not increase. At the beginning, it growing rapidly, then gradually slowed down the rate of change

We consider the evaporation between air and water [2]:

$$\mathrm{d}Q_1 = \beta(p_2 - p_1)\mathrm{d}F \tag{2}$$

where Q_1 is the evaporation between the air and water, F – the water surface area in contact with the air, p_2 – the water saturated layer, and p_1 – the water vapor partial pressure in wet air.

$$\mathrm{d}Q_2 = \alpha(\theta_2 - \theta_1)\mathrm{d}F \tag{3}$$

where Q_2 is the water face air convection cooling, θ_1 – the water surface temperature, and θ_1 – the air dry bulb's humidity. The rate of heat water's change in the bathtub is proportional to the heat increasing rate of the hot water. The rate of heat water's change in the bathtub is inversely proportional to the rate of atmosphere's the cooling and the rate of water cooling of air. The Q is the Water energy and fig. 3 shows the flow of energy.

We can establish the model:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = C_1 \rho v_c [T_1 - T(t)] - 2\lambda_1 A_1 \frac{T(t) - T_{\text{air}}}{b} - 2\lambda_1 A_2 \frac{T(t) - T_{\text{air}}}{b} \tag{4}$$

where λ_2 is the thermal conductivity of water, and λ_1 – the ceramic thermal conductivity, b_1 – the thick is the ness on the surface of the flat I, and b_2 – the thickness on the surface of the flat II. By eqs. (1) and (4), we get [3]:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = C_1 \rho v_c \left\{ T_1 - \left[\left(T_0 - T_\infty \right) \mathrm{e}^{-\frac{hA}{pcV}t} \right] \right\} - 2\lambda_1 (A_1 + A_2) \frac{\left[\left(T_0 - T_\infty \right) \mathrm{e}^{-\frac{hA}{pcV}t} \right] - T_{\text{air}}}{b} \tag{5}$$

After using MATLAB, we can get:

$$Q = 52 + 650T - \frac{19}{35 v_c \mathrm{e}^{280 v_c T}} \tag{6}$$

Given a determined value to water flow rate ($v_C = 10^{-5}$ m), with the aid of MATLAB we get:

$$Q = 52 + 650T - \frac{380000}{7\mathrm{e}^{7T/2500}} \tag{7}$$

Although the equation of energy and time is not a linear one, but it is closed to the linear function which is shown as fig. 4.

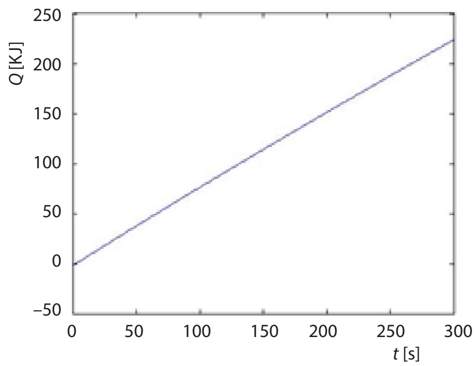

Figure 4. The time and the energy of the water in the bathtub

Stable phase's energy conservation model

Stable phase's energy conservation model is based on the model of the rate of change. While the 4th PDE value equals to zero, the insider and the outside of the bathtub system reach a new stable state:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = C_1 \rho v_c [T_1 - T(t)] - 2\lambda_1 A_1 \frac{T(t) - T_{\text{air}}}{b} - 2\lambda_1 A_2 \frac{T(t) - T_{\text{air}}}{b} = 0 \tag{8}$$

Figure 5 shows the speed needed of different temperature water to keep different stable state.

In the stable state, the speed of the input water is depended on the temperature of the input water. For example, if you use 45 °C water to keep the state rather than 55 °C water, you need more faster speed of water. Also, if you use the same temperature to keep different temperature stable state, the higher temperature state will cost more water per second.

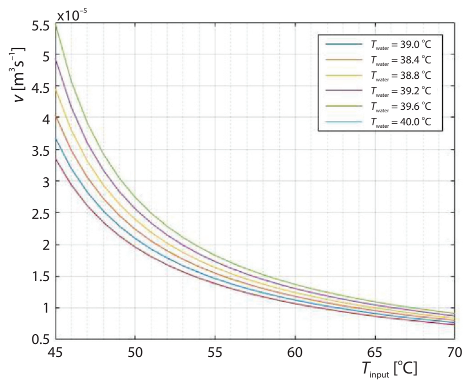

Figure 5. The speed needed of different temperature water to keep different stable state

The influence caused by the bubble bath

According to the knowledge of fluid mechanics, the fluid-flowing on the surface of the solid have two different nature of the areas.

A flow is far away from the solid surface area which is far away from the surface and have a small velocity gradient. Because there are few cohesive force in the flow, it can be handled as ideal fluid-flow. Another area is a thin layer of fluid which is near the solid surface.

The temperature field of boundary-layer is determined by velocity field. So if we want to solve the temperature field, we should solve the velocity field. According to the knowledge of fluid mechanics, we can establish a ternary steady flow equation of an incompressible fluid. Figure 6 shows the fluid through the plate of boundary-layer.

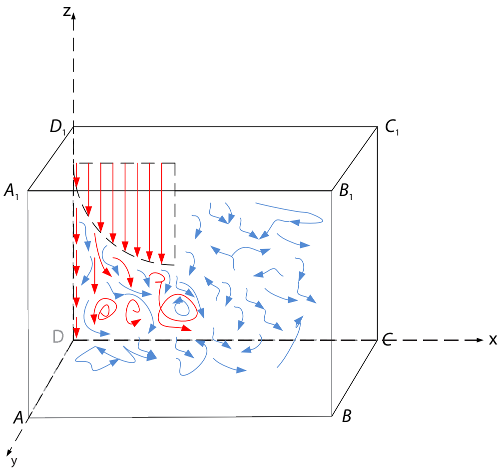

Figure 6. Fluid through the plate of boundary-layer

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{9}$$

In the eq. (9), u, v, and w, means the fluid velocity's components at the direction of x, y, and z.

Differential equations of forced convection boundary-layer with three parameters:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=w\frac{\partial^2 u}{\partial z^2} \tag{10}$$

Due to the fluid's diffusion momentum in the direction of z, there are fluid's diffusion momentum in the direction of x. The same as fluid's diffusion momentum in the direction of z, causing fluid's diffusion momentum in the direction of y.

The equation is:

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=w\frac{\partial^2 v}{\partial z^2} \tag{11}$$

Under the condition of steady-state's convective heat transfer, energy differential equation is expressed. The heat from the thermal convection taking into the control body is equal the heat leaving the heat control body:

$$u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}+w\frac{\partial t}{\partial z}=w\frac{\partial^2 t}{\partial y^2} \tag{12}$$

Comprehensive, the equation of fluid which through the tablet and transfer heat is [3]:

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0 \\ u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}+w\dfrac{\partial u}{\partial z}=w\dfrac{\partial^2 u}{\partial z^2} \\ u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}+w\dfrac{\partial v}{\partial z}=w\dfrac{\partial^2 u}{\partial z^2} \\ u\dfrac{\partial t}{\partial x}+v\dfrac{\partial t}{\partial y}+w\dfrac{\partial t}{\partial z}=c\dfrac{\partial^2 t}{\partial y^2} \\ h_{cx}=-\dfrac{\lambda\left(\dfrac{\partial t}{\partial y}\right)\Big|_{y=0}}{t_{wx}-t_\infty} \\ y=0: u=0, v=0, w=k, t=t_w \\ y=\infty: u=u_\infty, v=v_\infty, t=t_\infty \end{cases} \tag{13}$$

To solve PDE, we use the Laplace operator:

$$\Delta=\frac{\partial^2}{\partial x^2_1}+\frac{\partial^2}{\partial x^2_2}+\cdots\frac{\partial^2}{\partial x^2_n} \tag{14}$$

$$\nabla=e_1\frac{\partial}{\partial x_1}+e_2\frac{\partial}{\partial x_2}+\cdots e_n\frac{\partial}{\partial x_n} \tag{15}$$

Change the equation:

$$\begin{bmatrix} 1 & 1 & 1 \\ u & v & w \\ u & v & w \\ u & v & w \end{bmatrix}\frac{\partial}{\partial t}\begin{bmatrix} u & u & v & t \\ v & u & v & t \\ w & u & v & t \end{bmatrix}=\frac{\partial}{\partial t}\begin{bmatrix} 0 \\ w\dfrac{\partial u}{\partial z} \\ w\dfrac{\partial v}{\partial z} \\ c\dfrac{\partial t}{\partial y} \end{bmatrix} \tag{16}$$

There is a situation, become much more complicated, when we consider that the human body is in the bathtub. On the one hand, humans are creatures of constant temperature, who can be regarded as a constant heat source. On the other hand, the water temperature is significantly higher than the temperature of the skin surface. As a result, the skin will absorb the heat from the water in the bathtub. However, it is difficult for us to directly gain the data on heat transfer between the water and the skin.

Under the requirement of the bathtub system being in a steady-state, we ignore the human body's self regulating of skin and viscus, and think that the heat quantity into the water and out of the skin surface is approximately equal; otherwise humans will no longer be the warm blooded animal:

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=c\frac{\partial^2 v}{\partial z^2} \tag{17}$$

We gain the data on heat transfer between the skin and the air which is between water and skin.

The heat being produced by the metabolism of the human body, if ignore tiny heat released from breathing and mechanical movement, is mainly by convection and sweating all the way to release and maintain normal thermal balance with heat transfer from the water in the bathtub [4]:

$$Q = Q_d + Q_e = h_c(t_s - t_a) + kh_c(x_s - x_a)A_s \tag{18}$$

where K is the wet exchange coefficient, x_s – the skin surface mean mixing ratio, x_a – the mixing ratio of the surrounding environment, and A_s – the effective heat release area of the human body.

Calculating the complex form of body surface convective heat transfer coefficient, we usually put the shape simplification for ordinary solid geometry graphics, take the cylinder diameter which is equivalent to the humans convective heat transfer coefficient, and regard the data related to the cylinder as the humans convective heat transfer coefficient.

When the human body is considered as a model, we can divide it into several parts, each of which is the same size of the cylinder, as a collection of these cylinders. To this end, when the human body is regard as several representative diameter of the cylinder, we can ignore the breath or other small heat, and the sum of the parts of heat is used the following formula to calculate [4]:

$$Q = \sum Q_{di} + \sum Q_{ei} = \sum_{i-1}^{n} h_{ci}(t_{si} - t_a)A_{si} + \sum_{i-1}^{n} kh_{ci}(x_{si} - x_a)A_{si} \tag{19}$$

where x_{si} is the mixing ratio of parts of the human body.

Next, we derive the formula for the convective heat transfer coefficient. When people are in the bathtub, the water temperature was significantly higher than the skin's, and most of the skin is in the heated state. Physiological response of blood concentrated on the surface of the body, forming parts close to the skin surface and approximately equal temperature. There is almost no difference in temperature, and the flow and the heat radiation is almost non-existent, either. In addition, we assume that the skin surface of the mixed ratio of roughly equal, and draw the following relationship according to the heat balance formula between the body and the surrounding environment and the sum of the heat of each part [4]:

$$Q = \sum_{i-1}^{n} kh_{ci}(x_{si} - x_a)A_{si} \tag{20}$$

Based on the previous assumptions, we can deduce the formula of convection heat transfer coefficient above the average of the surface of the human body:

$$h_c = \sum_{i=1}^{n} h_{ci}\frac{A_{si}}{A_s} \tag{21}$$

The modified model considering the influence of body behavior

This formula expresses the heat transfer coefficient of each part and the area ratio of the parts to the average value of the average heat transfer coefficient of the human body surface. After giving the area ratio of the total surface and its representative diameter, we can get a rough conclusion that the convection heat transfer coefficient of the human body is replaced by a sin-

Table 1. The diameter of the cylinder and the area of the skin from the humans

	Diameter [cm]	Skin area [m^2]
Head	20	0.2
Trunk	30	0.6
Upper arm	10	0.1
Lower arm	10	0.2
Thigh	20	0.4
Calf	10	0.3

gle circle diameter. So when we use such a cylinder model, we should consider the heat conduction problem for the human body in the bathtub.

For each part of the human body, we first refer to the cylinder model of Parker. They divided the body into the head, neck, trunk, thigh, leg, upper arm, lower arm and finger eight parts, and replaced by their respective cylinders. This paper is from the shape up segmentation, which synthesizes a part of our head and neck and our fingers merge into the lower arm. In this way, we put the human body into six cylinder combination, reference Parker's representative diameter and skin area value, and sum up into the following tab. 1.

Forced on the convection heat transfer of the model of the table of six parts of human value and Hllpert cylinder non-dimensional formula and further consideration of the nature of the effect of Oosthiuzen dimensionless formula of, we can represent the diameter under human body model of the convective heat transfer coefficient is obtained:

$$\frac{\text{Nu}}{\text{Nu}_{\text{for}}} = 1 + 0.18\left(\frac{\text{Gr}}{\text{Re}^2}\right) - 0.011\left(\frac{\text{Gr}}{\text{Re}^2}\right)^2 \tag{22}$$

where Nu is the natural mixed with forced convection Nusselt number, Nu_{for} – the Nusselt number when forced convection, Gr – the Grash of number, and Re – the Reynolds number.

In naked, the comfortable human body skin temperature is nearly 33.5 °C and the indoor air temperature is 26 °C, so we can deduce the human cylinder model of convection heat transfer coefficient formula:

$$h_c = \sqrt[3]{270V^3 + 23}\ (0.1 \le V \le 3.0) \tag{23}$$

where V is wind velocity.

The surface temperature of the skin:

$$t_s = \sum_{i=1}^{n} \frac{h_{ci}}{h_c} \frac{A_{si}}{A_s} t_{si} \tag{24}$$

This means that the skin temperature is a weighted average of the heat transfer coefficient and skin area ratio.

We can draw a new equation by combining the heat conservation model of the bathtub:

$$Q_1 + Q_2 + Q_d + Q_e = C_{\text{water}} \rho v_w (T_{\text{input}} - T_{\text{output}}) \tag{25}$$

Among them, due to the human body's access and water and air direct contact area change, we assume that the cross-sectional area of the human and the surface of the water is S:

$$Q_2 = \alpha(xy - S)(T_{\text{water}} - T_{\text{air}}) \tag{26}$$

Equation is substituted into the eq. (25):

$$\sum_{i-1}^{n} h_{ci}(t_{si}-t_a)A_{si} + \sum_{i-1}^{n} kh_{ci}(x_{si}-x_a)A_{si} + 2\lambda_c xz\frac{T_{\text{water}}-T_{\text{air}}}{b} + $$
$$ + 2\lambda_c yz\frac{T_{\text{water}}-T_{\text{air}}}{b} + \alpha(xy-S)(T_{\text{water}}-T_{\text{air}}) = C_{\text{water}}\rho v_w(T_{\text{input}}-T_{\text{output}}) \tag{27}$$

Due to the human posture in the bath is different, next we consider different actions brought about by the impact of the heat sink. When a person moves from a half to a sitting (as in figs. 7 and 8).

In the first case, the body is lying in the bathtub, only exposed to the head, so the effective area of heat transfer with the air:

$$A_{si} = A_{\text{head}} \tag{28}$$

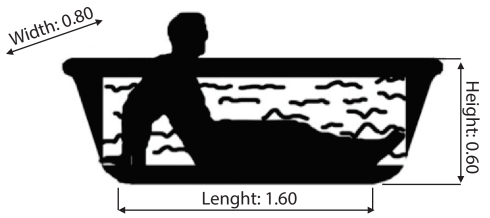

Figure 7. Lying in the bathtub

In the second case: the human body sitting in the bathtub, exposing the head, hands and a part of the trunk, the effective area:

$$A_{si} = A_{\text{head}} + h_{\text{arm}} + ah_{\text{trunk}} \tag{29}$$

At this time, due to reduce volume in water, so the water level will drop, the volume changes:

$$V = V_{\text{head}} + aV_{\text{trunk}} \tag{30}$$

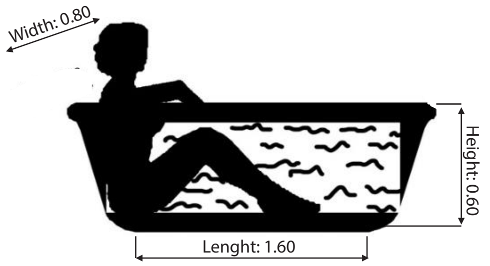

Figure 8. Setting in the bathtub

Consider the heat transfer of the bottom of the bathtub in the process of unsteady heat conduction, the temperature is gradually increased, and the heat transfer area is gradually expanded. It can be divided into three-stages:

Initial stage: in the heat transfer area, temperature changes are regard as the transmission region maintain the initial temperature of the human body. This phase is relatively short, and we can ignore the phase of this phase.

Normal stage: the period of time after the initial period to the new stability. The heat has spread to the human body, the skin temperature no longer subject to the initial temperature.

Stable stage: the heat, produced by the metabolism of the human body, if ignore the small heat released from breathing and mechanical movement, is mainly by convection and sweating all the way to release and maintain normal thermal balance with heat transfer from the water in the bathtub.

After a certain period of time, the human body inside and outside reaches a new stable state, which the water in the bath on the body's heat is equal to the body's heat on the air.

We decided to use the Biot number to choose the method between the Lumped parameter or look up [5]:

$$Bi_v = \frac{h\left(\frac{V}{A}\right)}{\lambda} < 0.1M \tag{31}$$

where M is a dimensionless number related to the geometry of the object, and also is the surface heat transfer coefficient. After we research the information, we find that $M = 1.22$. We use a long 1.75 m, wide 0.903 m, high for the 0.6 m of the rectangular bathtub to calculate that A = 3.995.Through calculating, it can be known that the temperature deviation of each point in the object is more than 5%. We cannot take the temperature of each point of the heat conduction body as the same t, and cannot be solved with the lumped parameter method, either. It should be solved by the method of check graph.

The influence caused by the bubble bath

When adding a bubble bath, the thermal insulation effect will increase a lot. We assume that the convection heat transfer coefficient of the bathtub is changed to the 1/10 of original.

Stability and sensitivity analysis

We used the finite difference method is to use a finite number of discrete grid of dots instead of a continuous range of definite solution [6].

We used difference method to obtain correct results with the results of the maximum relative error of 2.6034%.

We can see from the tab. 2, the error of the numerical solution with the exact solution is very small, barely visible in the figure above the numerical solution with the exact solution of differences.

Table 2. Numerical solution and exact solutions

Numerical solution	Exact solutions
0	0
4.40E-02	4.29E-02
8.38E-02	8.17E-02
1.15E-01	1.12E-01
1.36E-01	1.32E-01
1.43E-01	1.39E-01
1.36E-01	1.32E-01
1.15E-01	1.12E-01
8.38E-02	8.17E-02
4.40E-02	4.29E-02
0	1.70E-17

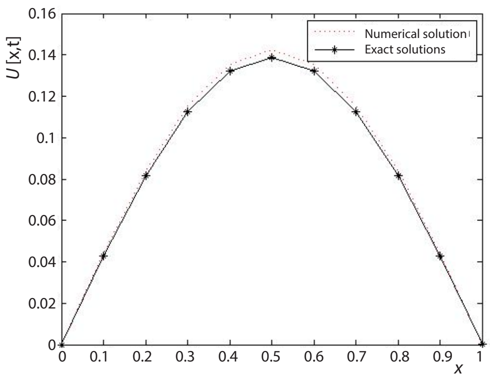

Figure 9. Numerical solution and exact solutions

Conclusions

Bathtub do not have secondary heating function, so about the bath temperature control, we can only use the open faucet, by hot water coming down the way to maintain the original temperature of the bath, or improve a certain temperature to the desired value. Figure 9 is a comparision of the two solutions drawn from the data in tab. 2.

When the body does not consider the bathtub, bathtub insulation effect is substantially decreased 5 °C per hour, but because of the volume of the bath often large (approximately 800 L), the flow rate of the faucet has a certain limit, so to maintain the original temperature comparison difficult.

When people enter the bath, because people have a larger skin area, it will take a considerable part of the heat, skin contact with the air will also accelerate the speed of the cooling water in the bathtub, except that human skin can accept the temperature range is limited (generally no higher than 70 °C), it cannot be poured into the water to maintain the water temperature is too high. For example, suppose that in the tub 35 °C, the one who wants to bathe 45 °C as soon as possible, if you want in the human acceptable range is increased by 10 °C, and in the normal flow rate, generally better long time. So take full advantage of coming down to maintain the temperature of hot water is required to spend a lot of time.

In complete with hot tub bath heated this case, human postures also a lot of changes, such as – get up to get something, hand clean the body and so on, this series of actions will accelerate the evaporation heat of people, resulting in bathtub the temperature of the system is reduced. Because even get up or do the next, resulting in an overflow part of the water, the water level of impact, so reducing the volume of water in the bathtub. So this is also difficult to maintain the water temperature.

In order to solve this problem is difficult to heat, and water conservation as a precondition, have been filled with water for a bath, our new strategy are:

- In the case of other conditions do not change, when the temperature is expected to reach the original temperature greater than when less than 5 °C and 10 °C, we recommend that you drain the tub half the volume of water, then start adding the hot water within the acceptable range of the human body. 2.
- In the case of other conditions are not changed, and when the temperature is expected to reach the original big temperature 10 °C, we recommend that the water in the bathtub drains through the whole row left, add hot water again.

Such strategies contrast, both less time and more water-saving.

References

[1] Zhang, Y., *Heat Transfer*, Southeast University Press, Nanjing, China, 2004
[2] Zhao, Z., Enthalpy Difference for Calculating Heat Dissipation on Water Surface and Its Use, *Journal of Hydraulic Engineering*, *130* (2004), 2, pp. 34-37
[3] Bozsoki, I., *et al*., Component Level Modelling of Heat Transfer During Vapour Phase Soldering with Finite Difference ADI Approach, *International Journal of Heat and Mass Tranfer*, *128* (2019), 1, pp. 562-569
[4] Shi, C., Heat Conduction Equation Finite Difference Method to Achieve the Matlab, *Journal of Xianyang Normal Univercity*, *24* (2009), 4, pp. 27-29
[5] Zhu, Q., Hen, X., Application of PDE Toolbox and Least Square Method in Heat Conduction of Metal, *Process Automation Instrumentation*, *32* (2011), 6, pp. 26-31
[6] Zhang, C., Application of Matlab in Analysis of Steady-State and Unsteady-State Heat Conduction, *Computers and Applied Chemistry*, *23* (2006), 10, pp. 986-990

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