AXISYMMETRIC STATIONARY HEAT CONDUCTION PROBLEM FOR HALF-SPACE WITH TEMPERATURE-DEPENDENT PROPERTIES

by

Dariusz M. PERKOWSKI*, Piotr SEBESTIANIUK, and Jakub AUGUSTYNIAK
Faculty of Mechanical Engineering, Bialystok University of Technology, Bialystok, Poland

Original scientific paper
https://doi.org/10.2298/TSCI181206109P

The study examines problems of heat conduction in a half-space with a thermal conductivity coefficient that is dependent on temperature. A boundary plane is heated locally in a circle zone at a given temperature as a function of radius. A solution is obtained for any function that describes temperature in the heating zone. Two special cases are investigated in detail, namely Case 1 with given constant temperature in the circle zone and Case 2 with temperature given as a function of radius, r. The temperature of the boundary on the exterior of the heating zone is assumed as zero. The Hankel transform method is applied to obtain a solution for the formulated problem. The effect of thermal properties on temperature distributions in the considered body is investigated. The obtained results were compared with finite element method model.

Key words: *temperature, heat flux, temperature-dependent conductivity, Kirchhoff transform, Hankel transform*

Introduction

The accuracy of distribution of temperature field is extremely important in the calculation of thermal stresses or residual stresses. It is well known that a few materials change mechanical and thermal properties under temperature change. In the aforementioned cases, the application of linear heat equation is not appropriate to describe the temperature field and heat flux vector. The theory of non-linear heat equation for materials with temperature-dependent properties appears as the most adequate to describe heat conduction problems and to use the obtained results to determine thermal stresses. One of the first articles on the modelling of thermoelastic materials with temperaturedependent properties were discussed in [1, 2]. The wide list of references connected with the aforementioned problems could be found in the monograph [3]. Evidently, the determination of a precise temperature field is necessary for solid mechanics problems as evidenced in [5-12].

Several existing studies consider thermal and residual stresses analyses. Various materials in engineering construction change thermomechanical properties, and heat conduction characteristics play an extremely important role in thermal and residual stress considerations, see [13-15]. The aforementioned studies deal with the exact solution of axisymmetric stationary conduction problems by considering the effect of thermo-sensitivity of materials on stress distributions.

* Corresponding author, e-mail: d.perkowski@pb.edu.pl

The solution to non-linear problems of heat conduction is typically difficult. The main stream of techniques that are applied to the problems include perturbation methods [7, 15]. However, certain problems in engineering or science cannot be solved via perturbation methods. Subsequently, nonperturbative methods were presented in [6, 16]. The control of the convergence region and rate of approximate series in the methods leads to difficulties. An analytical method for non-linear problems, namely the homotopy analysis method, was proposed in [17]. It enables the control and adjustment of the convergence region and rate of approximate series when necessary.

Changes in heat conduction properties relative to temperature can be found in publications based on experimental studies. They are also widely used in FEM analyzes that take into account the thermosensitivity of materials, eg in the case of testing braking systems [18, 19].

Linearisation is another well-known analytical approach to solve a non-linear heat conduction problem. One of the methods corresponds to the Kirchhoff integral transformation [4, 20].

In this study, the axisymmetric stationary heat conduction problem for half-space with temperature-dependent conductivity coefficient is considered. The boundary surface is assumed as locally heated by the given temperature in a circle zone with radius, a. The boundary surface on the exterior of the heating zone is maintained at 0 K. The solution is obtained for any function that describes the temperature boundary condition on the surface of the half-space and depends only on radius, r. Two special cases of the boundary temperature are considered in detail: with given constant temperature in the circle zone, or with given temperature as a function of radius, r. At infinity, it is assumed that the temperature tends to 0 K. The formulated problem is axisymmetric and stationary, with given dependency of the thermal conductivity coefficient on temperature, the considered problem is non-linear. With respect to the linearisation, the Kirchhoff integral transform method is applied, and a well-known Hankel transform method is then used for the solution of the problem.

Formulation and solution of the problem

The problem is formulated via the cylindrical co-ordinates (r, φ, z) such that axis z is perpendicular to the boundary plane of the half-space, fig. 1. Let $T(r,z) = \theta(r,z) - T_0$ denote the difference of temperature, θ, at the point (r,z) of half-space and a reference temperature, T_0, fig. 1. The thermal conductivity coefficient of the half-space depends on the temperature and is given:

$$k(T) = k_0 f(T) \tag{1}$$

where k_0 denote the thermal conductivity coefficient at the reference temperature, T_0. The function $f(T)$ describes dependence of thermal conductivity coefficient as a function of temperature, T.

On the boundary, surface boundary heating is assumed in the form of a given temperature that depends on the radius, r, and is given:

$$T(r, z=0) = T_g(r) = \vartheta_0(r)\mathrm{H}(a-r) \tag{2}$$

where $\vartheta_0(r)$ denotes a given temperature as a function of radius, r, in the circle zone with radius, a, and $\mathrm{H}(\bullet)$ denotes the Heaviside step function.

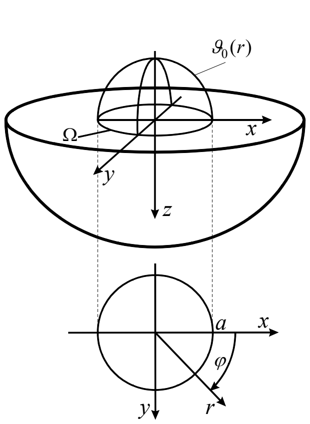


Figure 1. Scheme of the considered problem

Furthermore, it is assumed that the temperature at infinity tends to zero, and this assumption leads to the following condition:

$$T(r,z) \to 0, \text{ for } r^2 + z^2 \to \infty \tag{3}$$

The formulated problem denotes a stationary axisymmetric heat conduction problem, and the heat equation is expressed:

$$\frac{\partial}{\partial z}\left[k(T)\frac{\partial T}{\partial z}\right] + \frac{1}{r}\left\{\frac{\partial}{\partial r}\left[rk(T)\frac{\partial T}{\partial r}\right]\right\} = 0 \tag{4}$$

The components of the heat flux vector are given:

$$\mathbf{q}(r,z) = \left[q_r, 0, q_z\right] = \left[k(T)\frac{\partial T}{\partial r}, 0, -k(T)\frac{\partial T}{\partial z}\right] \tag{5}$$

Equation (4) denotes a non-linear stationary heat conduction equation. Existing studies propose the Kirchhoff approach for the linearisation of heat equation. This is expressed in the form [4, 20]:

$$\Psi = \int_0^T \frac{k(\vartheta)}{k_0} \mathrm{d}\vartheta \tag{6}$$

where $k_0 = k(0)$ denotes a thermal conductivity coefficient when temperature is equal to reference temperature T_0. Equation (4) is reduced to a linear Laplace equation:

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{\partial^2 \Psi}{\partial z^2} = 0 \tag{7}$$

Equation (7) is solved in conjunction with the relevant boundary conditions that also apply Kirchhoff's transformation. The Hankel transform method is applied to solve the formulated boundary value problem:

$$\tilde{f}(s,z) = \int_0^\infty f(r,z) r \mathrm{J}_0(sr) \mathrm{d}r \tag{8}$$

where $\mathrm{J}_0(\cdot)$ denotes the Bessel function [21].

Equation (7) in the Hankel transform space is reduced to the ODE with the constant coefficient given:

$$\frac{\mathrm{d}^2 \tilde{\Psi}(s,z)}{\mathrm{d}z^2} - s^2 \tilde{\Psi}(s,z) = 0 \tag{9}$$

The solution of eq. (9) that satisfies condition (3) is as follows:

$$\tilde{\Psi}(s,z) = A \exp(-sz) \tag{10}$$

where A denotes a constant that is determined from the heat boundary condition.

From eqs. (2) and (6) we can write the $\Psi(r, z=0) = \Psi_0(r)$:

$$\Psi_0(r) = \int_0^{T_g(r)} \frac{k(\vartheta)}{k_0} \mathrm{d}\vartheta \tag{11}$$

Equation (11) in Hankel integral transform space take the form:

$$\tilde{\Psi}_0(s) = \int_0^a r\Psi_0(r)\mathrm{J}_0(sr)\mathrm{d}r, \text{ for } r < a \tag{12}$$

Using eq. (12) in eq. (10) and applied inverse Hankel transformation the $\Psi(r,z)$ is expressed in the form:

$$\Psi(r,z) = \int_0^\infty s\exp(-sz)\mathrm{J}_0(rs)\mathrm{d}s\int_0^a \rho\Psi_0(\rho)\mathrm{J}_0(s\rho)\mathrm{d}\rho \tag{13}$$

The solution of considered problem it is given by eq. (13) and it have the character of the general solution for given *a priori* dependence of thermal conductivity coefficient as a function of temperature and given boundary temperature as a function of radius, *r*. The distribution of temperature, *T,* can be determined by Ψ, when the thermal conductivity is given *a priori*. The components of heat flux q_r, q_z are expressed by the potential Ψ:

$$q_r = k(T)\frac{\partial T}{\partial r} = k_0\frac{\partial \Psi}{\partial r} \tag{14}$$

$$q_z = -k(T)\frac{\partial T}{\partial z} = -k_0\frac{\partial \Psi}{\partial z} \tag{15}$$

Special cases

Let assume that the thermal conductivity coefficient is given:

$$k(T) = k_0(1+\beta T)^\alpha \tag{16}$$

where k_0, α, β denote given constants. This form of the thermal conductivity coefficient function is based on the experimental studies [21]. It is assumed that α has the non-negative integer values $\alpha = 0,1,2,\ldots$. Constant α describes the rate of these changes (linear growth at $\alpha = 1$ or higher order, *e.g* for $\alpha = 2$). Constant β describes the nature of the change in the heat transfer coefficient (increase or decrease in value as the temperature rises). For $\beta > 0$ material have better thermal conductivity as the temperature rises, and for $\beta < 0$ material conductivity properties deteriorate with increase in temperature. At $\beta = 0$ or $\alpha = 0$ thermal conductivity coefficient does not depend on temperature.

The thermal potential, Ψ, is defined by eq. (6) and the form of dependence of thermal conductivity coefficient in eq. (16) as a function of temperature is expressed:

$$\Psi(r,z) = \frac{[1+\beta T(r,z)]^{1+\alpha} - 1}{\beta(1+\alpha)} \tag{17}$$

The inverse dependence of the temperature *T*(Ψ) of thermal potential is Ψ:

$$T(r,z) = \frac{1}{\beta}\left[\sqrt[1+\alpha]{1+\beta(1+\alpha)\Psi(r,z)} - 1\right] \tag{18}$$

Subsequently, we satisfy the boundary surface condition of heating (2). Thermal potential (17) assuming that $z = 0$ is expressed:

$$\Psi_0(r) = \frac{[1+\beta\vartheta_0(r)\mathrm{H}(a-r)]^{1+\alpha} - 1}{\beta(1+\alpha)} \tag{19}$$

The boundary condition (19) in the Hankel transform space is expressed:

$$\tilde{\Psi}_0(s) = \int_0^a \frac{[1+\beta\vartheta_0(r)]^{1+\alpha} - 1}{\beta(1+\alpha)} r\mathrm{J}_0(sr)\mathrm{d}r, \ \ r < a \tag{20}$$

The solution of the problem in the Kirchhoff transform space could be written in the form:

$$\Psi(r,z) = \int_0^\infty \tilde{\Psi}_0(s)\exp(-sz)s\mathrm{J}_0(rs)\mathrm{d}s =$$

$$= \int_0^a \frac{[1+\beta\vartheta_0(\xi)]^{1+\alpha} - 1}{\beta(1+\alpha)} \xi\mathrm{J}_0(s\xi)\mathrm{d}\xi \int_0^\infty \exp(-sz)s\mathrm{J}_0(rs)\mathrm{d}s \tag{21}$$

The general solution of the formulated problem for the axisymmetric heat conduction problem for a half-space with temperature-dependent properties heated locally via a given temperature distribution on the boundary surface is given:

$$T(r,z) = \frac{1}{\beta}\left(\sqrt[1+\alpha]{1 + \int_0^\infty \exp(-sz)s\mathrm{J}_0(sr)\mathrm{d}s \int_0^a \left\{[1+\beta\vartheta_0(\xi)]^{1+\alpha} - 1\right\}\xi\mathrm{J}_0(s\xi)\mathrm{d}\xi} - 1\right) \tag{22}$$

Components of the heat flux vector are calculated:

$$q_r(r,z) = k(T)\frac{\partial T(r,z)}{\partial \breve{r}} = k_0\frac{\partial \Psi}{\partial r} = k_0\int_0^\infty s^2\tilde{\Psi}_0(s)\exp(-sz)\mathrm{J}_1(sr)\mathrm{d}s \tag{23}$$

$$q_z(\breve{r},\breve{z}) = -k(T)\frac{\partial T(r,z)}{\partial \breve{z}} = -k_0\frac{\partial \Psi}{\partial z} = k_0\int_0^\infty s^2\tilde{\Psi}_0(s)\exp(-sz)\mathrm{J}_0(sr)\mathrm{d}s \tag{24}$$

The shape of the temperature boundary condition in considered cases plays an important role in contact mechanics [22, 23].In the study, two cases of the boundary condition are considered as follows:

– Case 1

$$T(r,z=0) = \theta_0\mathrm{H}(a-r) \tag{25}$$

– Case 2

$$T(r,z=0) = \theta_0 a^{-1}\sqrt{a^2 - r^2}\,\mathrm{H}(a-r) \tag{26}$$

For example, the constant temperature is assumed in the framework of contact problem with preheated punch and Case 2 is used for contact problem with rotating punch.

We denote the dimensionless co-ordinate system $(\breve{r},\breve{z})$ as related to radius a of the heating area. This is expressed:

$$\breve{r} = \frac{r}{a}, \ \ \breve{z} = \frac{z}{a} \tag{27}$$

The temperature distribution in dimensionless co-ordinates is expressed as follows in both cases:

– Case 1

$$\breve{T}(\breve{r},\breve{z})=\frac{1}{\breve{\beta}}\left(\sqrt[1+\alpha]{1+\left[(1+\breve{\beta})^{1+\alpha}-1\right]\int\limits_0^\infty \exp(-\breve{s}\breve{z})\mathrm{J}_1(\breve{s})\mathrm{J}_0(\breve{s}\breve{r})\mathrm{d}\breve{s}}-1\right) \tag{28}$$

– Case 2

$$\breve{T}(\breve{r},\breve{z})=\frac{1}{\breve{\beta}}\left(\sqrt[1+\alpha]{1+\int\limits_0^\infty\left[\sum_{k=0}^{1+\alpha}\binom{1+\alpha}{k}\breve{\beta}^k\left(\frac{2}{\breve{s}}\right)^{\frac{k}{2}}\Gamma\left(\frac{k}{2}+1\right)\mathrm{J}_{\frac{k}{2}+1}(\breve{s})-\mathrm{J}_1(\breve{s})\right]\exp(-\breve{s}\breve{z})\mathrm{J}_0(\breve{s}\breve{r})\mathrm{d}\breve{s}}-1\right) \tag{29}$$

where dimensionless temperature is related to the maximum value of temperature in the considered boundary condition $\breve{T}=T/\theta_0$. Specifically, $\Gamma(\cdot)$ denotes a gamma function [14].

Components of the heat flux vector are calculated as follows:

– Case 1

$$\frac{q_r(\breve{r},\breve{z})a}{k_0\theta_0}=-\frac{[(1+\breve{\beta})^{1+\alpha}-1]}{\breve{\beta}(1+\alpha)}\int\limits_0^\infty \breve{s}^2\exp(-\breve{s}\breve{z})\mathrm{J}_1(\breve{s})\mathrm{J}_1(\breve{s}\breve{r})\mathrm{d}\breve{s} \tag{30}$$

$$\frac{q_z(\breve{r},\breve{z})a}{k_0\theta_0}=\frac{[(1+\breve{\beta})^{1+\alpha}-1]}{\breve{\beta}(1+\alpha)}\int\limits_0^\infty \breve{s}^2\exp(-\breve{s}\breve{z})\mathrm{J}_1(\breve{s})\mathrm{J}_0(\breve{s}\breve{r})\mathrm{d}\breve{s} \tag{31}$$

– Case 2

$$\frac{q_r(\breve{r},\breve{z})a}{k_0\theta_0}=-\frac{1}{\breve{\beta}(1+\alpha)}\int\limits_0^\infty\left[\sum_{k=0}^{1+\alpha}\binom{1+\alpha}{k}\breve{\beta}^k\left(\frac{2}{\breve{s}}\right)^{\frac{k}{2}}\Gamma\left(\frac{k}{2}+1\right)\mathrm{J}_{\frac{k}{2}+1}(\breve{s})-\mathrm{J}_1(\breve{s})\right]\cdot$$
$$\cdot\breve{s}^2\exp(-\breve{s}\breve{z})\mathrm{J}_1(\breve{s}\breve{r})\mathrm{d}\breve{s} \tag{32}$$

$$\frac{q_z(\breve{r},\breve{z})a}{k_0\theta_0}=\frac{1}{\breve{\beta}(1+\alpha)}\int\limits_0^\infty\left[\sum_{k=0}^{1+\alpha}\binom{1+\alpha}{k}\breve{\beta}^k\left(\frac{2}{\breve{s}}\right)^{\frac{k}{2}}\Gamma\left(\frac{k}{2}+1\right)\mathrm{J}_{\frac{k}{2}+1}(\breve{s})-\mathrm{J}_1(\breve{s})\right]\cdot$$
$$\cdot\breve{s}^2\exp(-\breve{s}\breve{z})\mathrm{J}_0(\breve{s}\breve{r})\mathrm{d}\breve{s} \tag{33}$$

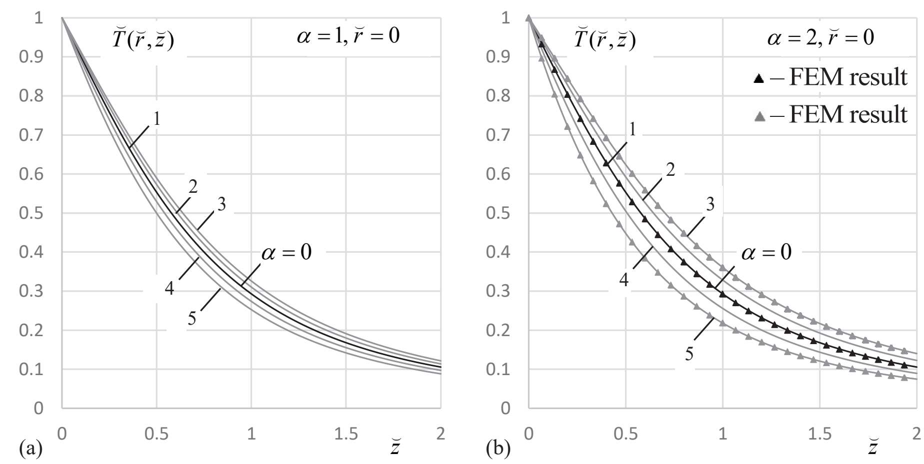
It is easy to show the transition to a medium that is not thermo-sensitive (for $\alpha = 0$) and the results reduce to a classic solutions for non-thermo-sensitive materials.

Numerical analysis and discussion of results

The analysis of the original relations in Case 1 and Case 2 indicates that the solution in the problem involving the modelling the thermo-sensitive half-space depends on two parameters, namely β and α. In addition the obtained results were compared with results from simple finite element method (FEM) model, created using COMSOL Multiphysics. Results from FEM analyses were presented in the form of temperature distributions (grey and black triangles in figs. 2(b) and 5(b).

Case 1

Figure 2 shows the dimensionless distribution of temperature as a function of coefficient β for two cases of assumed temperature, namely $\theta_0 = 300\,\mathrm{K}; 600\,\mathrm{K}$ and $\alpha = 0,1,2$. With respect to $\alpha = 0$, see fig. 2(a), we obtained results for the half-space in which properties are constant under effect of temperature, and it partially verifies the obtained solution. The solutions for $\alpha = 1,2$ were shown using grey lines. As shown in fig. 2, when the parameter β corresponds to a positive value, the thermal conductivity coefficient increases under effect of temperature. An analogical conclusion is formulated for negative value of parameter β wherein the thermal conductivity coefficient decreases with increases in the temperature. Figure 2(a) shows the distribution of temperature as a function of depth, and it is observed that the temperature tends to zero with depth increase. Figure 2(b) shows a high consistency of the results from the FEM analysis (marked as grey and black triangles) with obtained analytical solutions.



Legend:

$1 - \alpha = 0$, $2 - \beta = 0.000588\ \mathrm{K}^{-1}$, $\theta_0 = 300$ K, $3 - \beta = 0.000588\ \mathrm{K}^{-1}$, $\theta_0 = 600$ K,
$4 - \beta = -0.000588\ \mathrm{K}^{-1}$ $\theta_0 = 300$ K, $5 - \beta = 0.000588\ \mathrm{K}^{-1}$, $\theta_0 = 600$ K

Figure 2. Dimensionless temperature, $\breve{T}$, as a function of depth $\breve{z}$: (a) $\alpha = 1$ and (b) $\alpha = 2$

As shown in results presented in fig. 2, it is observed that the difference between the solution for body with constant thermal conductivity coefficient ($\alpha = 0$) and thermo-sensitive materials consider higher values for $\alpha = 2$ when compared to the solution for $\alpha = 1$ *i. e.* linear dependence of thermal conductivity coefficient. On the basis of the results presented in fig. 2, it can be stated that the influence of changes in the thermal conductivity properties is significant. This can be confirmed by experimental studies [18, 19]. These changes reach up to several dozen percent. Differences in temperature values between the thermo-sensitive material and the material with constant thermal conductivity coefficient ($\alpha = 0$) decrease with depth. Figures 3 and 4 show the distributions of heat flux component q_z and q_r, respectively. The distributions of q_z and q_r are shown at two depths for $\breve{z} = 0.05; 0.25$. Figure 3 shows the effect of parameter α on the distribution of the q_z component of the heat flux vector. Thus, larger differences are observed in the case when $\alpha = 2$, and it decreases with depth when compared to the solution for the constant thermal conductivity coefficient for $\alpha = 0$.

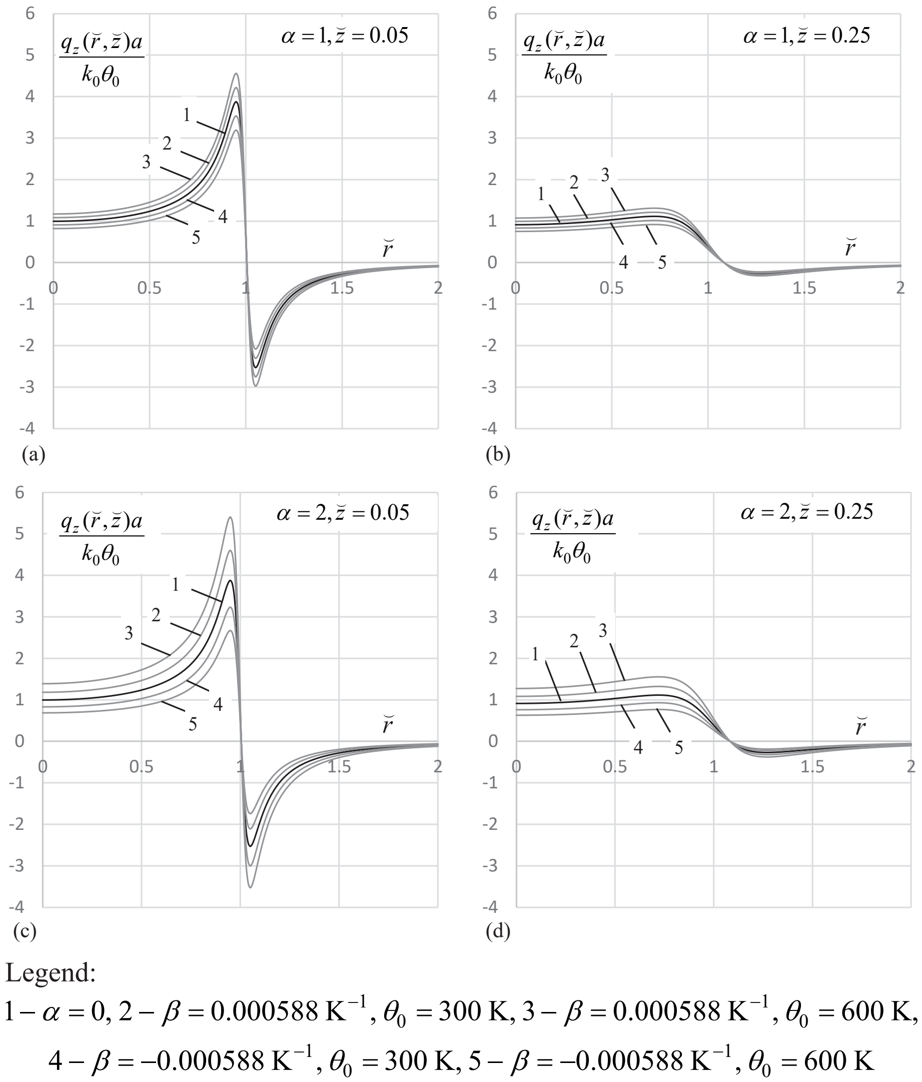
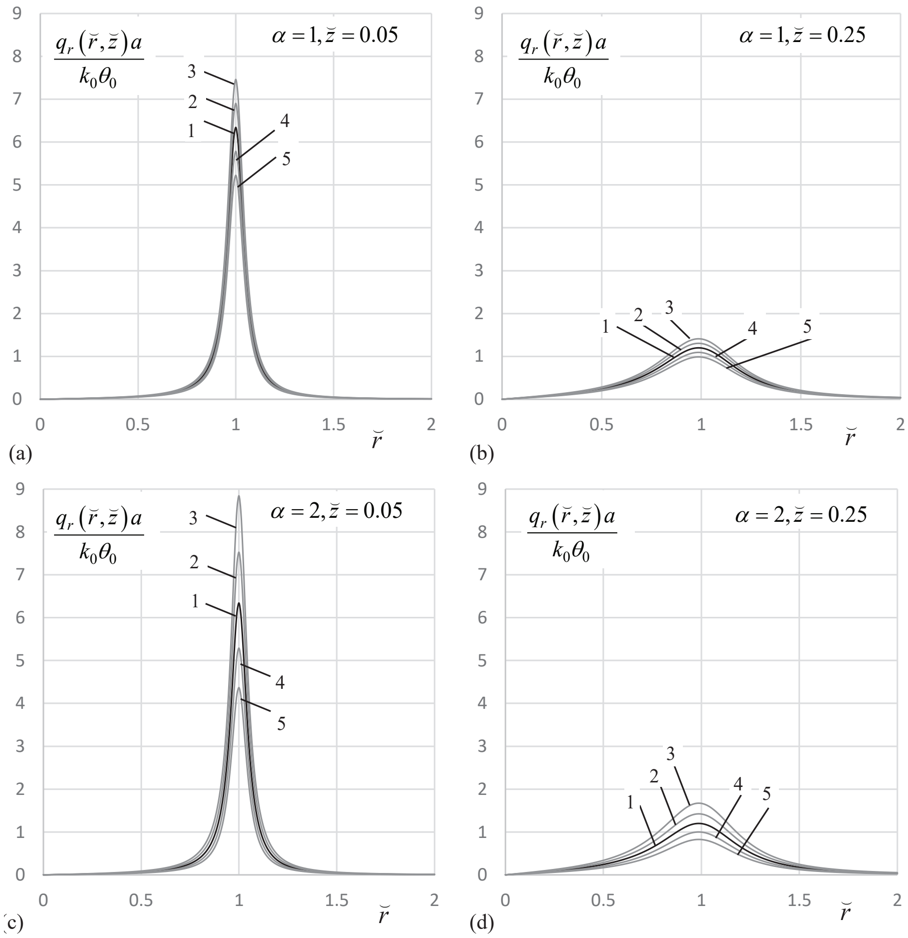


Figure 3. Dimensionless heat flux vector component, q_z, as a function of radius $\breve{r}$; (a) and (b) $\alpha = 1$, (c) and (d) $\alpha = 2$

Analogical conclusion of the effect of parameter α on the distribution of the radial component of heat flux vector is shown in fig. 4. It indicates that the larger differences are observed in the case when the $\alpha = 2$ (as denoted by grey lines), and it decreases with the depth when compared to the solution for the constant thermal conductivity coefficient for $\alpha = 0$ (as denoted by black line).

Case 2

In this case, we consider the heat boundary condition with distribution of temperature as a function of radius as given by eq. (26). The analysis of the obtained results is performed in an analogical manner as in Case 1. Initially, we begin to analyse temperature as a function of parameter α, and three cases are considered, namely $\alpha = 0;1;2$. Figure 5 shows the distribution



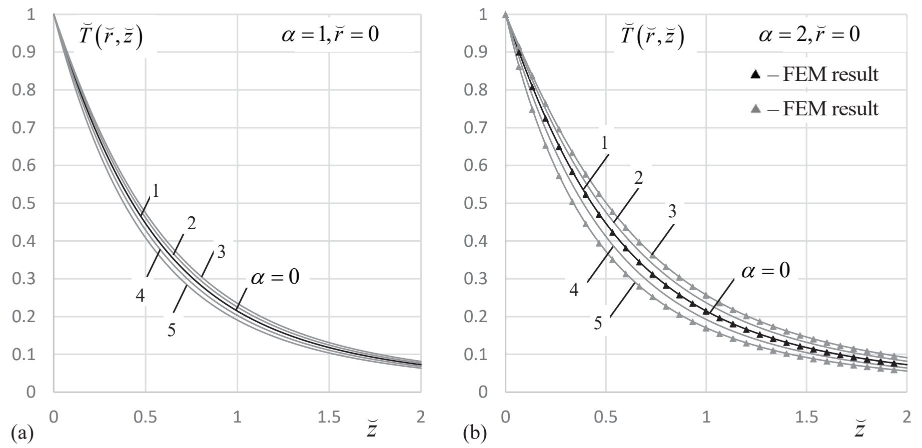
Legend:

$1-\alpha=0$, $2-\beta=0.000588\,\mathrm{K}^{-1}$, $\theta_0=300\,\mathrm{K}$, $3-\beta=0.000588\,\mathrm{K}^{-1}$, $\theta_0=600\,\mathrm{K}$,
$4-\beta=-0.000588\,\mathrm{K}^{-1}$, $\theta_0=300\,\mathrm{K}$, $5-\beta=-0.000588\,\mathrm{K}^{-1}$, $\theta_0=600\,\mathrm{K}$

Figure 4. Dimensionless heat flux vector component, q_r, as a function of radius $\breve{r}$

of temperature at the centre of heating zone for $\breve{r}=0$ as a function of depth $\breve{z}$. Comparing figs. 5(a) with 5(b) the differences in the values of the component of the heat flux vector between the materials are higher for the materials with a higher α and decrease with the distance from the heating zone. Figure 5(b) also shows a high consistency of the results from the FEM analysis (marked as grey and black triangles) with obtained analytical solutions.

In results shown in fig. 5, we observe analogical behaviour similar to that in Case 1, see fig. (2) wherein the difference between the solution for body with constant thermal conductivity coefficient and thermo-sensitive materials considers higher values for $\alpha = 2$ when compared to the solution for $\alpha = 1$ (*i. e.* linear dependence of the thermal conductivity coefficient). Figures 6 and 7 show the distributions of heat flux components q_z and q_r, respectively. Distributions of q_z and q_r are depicted on two depths for $\breve{z}=0.05; 0.25$.



Legend:

$1-\alpha=0,\ 2-\beta=0.000588\,\mathrm{K}^{-1},\ \theta_0=300\,\mathrm{K},\ 3-\beta=0.000588\,\mathrm{K}^{-1},\ \theta_0=600\,\mathrm{K},$

$4-\beta=-0.000588\,\mathrm{K}^{-1},\ \theta_0=300\,\mathrm{K},\ 5-\beta=-0.000588\,\mathrm{K}^{-1},\ \theta_0=600\,\mathrm{K}$

Figure 5. Dimensionless temperature, $\breve{T}$, as a function of depth $\breve{z}$: (a) $\alpha = 1$ and (b) $\alpha = 2$

Initially, we analyse the effect of parameter α on the distribution of the q_z component of heat flux vector. Thus, we observe larger differences in the case when $\alpha = 2$, and it decreases with depth when compared to the solution for a constant thermal conductivity coefficient for $\alpha = 0$. Furthermore, the second parameter that affects the distribution of heat flux corresponds to parameter β. The effect of parameter β on component q_z of the heat flux vector is shown in fig. 6. The highest difference for the values of q_z is visible in the centre of the heating zone where the maximum temperature value changes from 300 K to 600 K. We assume that parameter β considers positive value, and thus the temperature increases when compared with that of the body with a constant thermal conductivity coefficient.

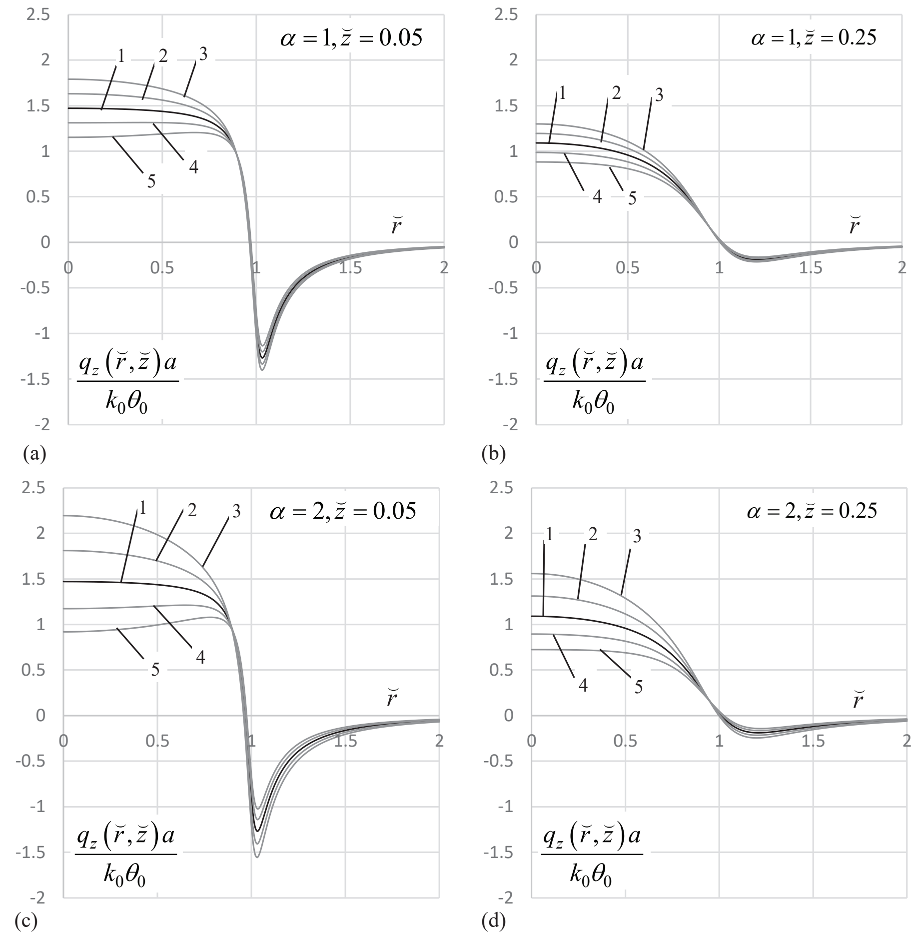
The effect of parameter α on the distribution of radial component of heat flux vector is shown in fig. 7 and indicates that larger differences are observed in the case when $\alpha = 2$ (as denoted by grey lines). It decreases with depth when compared to the solution for constant thermal conductivity coefficient for $\alpha = 0$ (as denoted by the black line).

Knowledge of the effect of temperature on the distribution of the change in the thermal conductivity coefficient $k(\breve{T})/k_0$ is interesting and useful. This is shown in fig. 8.

Figure 8 shows the effect of temperature on the thermal conductivity coefficient. The heat affected zone of temperature to the thermal conductivity coefficient increases with increases in parameter β. This knowledge is very important in the analysis of thermal stresses.

Conclusions

In the study, the problem of heat conduction in the half-space with the thermal conductivity coefficient based on temperature was investigated. Two cases of boundary heating were considered, namely Case 1 (with given constant temperature in the circle zone) and Case 2 (with given temperature as a function of radius, r). The temperature of the boundary on the exterior of the heating zone was assumed as zero. A solution in analytical form of Hankel integrals was obtained. A numerical analysis was used to investigate the effect of thermal conductivity



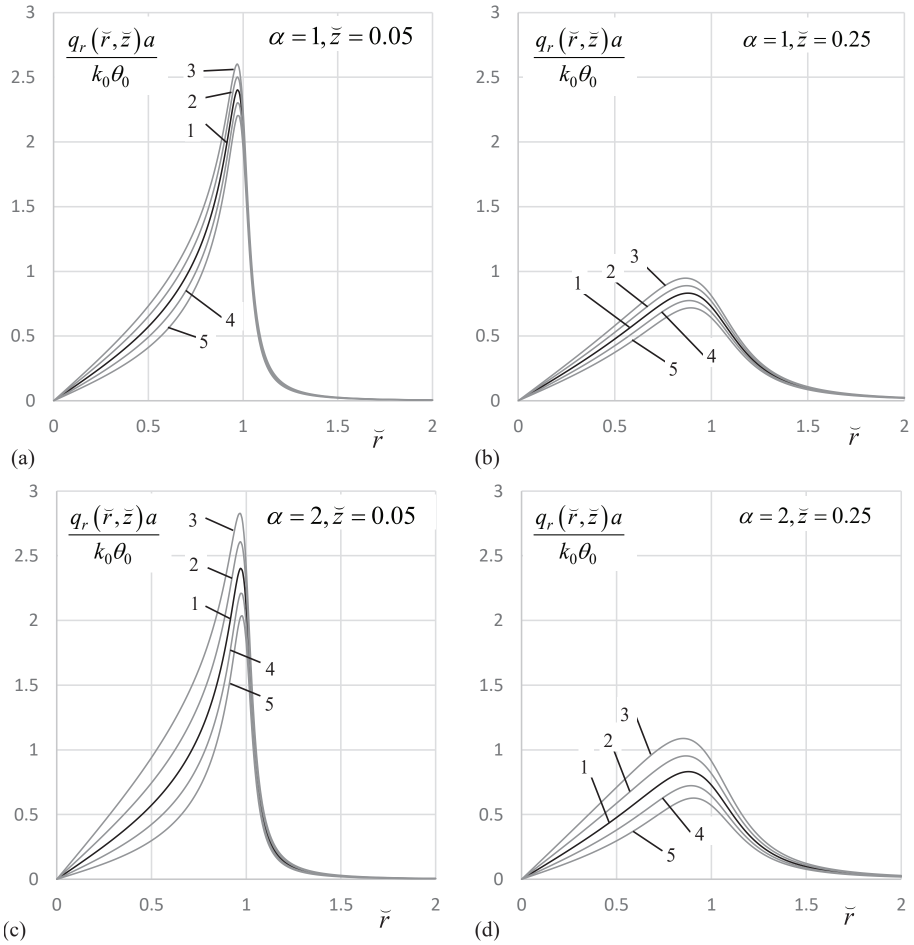
Legend:

$1-\alpha = 0$, $2-\beta = 0.000588\,\mathrm{K}^{-1}$, $\theta_0 = 300\,\mathrm{K}$, $3-\beta = 0.000588\,\mathrm{K}^{-1}$, $\theta_0 = 600\,\mathrm{K}$,

$4-\beta = -0.000588\,\mathrm{K}^{-1}$, $\theta_0 = 300\,\mathrm{K}$, $5-\beta = -0.000588\,\mathrm{K}^{-1}$, $\theta_0 = 600\,\mathrm{K}$

Figure 6. Dimensionless heat flux vector component, q_z, as a function of radius $\breve{r}$; (a) and (b) $\alpha = 1$; (c) and (d) $\alpha = 2$

coefficient of materials with temperature-dependent properties on the temperature and heat flux distribution. In both cases of the temperature boundary heating condition with increases in the parameter β, the effect of locally heating exceeded that of the analogical problem in the case of body with constant thermal conductivity coefficient ($\beta = 0$ or $\alpha = 0$). The same analogical conclusion was formulated about the parameter α wherein increases in α increase the temperature and heat flux for a positive value of parameter β. The obtained solution is in general form and it can be used with any form of thermal conductivity coefficient $k(T)$ and temperature T at the boundary. Presented approach concerns the problem of heat conduction in the half-space with the thermal conductivity coefficient based on temperature and has an exact solution character. Through the linearization of the equation, the problem has been reduced to a linear problem whose solutions can be used to validate other methods *e. g.* FEM models. It is expected that the



Legend:

$1-\alpha = 0, 2-\beta = 0.000588\,\text{K}^{-1}, \theta_0 = 300\,\text{K}, 3-\beta = 0.000588\,\text{K}^{-1}, \theta_0 = 600\,\text{K},$

$4-\beta = -0.000588\,\text{K}^{-1}, \theta_0 = 300\,\text{K}, 5-\beta = -0.000588\,\text{K}^{-1}, \theta_0 = 600\,\text{K}$

Figure 7. Dimensionless heat flux vector component, q_r, as a function of radius $\breve{r}$; (a) and (b) $\alpha = 1$, (c) and (d) $\alpha = 2$

results of the study can be used to design element construction with thermo-sensitive materials and further in the analysis of thermal stresses for local axisymmetric heating. Furthermore, the solution is used to determine the zone in which the temperature affects the thermal conductivity coefficient. Solution is in general form and it is shown in the examples.

Acknowledgment

The study was performed within the project *Selected problems of thermomechanics for materials with temperature-dependent properties*. The project was financed by the National Science Centre awarded based on the decision number DEC2013/11/D/ST8/03428.

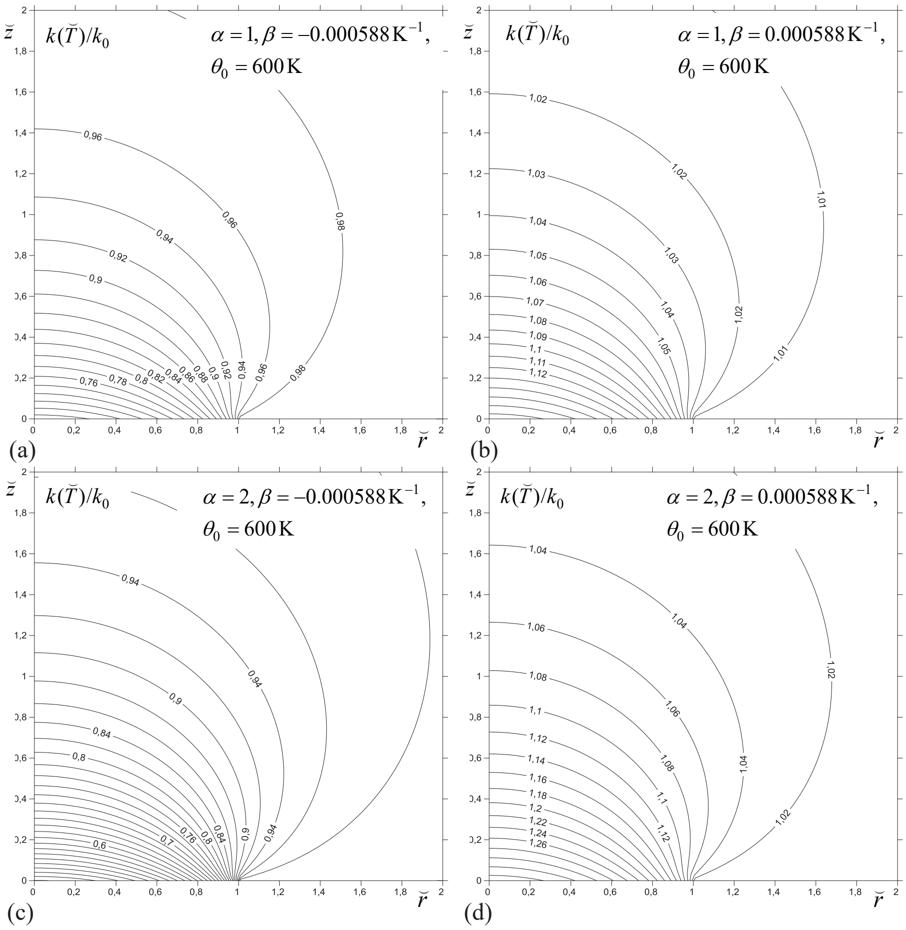


Figure 8. Dimensionless isoline of the value of the thermal conductivity coefficient $k(\breve{T})/k_0$; (a) and (b) $\alpha = 1$, (c) and (d) $\alpha = 2$

Nomenclature

a – radius of heating zone, [m]
H (•) – Hankel function
$J_n(\bullet)$ – Bessel function of the n^{th} kind
$k(T)$ – coefficient of heat conductivity of the half-space, [Wm^{-1}K^{-1}]
k_0 – coefficient of heat conductivity of the half-space for the reference temperature, [Wm^{-1}K^{-1}]
$\mathbf{q}(r, z)$ – heat flux vector, [W]
$T(r,z)$ – temperature in the half-space, [K]
T_0 – reference temperature, [K]
x, y, z – cylindrical co-ordinate system

Greek symbols

α, β – constants related to heat conductivity coefficient function
$\Gamma(\bullet)$ – Gamma function
$\Psi(r, z)$ – thermal potential function
ϑ – function of temperature in circle zone

References

[1] Nowinski, J. L., Thermoelastic Problem for an Isotropic Sphere with Temperature Dependent Properties, *ZAMP*, *10* (1959), Nov., pp. 565-575
[2] Nowinski, J. L., A Betti-Rayleigh Theorem for Elastic Bodies Exhibiting Temperature Dependent Properties, *Applied Scientific Research*, *9* (1960), Jan., 429

[3] Nowinski, J. L., *Theory of Thermoelasticity with Applications*, Sijthoff and Noordhoff Int. Publ., Alphen aan den Rijn, 1978 (Eds. Hahn, D. W., Ozisik Necat, M.,) *Heat conduction*, John Wiley & Sons, Inc, New York, USA, 2012
[4] Hata, T., Thermoelastic Problem for a Griffith Crack in a Plate with Temperature-Dependent Properties under a Linear Temperature Distribution, *Journal of Thermal Stresses, 2* (1979), 3-4, pp. 353- 366
[5] Hata, T., Thermoelastic Problem for a Griffith Crack in a Plate Whose Shear Modulus is an Exponential Function of the Temperature, *ZAMM, 61* (1981), 2, pp. 81-87
[6] He, J. H., *Non-Perturbative Method for Strongly Nonlinear Problems*, Dissertation De Verlage in Internt GmbH, Berlin, 2006
[7] Lyapunov, A. M., *General Problem on Stability of Motion* (English translation), Taylor and Francis, (Original work published 1892), London, UK, 1992
[8] Matysiak, S. J., Wave Fronts in Elastic Media with Temperature Dependent Properties, *Applied Scientific Research, 45*, (1988), June, pp. 97-106
[9] Yevtushenko, A. A., Grzes, P., Axisymmetric Finite Element Model for the Calculation of Temperature at Braking for Thermosensitive Materials of a Pad and a Disc, *Numerical Heat Transfer, Section A Applications, 62* (2012), 3, pp. 211-230
[10] Adamowicz, A. Grzes, P., Three-Dimensional Fe Model for Calculation of Temperature of a Thermosensitive Disc, *Applied Thermal Engineering, 50* (2013), 1, pp. 572-581
[11] Thuresson, D., Influence of Material Properties on Sliding Contact Braking Applications, *Wear, 257* (2004), 5-6, pp. 451-460
[12] Noda, N., Thermal Stresses in Materials with Temperature-Dependent Properties, *Appl. Mech. Rev, 44* (2009), 9, pp. 383-397
[13] Zhu, X. K., Chao, Y. J., Effects of Temperature-Dependent Material Properties on Welding Simulation, *Computers and Structures, 80* (2002), 11, pp. 967-976
[14] Cole, J. D., *Perturbation Methods in Applied Mathematics*, Waltham Massachusetts: Blaisdell Publishing Company, Waltham, Mass., USA, 1968
[15] Nayfeh, A. H., *Perturbation Methods*, John Wiley and Sons, New York, USA, 2000
[16] Mosayebidorcheh, S., *et al.*, Approximate Solution of the Nonlinear Heat Transfer Equation of a Fin with the Power-Law Temperature-Dependent Thermal Conductivity and Heat Transfer Coefficient, *Propulsion and Power Research, 3* (2014), 1, pp. 41-47
[17] Weigand, B., *Analytical Methods for Heat Transfer and Fluid Flow Problems*, Springer-Verlag, Heidelberg, Berlin, 2010
[18] Tillman, A. R., *et al*., Identification of Temperature Dependent Thermal Properties of Solid Materials, *J. Braz. Soc. Mech. Sci. Eng., 30* (2008), 4, pp. 269-278
[19] Schreiber, E., *et al*., *Elastic Cconstatnts and Ttheir Measurement*, McGrawHill, New York, USA, 1973, p 196
[20] Chichinadze, A. V., *Calculation and Investigation of External Friction During Braking* (in Russian), Nauka, Moscow, 1967
[21] Gradshteyn, I. S. Ryzhik, I. M., *Tables of Integrals, Series and Products,* Academic Press, New York, USA, 1965
[22] Kulchytsky-Zhyhailo, R., *Spatial Issues of Thermoelasticity* (in Polish), Publishing House of Bialystok University of Technology, Bialystok, Poland, 2002
[23] Barber, J. R., Ciavarella, M., Contact Mechanics, *Int. J. Solids Structures, 37* (2000), pp. 29-43

Paper submitted: December 6, 2018
Paper revised: March 12, 2019
Paper accepted: March 20, 2019
© 2020 Society of Thermal Engineers of Serbia
Published by the Vinča Institute of Nuclear Sciences, Belgrade, Serbia.
This is an open access article distributed under the CC BY-NC-ND 4.0 terms and conditions