SOME SPECIAL CURVES BELONGING TO MANNHEIM CURVES PAIR

by

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In this paper, we investigate special Smarandache curves with regard to Sabban frame for Mannheim partner curve spherical indicatrix. We create Sabban frame belonging to this curves. Smarandache curves are explained by taking position vector as Sabban vectors belonging to this curves. Then, we calculate geodesic curvatures of this Smarandache curves. Found results are expressed depending on the Mannheim curve.

Key words: *Mannheim curve pair, Smarandache curve, Sabban frame, geodesic curvature*

Introduction

In differential geometry, special curves have an important role. One of these curves Mannheim curves. Mannheim curve was firstly defined by A. Mannheim in 1878. Any curve can be a Mannheim curve if and only if $\lambda = \kappa/(\kappa^2 + \tau^2)$, λ is a non-zere constant, where curvature of curve is κ and curvature of torsion is τ. After a time, Mannheim curve was redefined by Liu and Wang [1]. According to this new definition, if principal normal vector of first curves and binormal vector of second curves are linearly dependent, the first curve is named as Mannheim curve, and the second curve is named as Mannheim partner curve. We can found many studies in literature about Mannheim curves [2, 3]. A regular curve in Minkowski space-time whose position vector is composed by Frenet frame vectors is called a Smarandache curve [4]. Special Smarandache curves have been studied by some authors [5-10]. Taskopru and Tosun [11] studied special Smarandache curves according to Sabban frame on S^2. Senyurt and Caliskan [12] investigated special Smarandache curves in terms of Sabban frame of spherical indicatrix curves and they gave some characterization of Smarandache curves. We investigated special Smarandache curves belonging to Sabban frame drawn on the surface of the sphere by Darboux vector of involute and Bertrand partner curves [13, 14]. Let $\alpha : I \to E^3$ be a unit speed curve, we define the quantities of the Frenet frame and Frenet formulae, respectively [15]:

$$T(s) = \alpha'(s), \quad N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \quad B(s) = T(s) \wedge N(s) \tag{1}$$

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$$T'(s)=\kappa(s)N(s),\quad N'(s)=-\kappa(s)T(s)+\tau(s)B(s),\quad B'(s)=-\tau(s)N(s) \tag{2}$$

Let α,α^* be Mannheim pair curve and Frenet apparatus be $\{T(s),N(s),B(s),\kappa(s),\tau(s)\}$ and $\{T^*(s),N^*(s),B^*(s),\kappa^*(s),\tau^*(s)\}$, respectively. The relation between the Frenet apparatus are [3]:

$$T^*=\cos\theta T-\sin\theta B,\quad N^*=\sin\theta T+\cos\theta B,\quad B^*=N,\quad \kappa^*=\frac{\kappa\theta'}{\lambda\tau\|W\|},\quad \tau^*=\frac{\kappa}{\lambda\tau} \tag{3}$$

Let $\gamma:I\to S^2$ be a unit speed spherical curve. We denote s as the arc-length parameter of γ. Let us denote:

$$\gamma(s)=\gamma(s),\quad t(s)=\gamma'(s),\quad d(s)=\gamma(s)\wedge t(s) \tag{4}$$

The $\{\gamma(s),t(s),d(s)\}$ frame is called the Sabban frame of γ on S^2. Then we have the following spherical Frenet formulae of γ:

$$\gamma'(s)=t(s),\quad t'(s)=-\gamma(s)+\kappa_g(s)d(s),\quad d'(s)=-\kappa_g(s)t(s) \tag{5}$$

where κ_g is called the geodesic curvature of the curve γ on S^2 which is:

$$\kappa_g(s)=\langle t'(s),d(s)\rangle \tag{6}$$

Main results

In this section, we investigate special Smarandache curves created by Sabban frame such as, $\{T^*,T_{T^*},T^*\wedge T_{T^*}\}$, $\{N^*,T_{N^*},N^*\wedge T_{N^*}\}$ and $\{B^*,T_{B^*},B^*\wedge T_{B^*}\}$. We will find some results. These results will be expressed depending on the Mannheim curve. Let as find results on this Smarandache curves. $\alpha_{T^*}(s_{T^*})=T^*(s^*)$, $\alpha_{N^*}(s_{N^*})=N^*(s^*)$, and $\alpha_{B^*}(s_{B^*})=B^*(s^*)$ be a regular spherical curves on S^2. The Sabban frames of spherical indicatrix belonging to Mannheim partner curve are:

$$T^*=T^*,\quad T_{T^*}=N^*,\quad T^*\wedge T_{T^*}=B^* \tag{7}$$

$$N^*=N^*,\quad T_{N^*}=-\cos\phi^*T^*+\sin\phi^*B^*,\quad N^*\wedge T_{N^*}=\sin\phi^*T^*+\cos\phi^*B^* \tag{8}$$

$$B^*=B^*,\quad T_{B^*}=-N^*,\quad B^*\wedge T_{B^*}=T^* \tag{9}$$

From the eq. (5), the spherical Frenet formulae of (T^*), (N^*), and (B^*) are, respectively:

$$T^{*\prime}=T_{T^*},\quad T_{T^*}'=-T^*+\frac{\tau^*}{\kappa^*}T^*\wedge T_{T^*},\quad (T^*\wedge T_{T^*})'=-\frac{\tau^*}{\kappa^*}T_{T^*} \tag{10}$$

$$N^{*\prime}=T_{N^*},\quad T_{N^*}'=-N^*+\frac{\phi^{*\prime}}{\|W^*\|}N^*\wedge T_{N^*},\quad (N^*\wedge T_{N^*})'=-\frac{\phi^{*\prime}}{\|W^*\|}T_{N^*} \tag{11}$$

$$B^{*\prime}=T_{B^*},\quad T_{B^*}'=-B^*+\frac{\kappa^*}{\tau^*}B^*\wedge T_{B^*},\quad (B^*\wedge T_{B^*})'=-\frac{\kappa^*}{\tau^*}T_{B^*} \tag{12}$$

Using the eq. (6) the geodesic curvatures of (T^*), (N^*), and (B^*) are:

$$\kappa_g^{T^*}=\frac{\tau^*}{\kappa^*},\quad \kappa_g^{N^*}=\frac{\phi^{*\prime}}{\|W^*\|}\quad\text{and}\quad \kappa_g^{B^*}=\frac{\kappa^*}{\tau^*} \tag{13}$$

Definition 1. Let (T^*) be curve of α^* and let T^* and T_{T^*} be unit vectors of (T^*). In this case β_1-Smarandache curve is defined:

$$\beta_1(s) = \frac{1}{\sqrt{2}}(T^* + T_{T^*}) \tag{14}$$

Theorem 2. The $\kappa_g^{\beta_1}$ geodesic curvature belonging to β_1-Smarandache curve of the Mannheim curve is:

$$\kappa_g^{\beta_1} = \frac{1}{\left[2+\left(\frac{\|W\|}{\theta'}\right)^2\right]^{\frac{5}{2}}}\left(\frac{\|W\|}{\theta'}\bar{\lambda}_1 + \frac{\|W\|}{\theta'}\bar{\lambda}_2 + 2\bar{\lambda}_3\right)$$

where

$$\bar{\lambda}_1 = -2-\left(\frac{\|W\|}{\theta'}\right)^2 + \left(\frac{\|W\|}{\theta'}\right)'\left(\frac{\|W\|}{\theta'}\right), \quad \bar{\lambda}_2 = -2-3\left(\frac{\|W\|}{\theta'}\right)^2 - \left(\frac{\|W\|}{\theta'}\right)^4 - \left(\frac{\|W\|}{\theta'}\right)'\left(\frac{\|W\|}{\theta'}\right)$$

$$\bar{\lambda}_3 = 2\left(\frac{\|W\|}{\theta'}\right) + \left(\frac{\|W\|}{\theta'}\right)^3 + \left(\frac{\|W\|}{\theta'}\right)'$$

Proof. The $\beta_1(s_{T^*}) = (T^* + T_{T^*})/2^{1/2}$ and from the eq. (7), we can write:

$$\beta_1(s^*) = \frac{1}{\sqrt{2}}(T^* + N^*) \tag{15}$$

Differentiating eq. (15), $T_{\beta_1}(s^*)$ is:

$$T_{\beta_1}(s^*) = \frac{1}{\sqrt{2+\left(\frac{\tau^*}{\kappa^*}\right)^2}}\left(-T^* + N^* + \frac{\tau^*}{\kappa^*}B^*\right) \tag{16}$$

Considering the eqs. (15) and (16), it is ease to see that:

$$(\beta_1 \wedge T_{\beta_1})(s^*) = \frac{1}{\sqrt{4+2\left(\frac{\tau^*}{\kappa^*}\right)^2}}\left(\frac{\tau^*}{\kappa^*}T^* - \frac{\tau^*}{\kappa^*}N^* + 2B^*\right) \tag{17}$$

Differentiating eq. (16), T_{β_1}' vector is:

$$T_{\beta_1}'(s^*) = \frac{\sqrt{2}}{\left[2+\left(\frac{\tau^*}{\kappa^*}\right)^2\right]^2}\left(\lambda_1 T^* + \lambda_2 N^* + \lambda_3 B^*\right) \tag{18}$$

where

$$\begin{cases} \lambda_1 = -2 - \left(\dfrac{\tau^*}{\kappa^*}\right)^2 + \left(\dfrac{\tau^*}{\kappa^*}\right)'\left(\dfrac{\tau^*}{\kappa^*}\right) \\ \lambda_2 = -2 - 3\left(\dfrac{\tau^*}{\kappa^*}\right)^2 - \left(\dfrac{\tau^*}{\kappa^*}\right)^4 - \left(\dfrac{\tau^*}{\kappa^*}\right)'\left(\dfrac{\tau^*}{\kappa^*}\right) \\ \lambda_3 = 2\left(\dfrac{\tau^*}{\kappa^*}\right) + \left(\dfrac{\tau^*}{\kappa^*}\right)^3 + \left(\dfrac{\tau^*}{\kappa^*}\right)' \end{cases} \tag{19}$$

From the eqs. (6), (17), and (18), $\kappa_g^{\beta_1}$ geodesic curvature of $\beta_1(s^*)$ is:

$$\kappa_g^{\beta_1} = \frac{1}{\left[2 + \left(\dfrac{\tau^*}{\kappa^*}\right)^2\right]^{\frac{5}{2}}}\left(\frac{\tau^*}{\kappa^*}\lambda_1 + \frac{\tau^*}{\kappa^*}\lambda_2 + 2\lambda_3\right) \tag{20}$$

Substituting the eq. (3) into eqs. (15)-(18), Sabban apparatus of the β_1-Smarandache curve for Mannheim curve are found:

$$\beta_1(s) = \frac{1}{\sqrt{2}}\left[(\sin\theta + \cos\theta)T + (\cos\theta - \sin\theta)B\right]$$

$$T_{\beta_1}(s) = \frac{\theta'(\sin\theta - \cos\theta)}{\sqrt{\|W\|^2 + 2\theta'^2}}T + \frac{\|W\|}{\sqrt{\|W\|^2 + 2\theta'^2}}N + \frac{\theta'(\cos\theta + \sin\theta)}{\sqrt{\|W\|^2 + 2\theta'^2}}B$$

$$(\beta_1 \wedge T_{\beta_1})(s) = \frac{\|W\|(\cos\theta + \sin\theta)}{\sqrt{2\|W\|^2 + 4\theta'^2}}T + \frac{2\theta'}{\sqrt{2\|W\|^2 + 4\theta'^2}}N + \frac{\|W\|(\cos\theta - \sin\theta)}{\sqrt{2\|W\|^2 + 4\theta'^2}}B$$

$$T_{\beta_1}'(s) = \frac{(\theta')^4\sqrt{2}(\overline{\lambda}_1\cos\theta + \overline{\lambda}_2\sin\theta)}{(\|W\|^2 + 2\theta'^2)^2}T + \frac{(\theta')^4\sqrt{2}\overline{\lambda}_3}{(\|W\|^2 + 2\theta'^2)^2}N + \frac{(\theta')^4\sqrt{2}(\overline{\lambda}_2\cos\theta - \overline{\lambda}_1\sin\theta)}{(\|W\|^2 + 2\theta'^2)^2}B$$

and $\kappa_g^{\beta_1}$ geodesic curvature is:

$$\kappa_g^{\beta_1} = \frac{1}{\left[2 + \left(\dfrac{\|W\|}{\theta'}\right)^2\right]^{\frac{5}{2}}}\left(\frac{\|W\|}{\theta'}\overline{\lambda}_1 + \frac{\|W\|}{\theta'}\overline{\lambda}_2 + 2\overline{\lambda}_3\right)$$

Definition 3. Let (T^*) be curve of α^* and let T_{T^*} and $T^* \wedge T_{T^*}$ be unit vectors of (T^*). In this case β_2-Smarandache curve is defined:

$$\beta_2(s) = \frac{1}{\sqrt{2}}\left(T_{T^*} + T^* \wedge T_{T^*}\right) \tag{21}$$

Theorem 4. The $\kappa_g^{\beta_2}$ geodesic curvature belonging to β_2-Smarandache curve of the Mannheim curve is:

$$\kappa_g^{\beta_2} = \frac{1}{\left[1+2\left(\frac{\|W\|}{\theta'}\right)^2\right]^{\frac{5}{2}}}\left(2\frac{\|W\|}{\theta'}\overline{\varepsilon}_1 - \overline{\varepsilon}_2 + \overline{\varepsilon}_3\right)$$

where

$$\overline{\varepsilon}_1 = \left(\frac{\|W\|}{\theta'}\right) + 2\left(\frac{\|W\|}{\theta'}\right)^3 + 2\left(\frac{\|W\|}{\theta'}\right)'\left(\frac{\|W\|}{\theta'}\right), \quad \overline{\varepsilon}_2 = -1 - 3\left(\frac{\|W\|}{\theta'}\right)^2 - 2\left(\frac{\|W\|}{\theta'}\right)^4 - \left(\frac{\|W\|}{\theta'}\right)'$$

$$\overline{\varepsilon}_3 = -\left(\frac{\|W\|}{\theta'}\right)^2 - 2\left(\frac{\|W\|}{\theta'}\right)^4 + \left(\frac{\|W\|}{\theta'}\right)'$$

Proof. From the eqs. (3), (7), and (21), we have:

$$\beta_2(s) = \frac{1}{\sqrt{2}}(\sin\theta T + N + \cos\theta B)$$

Herein, if Sabban apparatus are calculated, we have:

$$T_{\beta_2}(s) = \frac{-\theta'\cos\theta - \|W\|\sin\theta}{\sqrt{2\|W\|^2 + \theta'^2}}T + \frac{\|W\|}{\sqrt{2\|W\|^2 + \theta'^2}}N + \frac{\theta'\sin\theta - \|W\|\cos\theta}{\sqrt{2\|W\|^2 + \theta'^2}}B$$

$$T_{\beta_2}'(s) = \frac{(\theta')^4\sqrt{2}(\overline{\varepsilon}_2\sin\theta + \overline{\varepsilon}_1\cos\theta)}{2(\|W\|^2 + \theta'^2)^2}T + \frac{(\theta')^4\overline{\varepsilon}_3\sqrt{2}}{(2\|W\|^2 + \theta'^2)^2}N + \frac{(\theta')^4\sqrt{2}(\overline{\varepsilon}_2\cos\theta - \overline{\varepsilon}_1\sin\theta)}{(2\|W\|^2 + \theta'^2)^2}B$$

$$(\beta_2 \wedge T_{\beta_2})(s) = \frac{2\|W\|\cos\theta - \theta'\sin\theta}{\sqrt{4\|W\|^2 + 2\theta'^2}}T + \frac{\theta'}{\sqrt{4\|W\|^2 + 2\theta'^2}}N - \frac{2\|W\|\sin\theta + \theta'\cos\theta}{\sqrt{4\|W\|^2 + 2\theta'^2}}B$$

$\kappa_g^{\beta_2}$ geodesic curvature is:

$$\kappa_g^{\beta_2} = \frac{1}{\left[1+2\left(\frac{\|W\|}{\theta'}\right)^2\right]^{\frac{5}{2}}}\left(2\frac{\|W\|}{\theta'}\overline{\varepsilon}_1 - \overline{\varepsilon}_2 + \overline{\varepsilon}_3\right)$$

Definition 5. Let (T^*) be curve of α^* and let T^*, T_{T^*}, and $T^* \wedge T_{T^*}$ be unit vectors of (T^*). In this case β_3-Smarandache curve is defined:

$$\beta_3(s) = \frac{1}{\sqrt{3}}(T^* + T_{T^*} + T^* \wedge T_{T^*}) \tag{22}$$

Theorem 6. The $\kappa_g^{\beta_3}$ geodesic curvature belonging to β_3-Smarandache curve of the Mannheim curve is:

$$\kappa_g^{\beta_3}=\frac{1}{4\sqrt{2}\left[1+\frac{\phi'}{\|W\|}+\left(\frac{\phi'}{\|W\|}\right)^2\right]^{\frac{5}{2}}}\left[\left(2\frac{\phi'}{\|W\|}-1\right)\overline{\varphi}_1-\left(1+\frac{\phi'}{\|W\|}\right)\overline{\varphi}_2+\left(2-\frac{\phi'}{\|W\|}\right)\overline{\varphi}_3\right]$$

where

$$\begin{cases}\overline{\varphi}_1=-2+4\left(\frac{\|W\|}{\theta'}\right)-4\left(\frac{\|W\|}{\theta'}\right)^2+2\left(\frac{\|W\|}{\theta'}\right)^3+\left(\frac{\|W\|}{\theta'}\right)'\left(2\frac{\|W\|}{\theta'}-1\right)\\ \overline{\varphi}_2=-2+2\left(\frac{\|W\|}{\theta'}\right)-4\left(\frac{\|W\|}{\theta'}\right)^2+\left(\frac{\|W\|}{\theta'}\right)^3-2\left(\frac{\|W\|}{\theta'}\right)^4-\left(\frac{\|W\|}{\theta'}\right)'\left(1+\frac{\|W\|}{\theta'}\right)\\ \overline{\varphi}_3=2\left(\frac{\|W\|}{\theta'}\right)-4\left(\frac{\|W\|}{\theta'}\right)^2+4\left(\frac{\|W\|}{\theta'}\right)^3-2\left(\frac{\|W\|}{\theta'}\right)^4+\left(\frac{\|W\|}{\theta'}\right)'\left(2-\frac{\|W\|}{\theta'}\right)\end{cases}$$

Proof. From the eqs. (3), (7), and (22), we have:

$$\beta_3(s)=\frac{1}{\sqrt{3}}\left[(\sin\theta+\cos\theta)T+N+(\cos\theta-\sin\theta)B\right]$$

Then if Sabban apparatus are calculated, we have:

$$T_{\beta_3}(s)=\frac{\theta'\cos\theta-(\|W\|-\theta')\sin\theta}{\sqrt{2(\|W\|^2-\|W\|\theta'+\theta'^2)}}T+\frac{\|W\|}{\sqrt{2(\|W\|^2-\|W\|\theta'+\theta'^2)}}N-\frac{\theta'\sin\theta+(\|W\|-\theta')\cos\theta}{\sqrt{2(\|W\|^2-\|W\|\theta'+\theta'^2)}}B$$

$$T_{\beta_3}{}'(s)=\frac{(\theta')^4\sqrt{3}(\overline{\varphi}_2\sin\theta+\overline{\varphi}_1\cos\theta)}{4(\|W\|^2-\|W\|\theta'+\theta'^2)^2}T+\frac{(\theta')^4\sqrt{3}\overline{\varphi}_3}{4(\|W\|^2-\|W\|\theta'+\theta'^2)^2}N+$$

$$+\frac{(\theta')^4\sqrt{3}(\overline{\varphi}_2\cos\theta-\overline{\varphi}_1\sin\theta)}{4(\|W\|^2-\|W\|\theta'+\theta'^2)^2}B$$

$$(\beta_3\wedge T_{\beta_3})(s)=\frac{(2\|W\|-\theta')\cos\theta-(\|W\|+\theta')\sin\theta}{\sqrt{6\|W\|^2-6\|W\|\theta'+6\theta'^2}}T+\frac{2\theta'-\|W\|}{\sqrt{6\|W\|^2-6\|W\|\theta'+6\theta'^2}}N-$$

$$-\frac{(2\|W\|-\theta')\sin\theta+(\|W\|+\theta')\cos\theta}{\sqrt{6\|W\|^2-6\|W\|\theta'+6\theta'^2}}B$$

$\kappa_g^{\beta_3}$ geodesic curvature is:

$$\kappa_g^{\beta_3}=\frac{1}{4\sqrt{2}\left[1+\frac{\|W\|}{\theta'}+\left(\frac{\|W\|}{\theta'}\right)^2\right]^{\frac{5}{2}}}\left[\left(2\frac{\|W\|}{\theta'}-1\right)\overline{\varphi}_1-\left(1+\frac{\|W\|}{\theta'}\right)\overline{\varphi}_2+\left(2-\frac{\|W\|}{\theta'}\right)\overline{\varphi}_3\right]$$

Definition 7. Let (N^*) be curve of α^* and let N^* and T_{N^*} be unit vectors of (N^*). In this case β_4-Smarandache curve is defined:

$$\beta_4(s) = \frac{1}{\sqrt{4}}(N^* + T_{N^*}) \tag{23}$$

Theorem 8. The $\kappa_g^{\beta_4}$ geodesic curvature belonging to β_4-Smarandache curve of the Mannheim curve is:

$$\kappa_g^{\beta_4} = \frac{1}{\left(2+\eta^2\right)^{\frac{5}{2}}}\left(\eta\overline{\chi}_1 - \eta\overline{\chi}_2 + 2\overline{\chi}_3\right)$$

where

$$\eta = \left[\frac{\|W\|}{\sqrt{(\theta')^2 + \|W\|^2}}\right]' \frac{\lambda\tau}{\theta'}\csc\theta$$

and

$$\overline{\chi}_1 = -2 - \eta^2 + \eta'\eta, \quad \overline{\chi}_2 = -2 - 3\eta^2 - \eta^4 - \eta'\eta, \quad \overline{\chi}_3 = 2\eta + \eta^3 + \eta'$$

Proof. From the eqs. (3), (8), and (23), we have:

$$\beta_4(s) = \frac{\sqrt{\theta'^2 + \|W\|^2}\sin\theta - \theta'\cos\theta}{\sqrt{2\theta'^2 + 2\|W\|^2}}T + \frac{\|W\|}{\sqrt{2\theta'^2 + 2\|W\|^2}}N + \frac{\theta'\sin\theta + \sqrt{\theta'^2 + \|W\|^2}\cos\theta}{\sqrt{2\theta'^2 + 2\|W\|^2}}B$$

If Sabban apparatus are calculated, we have:

$$T_{\beta_4}(s) = \frac{(\eta\|W\| - \theta')\cos\theta - \sqrt{\|W\|^2 + \theta'^2}\sin\theta}{\sqrt{\|W\|^2 + \theta'^2}\sqrt{2+\eta^2}}T + \frac{\eta\theta' + \|W\|}{\sqrt{\|W\|^2 + \theta'^2}\sqrt{2+\eta^2}}N -$$
$$-\frac{(\eta\|W\| + \theta')\sin\theta + \sqrt{\|W\|^2 + \theta'^2}\cos\theta}{\sqrt{\|W\|^2 + \theta'^2}\sqrt{2+\eta^2}}B$$

$$T_{\beta_4}'(s) = \frac{(\overline{\chi}_3\|W\| - \overline{\chi}_2\theta')\sqrt{2}\cos\theta + \sqrt{2\|W\|^2 + 2\theta'^2}\,\overline{\chi}_1\sin\theta}{(2+\eta^2)^2\sqrt{\|W\|^2 + \theta'^2}}T + \frac{(\overline{\chi}_3\theta' + \overline{\chi}_2\|W\|)\sqrt{2}}{(2+\eta^2)^2\sqrt{\|W\|^2 + \theta'^2}}N +$$
$$+\frac{(\overline{\chi}_2\theta' - \overline{\chi}_3\|W\|)\sqrt{2}\sin\theta + \sqrt{2\|W\|^2 + 2\theta'^2}\,\overline{\chi}_1\cos\theta}{(2+\eta^2)^2\sqrt{\|W\|^2 + \theta'^2}}B$$

$$(\beta_4 \wedge T_{\beta_4})(s) = \frac{(2\|W\| - \eta\theta')\cos\theta - \eta\sqrt{\|W\|^2 + \theta'^2}\sin\theta}{\sqrt{4+2\eta^2}\sqrt{\|W\|^2 + \theta'^2}}T + \frac{2\theta' - \eta\|W\|}{\sqrt{4+2\eta^2}\sqrt{\|W\|^2 + \theta'^2}}N +$$

$$+\frac{(\eta\theta'-2\|W\|)\sin\theta-\eta\sqrt{\|W\|^2+\theta'^2}\cos\theta}{\sqrt{4+2\eta^2}\sqrt{\|W\|^2+\theta'^2}}B$$

$\kappa_g^{\beta_4}$ geodesic curvature is:

$$\kappa_g^{\beta_4}=\frac{1}{\left(2+\eta^2\right)^{\frac{5}{2}}}\left(\eta\overline{\chi}_1-\eta\overline{\chi}_2+2\overline{\chi}_3\right)$$

Definition 9. Let (N^*) be curve of α^* and let T_{N^*} and $N^*\wedge T_{N^*}$ be unit vectors of (N^*). In this case β_5-Smarandache curve is defined:

$$\beta_5(s)=\frac{1}{\sqrt{2}}(T_{N^*}+N^*\wedge T_{N^*}) \tag{24}$$

Theorem 10. The $\kappa_g^{\beta_5}$ geodesic curvature belonging to β_5-Smarandache curve of the Mannheim curve is:

$$\kappa_g^{\beta_5}=\frac{1}{\left(2+\eta^2\right)^{\frac{5}{2}}}(2\eta\overline{\delta}_1-\overline{\delta}_2+\overline{\delta}_3)$$

where

$$\overline{\delta}_1=\eta+2\eta^3+2\eta'\eta,\quad \overline{\delta}_2=-1-3\eta^2-2\eta^4-\eta',\quad \overline{\delta}_3=-\eta^2-2\eta^4+\eta'$$

Proof. From the eqs. (3), (8), and (24), we have:

$$\beta_5(s)=\frac{(\|W\|-\theta')\cos\theta}{\sqrt{2\theta'^2+2\|W\|^2}}T+\frac{\theta'+\|W\|}{\sqrt{2\theta'^2+2\|W\|^2}}N+\frac{(\theta'-\|W\|)\sin\theta}{\sqrt{2\theta'^2+2\|W\|^2}}B$$

Here, if Sabban apparatus are calculated, we have:

$$T_{\beta_5}(s)=\frac{\eta(\|W\|+\theta')\cos\theta-\sqrt{\|W\|^2+\theta'^2}\sin\theta}{\sqrt{1+2\eta^2}\sqrt{\|W\|^2+\theta'^2}}T+\frac{\eta(\theta'-\|W\|)}{\sqrt{1+2\eta^2}\sqrt{\|W\|^2+\theta'^2}}N-$$

$$-\frac{\eta(\|W\|+\theta')\sin\theta+\sqrt{\|W\|^2+\theta'^2}\cos\theta}{\sqrt{1+2\eta^2}\sqrt{\|W\|^2+\theta'^2}}B$$

$$T_{\beta_5}'(s)=\frac{(\overline{\delta}_3\|W\|-\overline{\delta}_2\theta')\sqrt{2}\cos\theta+\overline{\delta}_1\sqrt{2\|W\|^2+2\theta'^2}\sin\theta}{(1+2\eta^2)^2\sqrt{\theta'^2+\|W\|^2}}T+\frac{(\overline{\delta}_3\theta'+\overline{\delta}_2\|W\|)\sqrt{2}}{(1+2\eta^2)^2\sqrt{\theta'^2+\|W\|^2}}N+$$

$$+\frac{(\overline{\delta}_2\theta'-\overline{\delta}_3\|W\|)\sqrt{2}\sin\theta+\overline{\delta}_1\sqrt{2\|W\|^2+2\theta'^2}\cos\theta}{(1+2\eta^2)^2\sqrt{\theta'^2+\|W\|^2}}B$$

$$(\beta_5 \wedge T_{\beta_5})(s) = \frac{(\|W\| + \theta')\cos\theta + 2\eta\sqrt{\|W\|^2 + \theta'^2}\sin\theta}{\sqrt{2+4\eta^2}\sqrt{\|W\|^2 + \theta'^2}}T + \frac{\theta' - \|W\|}{\sqrt{2+4\eta^2}\sqrt{\|W\|^2 + \theta'^2}}N + $$
$$+\frac{2\eta\sqrt{\|W\|^2 + \theta'^2}\cos\theta - (\|W\| + \theta')\sin\theta}{\sqrt{2+4\eta^2}\sqrt{\|W\|^2 + \theta'^2}}B$$

$\kappa_g^{\beta_5}$ geodesic curvature is:

$$\kappa_g^{\beta_5} = \frac{1}{\left(2+\eta^2\right)^{\frac{5}{2}}}(2\eta\overline{\delta}_1 - \overline{\delta}_2 + \overline{\delta}_3)$$

Definition 11. Let (N^*) be curve of α^* and let N^*, T_{N^*}, and $N^* \wedge T_{N^*}$ be unit vectors of (N^*). In this case β_6-Smarandache curve is defined:

$$\beta_6(s) = \frac{1}{\sqrt{3}}(N^* + T_{N^*} + N^* \wedge T_{N^*}) \tag{25}$$

Theorem 12. The $\kappa_g^{\beta_6}$ geodesic curvature belonging to β_6-Smarandache curve of the Mannheim curve is:

$$\kappa_g^{\beta_6} = \frac{(2\eta-1)\overline{\rho}_1 - (1+\eta)\overline{\rho}_2 + (2-\eta)\overline{\rho}_3}{4\sqrt{2}\left(1-\eta+\eta^2\right)^{\frac{5}{2}}}$$

where

$$\overline{\rho}_1 = -2 + 4\eta - 4\eta^2 + 2\eta^3 + 2\eta'(2\eta-1),\ \ \overline{\rho}_2 = -2 + 2\eta - 4\eta^2 + 2\eta^3 - 2\eta^4 - \eta'(1+\eta)$$
$$\overline{\rho}_3 = 2\eta - 4\eta^2 + 4\eta^3 - 2\eta^4 + \eta'(2-\eta)$$

Proof. From the eqs. (3), (8), and (25), we have:

$$\beta_6(s) = \frac{(\|W\| - \theta')\cos\theta + \sqrt{\theta'^2 + \|W\|^2}\sin\theta}{\sqrt{3\theta'^2 + 3\|W\|^2}}T + \frac{\theta' + \|W\|}{\sqrt{3\theta'^2 + 3\|W\|^2}}N + $$
$$+\frac{(\theta' - \|W\|)\sin\theta + \sqrt{\theta'^2 + \|W\|^2}\cos\theta}{\sqrt{3\theta'^2 + 3\|W\|^2}}B$$

Herein, if Sabban apparatus are calculated, we have:

$$T_{\beta_6}(s) = \frac{(\eta\|W\| - (1-\eta)\theta')\cos\theta - \sqrt{\|W\|^2 + \theta'^2}\sin\theta}{\sqrt{2(1-\eta+\eta^2)}\sqrt{\|W\|^2 + \theta'^2}}T + \frac{\eta\theta' + (1-\eta)\|W\|}{\sqrt{2(1-\eta+\eta^2)}\sqrt{\|W\|^2 + \theta'^2}}N + $$
$$+\frac{\left[(1-\eta)\theta' - \eta\|W\|\right]\sin\theta - \sqrt{\|W\|^2 + \theta'^2}\cos\theta}{\sqrt{2(1-\eta+\eta^2)}\sqrt{\|W\|^2 + \theta'^2}}B$$

$$T_{\beta_6}{}'(s)=\frac{(\overline{\rho}_3\theta'-\overline{\rho}_2\|W\|)\sqrt{3}\cos\theta+\overline{\rho}_1\sqrt{3\|W\|^2+3\theta'^2}\sin\theta}{4(1-\eta+\eta^2)^2\sqrt{\theta'^2+\|W\|^2}}T+\frac{(\overline{\rho}_3\|W\|+\overline{\rho}_2\theta')\sqrt{3}}{4(1-\eta+\eta^2)^2\sqrt{\theta'^2+\|W\|^2}}N+$$
$$+\frac{(\overline{\rho}_2\|W\|-\overline{\rho}_3\theta')\sqrt{3}\sin\theta+\overline{\rho}_1\sqrt{3\|W\|^2+3\theta'^2}\cos\theta}{4(1-\eta+\eta^2)^2\sqrt{\theta'^2+\|W\|^2}}B$$

$$(\beta_6\wedge T_{\beta_6})(s)=\frac{\left[(2-\eta)\|W\|+(1+\eta)\theta'\right]\cos\theta+(2\eta-1)\sqrt{\|W\|^2+\theta'^2}\sin\theta}{\sqrt{6-6\eta+6\eta^2}\sqrt{\|W\|^2+\theta'^2}}T+$$
$$+\frac{(2-\eta)\theta'-(1+\eta)\|W\|}{\sqrt{6-6\eta+6\eta^2}\sqrt{\|W\|^2+\theta'^2}}N+\frac{(2\eta-1)\sqrt{\|W\|^2+\theta'^2}\cos\theta-\left[(2-\eta)\|W\|-(1+\eta)\theta'\right]\sin\theta}{\sqrt{6-6\eta+6\eta^2}\sqrt{\|W\|^2+\theta'^2}}B$$

$\kappa_g^{\beta_6}$ geodesic curvature is:

$$\kappa_g^{\beta_6}=\frac{(2\eta-1)\overline{\rho}_1-(1+\eta)\overline{\rho}_2+(2-\eta)\overline{\rho}_3}{4\sqrt{2}\left(1-\eta+\eta^2\right)^{\frac{5}{2}}}$$

Definition 13. Let (B^*) be curve of α^* and let T_{B^*} and $B^*\wedge T_{B^*}$ be unit vectors of (B^*). In this case β_7-Smarandache curve is defined:

$$\beta_7(s)=\frac{1}{\sqrt{2}}(T_{B^*}+T^*\wedge T_{B^*}) \tag{26}$$

Theorem 14. The $\kappa_g^{\beta_7}$ geodesic curvature belonging to β_7-Smarandache curve of the Mannheim curve is:

$$\kappa_g^{\beta_7}=\frac{1}{\left[1+2\left(\frac{\theta'}{\|W\|}\right)^2\right]^{\frac{5}{2}}}\left(2\frac{\theta'}{\|W\|}\overline{\psi}_1-\overline{\psi}_2+\overline{\psi}_3\right)$$

where

$$\overline{\psi}_1=\left(\frac{\theta'}{\|W\|}\right)+\left(\frac{\theta'}{\|W\|}\right)^3+2\left(\frac{\theta'}{\|W\|}\right)'\left(\frac{\theta'}{\|W\|}\right),\quad \overline{\psi}_2=-1-3\left(\frac{\theta'}{\|W\|}\right)^2-2\left(\frac{\theta'}{\|W\|}\right)^4-\left(\frac{\theta'}{\|W\|}\right)'$$
$$\overline{\psi}_3=-\left(\frac{\theta'}{\|W\|}\right)^2-2\left(\frac{\theta'}{\|W\|}\right)^4+\left(\frac{\theta'}{\|W\|}\right)'$$

Proof. From the eqs. (3), (9), and (26), we have:

$$\beta_7(s)=\frac{1}{\sqrt{2}}(-\sin\theta T+N-\cos\theta B)$$

If Sabban apparatus are calculated, we have:

$$T_{\beta_7}(s)=\frac{\theta'\cos\theta-\|W\|\sin\theta}{\sqrt{2\|W\|^2+\theta'^2}}T-\frac{\|W\|}{\sqrt{2\|W\|^2+\theta'^2}}N-\frac{\theta'\sin\theta+\|W\|\cos\theta}{\sqrt{2\|W\|^2+\theta'^2}}B$$

$$T_{\beta_7}'(s)=\frac{\|W\|^4\sqrt{2}(\overline{\omega}_3\cos\theta-\overline{\omega}_2\sin\theta)}{(2\|W\|^2+\theta'^2)^2}T+\frac{\|W\|^4\sqrt{2}\overline{\omega}_1}{(2\|W\|^2+\theta'^2)^2}N-\frac{\|W\|^4\sqrt{2}(\overline{\omega}_2\cos\theta+\overline{\omega}_3\sin\theta)}{(2\|W\|^2+\theta'^2)^2}B$$

$$(\beta_7\wedge T_{\beta_7})(s)=\frac{2\|W\|\cos\theta+\theta'\sin\theta}{\sqrt{4\|W\|^2+2\theta'^2}}T+\frac{\theta'}{\sqrt{4\|W\|^2+2\theta'^2}}N+\frac{\theta'\cos\theta-2\|W\|\sin\theta}{\sqrt{4\|W\|^2+2\theta'^2}}B$$

$\kappa_g^{\beta_7}$ geodesic curvature is:

$$\kappa_g^{\beta_7}=\frac{1}{\left[2+\left(\frac{\theta'}{\|W\|}\right)^2\right]^{\frac{5}{2}}}\left(\frac{\theta'}{\|W\|}\overline{\omega}_1-\frac{\theta'}{\|W\|}\overline{\omega}_2+2\overline{\omega}_3\right)$$

Definition 15. Let (B^*) be curve of α^* and let B^*, T_{B^*}, and $B^*\wedge T_{B^*}$ be unit vectors of (B^*). In this case β_8-Smarandache curve is defined:

$$\beta_8(s)=\frac{1}{\sqrt{3}}(B^*+T_{B^*}+B^*\wedge T_{B^*}) \tag{27}$$

Theorem 16. The $\kappa_g^{\beta_8}$ geodesic curvature belonging to β_8-Smarandache curve of the Mannheim curve is:

$$\kappa_g^{\beta_8}=\frac{1}{4\sqrt{2}\left[1+\frac{\theta'}{\|W\|}+\left(\frac{\theta'}{\|W\|}\right)^2\right]^{\frac{5}{2}}}\left[\left(2\frac{\theta'}{\|W\|}-1\right)\overline{\zeta}_1-\left(1+\frac{\theta'}{\|W\|}\right)\overline{\zeta}_2+\left(2-\frac{\theta'}{\|W\|}\right)\overline{\zeta}_3\right]$$

where

$$\begin{cases}\overline{\zeta}_1=-2+4\frac{\theta'}{\|W\|}+4\left(\frac{\theta'}{\|W\|}\right)-\left(\frac{\theta'}{\|W\|}\right)^2+2\left(\frac{\theta'}{\|W\|}\right)^3+\left(\frac{\theta'}{\|W\|}\right)'\left(2\frac{\theta'}{\|W\|}-1\right)\\ \overline{\zeta}_2=-2+2\frac{\theta'}{\|W\|}-4\left(\frac{\theta'}{\|W\|}\right)^2+\left(\frac{\theta'}{\|W\|}\right)^3-2\left(\frac{\theta'}{\|W\|}\right)^4-\left(\frac{\theta'}{\|W\|}\right)'\left(1+\frac{\theta'}{\|W\|}\right)\\ \overline{\zeta}_3=2\frac{\theta'}{\|W\|}-4\left(\frac{\theta'}{\|W\|}\right)^2+4\left(\frac{\theta'}{\|W\|}\right)^3-2\left(\frac{\theta'}{\|W\|}\right)^4+\left(\frac{\theta'}{\|W\|}\right)'\left(2-\frac{\theta'}{\|W\|}\right)\end{cases}$$

Proof. From the eqs. (3), (9), and (27), we have:

$$\beta_8(s)=\frac{1}{\sqrt{3}}\left[(\cos\theta-\sin\theta)T+N-(\cos\theta+\sin\theta)B\right]$$

Here if Sabban apparatus are calculated, we have:

$$T_{\beta_8}(s)=\frac{\theta'\cos\theta+(\theta'-\|W\|)\sin\theta}{\sqrt{2(\|W\|^2-\|W\|\theta'+\theta'^2)}}T-\frac{\|W\|}{\sqrt{2(\|W\|^2-\|W\|\theta'+\theta'^2)}}N-\frac{\theta'\sin\theta-(\theta'-\|W\|)\cos\theta}{\sqrt{2(\|W\|^2-\|W\|\theta'+\theta'^2)}}B$$

$$T_{\beta_8}{}'(s)=\frac{\|W\|^4\sqrt{3}(\overline{\zeta}_3\cos\theta-\overline{\zeta}_2\sin\theta)}{4(\|W\|^2-\|W\|\theta'+\theta'^2)^2}T+\frac{\|W\|^4\sqrt{3}\overline{\zeta}_1}{4(\|W\|^2-\|W\|\theta'+\theta'^2)^2}N-$$

$$-\frac{\|W\|^4\sqrt{3}(\overline{\zeta}_2\cos\theta+\overline{\zeta}_3\sin\theta)}{4(\|W\|^2-\|W\|\theta'+\theta'^2)^2}B$$

$$(\beta_8\wedge T_{\beta_8})(s)=\frac{(2\|W\|-\theta')\cos\theta+(\|W\|+\theta')\sin\theta}{\sqrt{6\|W\|^2-6\|W\|\theta'+6\theta'^2}}T+\frac{2\theta'-\|W\|}{\sqrt{6\|W\|^2-6\|W\|\theta'+6\theta'^2}}N+$$

$$+\frac{(\|W\|+\theta')\cos\theta-(2\|W\|-\theta')\sin\theta}{\sqrt{6\|W\|^2-6\|W\|\theta'+6\theta'^2}}B,$$

$\kappa_g^{\beta_8}$ geodesic curvature is:

$$\kappa_g^{\beta_8}=\frac{1}{4\sqrt{2}\left[1+\frac{\theta'}{\|W\|}+\left(\frac{\theta'}{\|W\|}\right)^2\right]^{\frac{5}{2}}}\left[\left(2\frac{\theta'}{\|W\|}-1\right)\overline{\zeta}_1-\left(1+\frac{\theta'}{\|W\|}\right)\overline{\zeta}_2+\left(2-\frac{\theta'}{\|W\|}\right)\overline{\zeta}_3\right]$$

Conclusion

In this paper, we studied Mannheim curve pairs. By moving the Frenet vectors of the Mannheim curve into the unit sphere, we formed the Sabban frame of this curve. We got Smarandache curves from the Sabban frame and calculated the geodesic curvature of obtained Smarandache curves. We expressed found results that depend on Frenet frame of the Mannheim curve pair.

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