A SPECIAL INTERPRETATION OF THE CONCEPT CONSTANT BREADTH FOR A SPACE CURVE

by

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The definition of curve of constant breadth in the literature is made by using tangent vectors, which are parallel and opposite directions, at opposite points of the curve. In this study, normal vectors of the curve, which are parallel and opposite directions are placed at the exit point of the concept of curve of constant breadth. In this study, on the concept of curve of constant breadth according to normal vector is worked. At the conclusion of the study, is obtained a system of linear differential equations with variable coefficients characterizing space curves of constant breadth according to normal vector. The coefficients of this system of equations are functions depend on the curvature and torsion of the curve. Then is obtained an approximate solution of this system by using the Taylor matrix collocation method. In summary, in this study, a different interpretation is made for the concept of space curve of constant breadth, the first time. Then this interpretation is used to obtain a characterization. As a result, this characterization we've obtained is solved.

Key words: *constant breadth, special curves in space, Taylor matrix collocation method*

Introduction

Journey of the concept of curve of constant breadth in the history goes back to the 1700s [1]. These curves are used in different fields such as kinematics, mechanical engineering, and com design [2]. These curves, which were previously defined in the plane, were moved to 3-D Euclidean space in 1914 [3]. A lot of studies have been done about this type of curve, which has been explained in n-dimensional Euclidean space, in different spaces [4-8].

Interpreting the geometric properties of a curve of constant breadth in any space is possible with differential equations or systems of equations [9, 10]. Generally, integral characterizations have been obtained, not solutions of differential equations of these curves in made works so far [11, 12].

In all studies, the concept of constant breadth is associated with the tangent vector of the curve [13, 14]. The question *can the concept of constant breadth be associated with the other two legs of the Frenet frame* is the starting point of this study. With this thought, the concept of constant breadth is examined according to the normal vector of the curve and a system of differential equations that characterizes these curves is obtained. In addition, an approximate solution of this obtained system is obtained and the curve is interpreted more clearly.

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Preliminaries

A simple closed C curve of constant breadth having opposite and parallel normals at opposite points is shown [15].

$$\vec{\alpha}^*(s) = \vec{\alpha}(s) + \lambda(s)\vec{t} + \mu(s)\vec{n} + \delta(s)\vec{b} \tag{1}$$

where $\vec{\alpha}$ and $\vec{\alpha}^*$ are the opposite points, $\vec{t}$, $\vec{n}$, $\vec{b}$ denote the unit tangent, normal and binormal at any $\vec{\alpha}$ point of the curve, respectively. The functions $\lambda(s)$, $\mu(s)$, $\delta(s)$ are the coefficients that determine the curve.

Definition 1. Let φ be the value of the angle between normal vector and a fixed direction at any $\vec{\alpha}(\mathrm{s})$ point on a C curve. The angle between $\vec{n}(\mathrm{s})$ and $\vec{n}(\mathrm{s}+\Delta\mathrm{s})$ normal vectors at the $\vec{\alpha}(\mathrm{s})$ and $\vec{\alpha}(\mathrm{s}+\Delta\mathrm{s})$ points of this C curve is called the contengency angle respect to the normal vector. If so, the first curvature according to the normal vector is defined:

$$\lim_{\Delta s\to 0}\frac{\Delta\varphi}{\Delta s} = \frac{\partial\varphi}{\partial s} = k_{n1}(s)$$

Therefore, the following equation is obvious for φ and s arc length parameter:

$$\varphi(s) = \int_0^s k_{n1}(s)\partial s$$

Theorem 2. Let C be a curve with unit speed from the C^4 class. This C curve has opposing and parallel $\vec{n}(\mathrm{s})$ and $\vec{n}^*$ normals at the opposite points $\vec{\alpha}(\mathrm{s})$ and $\vec{n}^*(\mathrm{s})$. If each beam joining the opposite points of the curve is a double tangent the curve is constant breadth according to normal vector and conversely if the curve is a curve of constant breadth according to the normal vector, each tangent of the curve is a double tangent.

Proof.

$$\vec{r} = \vec{\alpha}^*(\mathrm{s}) - \vec{\alpha}(\mathrm{s}) = \lambda(\mathrm{s})\vec{t} + \mu(\mathrm{s})\vec{n} + \delta(\mathrm{s})\vec{b}$$

Let $\vec{r}$ be the double tangent of the curve. In this case $\vec{r}.\vec{n}^* = -\vec{r}.\vec{n} = -\mu(\mathrm{s}) = 0$. Using this equality, the following equation system is obtained:

$$\left.\begin{aligned} \frac{\partial\lambda}{\partial s} &= \mu(\mathrm{s}) - \mathrm{g} \\ \frac{\partial\mu}{\partial s} &= -\lambda(s) + \rho\tau\delta(\mathrm{s}) \\ \frac{\partial\delta}{\partial s} &= -\rho\tau\mu(\mathrm{s}) \end{aligned}\right\}$$

The following statement is obtained from the 1st and 3rd equality of this system.

$$\lambda\frac{\partial\lambda}{\partial s} + \delta\frac{\partial\delta}{\partial s} = 0$$

Here $\lambda^2 + \delta^2 = \text{constant}$ is obtained. It is obvious that one of the curvature radius values at the opposite points of the curve will be equal to the negative of the other. On the contrary, the following equation can be written if the distance between the opposing points is fixed.

$$\lambda^2 + \mu^2 + \delta^2 = \text{constant}$$

Then $\lambda(\partial\lambda/\partial s)+\mu(\partial\mu/\partial s)+\delta(\partial\delta/\partial s)=0$ is obtained. Since $\mu(\mathrm{s})=0$, $\vec{r}$ is in the TB plane and is the double tangent of the curve.

Differential equation system characterizing curve of constant breadth according to normal

If the derivative of eq. (1) is taken according to the s arc length parameter, the following expression is found:

$$\frac{\partial\alpha^*}{\partial s}=\frac{\partial\alpha^*}{\partial s^*}\frac{\partial s^*}{\partial s}=\vec{t}^*\frac{\partial s^*}{\partial s}=$$

$$=\left(1+\frac{\partial\lambda}{\partial s}-\mu k_{n1}\right)\vec{t}+\left(\frac{\partial\mu}{\partial s}+\lambda k_{n1}-\delta k_{n2}\right)\vec{n}+\left(\frac{\partial\delta}{\partial s}+\mu k_{n2}\right)\vec{b}$$

Here k_{n2} is called the second curvature according to normal of the curve.

$$\frac{\partial s^*}{\partial s}\vec{t}^*=\left(1+\frac{\partial\lambda}{\partial s}-\mu k_{n1}\right)\vec{t}+\left(\frac{\partial\mu}{\partial s}+\lambda k_{n1}-\delta k_{n2}\right)\vec{n}+\left(\frac{\partial\delta}{\partial s}+\mu k_{n2}\right)\vec{b}$$

$$\frac{\partial s^*}{\partial s}\vec{t}^*=\left(1+\frac{\partial\lambda}{\partial s}-\mu k_{n1}\right)\vec{t}+\left(\frac{\partial\mu}{\partial s}+\lambda k_{n1}-\delta k_{n2}\right)\vec{n}+\left(\frac{\partial\delta}{\partial s}+\mu k_{n2}\right)\vec{b}$$

If the two sides of the equation are multiplied by $\dfrac{\partial s}{\partial\varphi}=\dfrac{1}{k_{n1}(\mathrm{s})}$ the following equation is obtained:

$$\frac{\partial s^*}{\partial\varphi}\vec{t}^*=\left(\frac{1}{k_{n1}}+\frac{\partial\lambda}{\partial\varphi}-\mu\right)\vec{t}+\left(\frac{\partial\mu}{\partial\varphi}+\lambda-\delta\frac{k_{n2}}{k_{n1}}\right)\vec{n}+\left(\frac{\partial\delta}{\partial\varphi}+\mu\frac{k_{n2}}{k_{n1}}\right)\vec{b}$$

Also, because $\partial\varphi/\partial s^*=k_{n1}^*$ and $k_{n1}^*=\|(\vec{t}^*)'\|$ the following equation occurs:

$$\vec{t}^*=k_{n1}^*\left[\left(\frac{1}{k_{n1}}+\frac{\partial\lambda}{\partial\varphi}-\mu\right)\vec{t}+\left(\frac{\partial\mu}{\partial\varphi}+\lambda-\delta\frac{k_{n2}}{k_{n1}}\right)\vec{n}+\left(\frac{\partial\delta}{\partial\varphi}+\mu\frac{k_{n2}}{k_{n1}}\right)\vec{b}\right]$$

$$\vec{t}^*=\left\|(\vec{t}^*)'\right\|\left[\left(\frac{1}{k_{n1}}+\frac{\partial\lambda}{\partial\varphi}-\mu\right)\vec{t}+\left(\frac{\partial\mu}{\partial\varphi}+\lambda-\delta\frac{k_{n2}}{k_{n1}}\right)\vec{n}+\left(\frac{\partial\delta}{\partial\varphi}+\mu\frac{k_{n2}}{k_{n1}}\right)\vec{b}\right]$$

In this equation, derivative is applied to both sides of the equation.

$$(\vec{t}^*)'=\left\|(\vec{t}^*)'\right\|\left[\left(\frac{1}{k_{n1}}+\frac{\partial\lambda}{\partial\varphi}-\mu\right)\vec{t}+\left(\frac{\partial\mu}{\partial\varphi}+\lambda-\delta\frac{k_{n2}}{k_{n1}}\right)\vec{n}+\left(\frac{\partial\delta}{\partial\varphi}+\mu\frac{k_{n2}}{k_{n1}}\right)\vec{b}\right]'$$

where $\|(\vec{t}^*)'\|=$ constant.

This final equation is regulated by the help of the Frenet equations using the equations:

$$\vec{n}^*=\frac{(\vec{t}^*)'}{\left\|(\vec{t}^*)'\right\|}\quad\text{and}\quad\vec{n}^*=-\vec{n}$$

And so the following equation system is obtained:

$$\left.\begin{array}{c} \lambda'' - (1+k_n 1)\mu' - k_{n1}\lambda + k_{n2}\delta = -\left(\dfrac{1}{k_{n1}}\right)' \\ \mu'' + (1+k_{n1})\lambda' - \left(\dfrac{k_{n2}}{k_{n1}} + k_{n2}\right)\delta' - \left(\dfrac{k_{n2}{}^2}{k_{n1}} + k_{n1}\right)\mu - \left(\dfrac{k_{n2}}{k_{n1}}\right)'\delta = -2 \\ \delta'' + \left(\dfrac{k_{n2}}{k_{n1}} + k_{n2}\right)\mu' + k_{n2}\lambda - \dfrac{k_{n2}{}^2}{k_{n1}}\delta + \left(\dfrac{k_{n2}}{k_{n1}}\right)'\mu = 0 \end{array}\right\} \tag{2}$$

This is a system of differential equations that characterizes curves of constant breadth according to normal vector.

Taylor matrix collocation method for solution of equation systems

In this section, Taylor matrix collocation method is explained for the solution of the systems of differential equations given in the general form:

$$\sum_{k=0}^{2}\sum_{j=1}^{3} P_{ij}^{k}(s)y_j^{(k)}(s) = g_i(s) \qquad i = 1,2,3 \tag{3}$$

This system can be written explicitly:

$$\sum_{k=0}^{2}\{P_{i1}^{k}(s)y_1^{(k)}(s) + P_{i2}^{k}(s)y_2^{(k)}(s) + P_{i3}^{k}(s)y_3^{(k)}(s)\} = g_i(s) \qquad i = 1,2,3$$

$$\text{For } i=1 \quad \sum_{k=0}^{2}\{P_{11}^{k}(s)y_1^{(k)}(s) + P_{12}^{k}(s)y_2^{(k)}(s) + P_{13}^{k}(s)y_3^{(k)}(s)\} = g_1(s)$$

$$\text{For } i=2 \quad \sum_{k=0}^{2}\{P_{21}^{k}(s)y_1^{(k)}(s) + P_{22}^{k}(s)y_2^{(k)}(s) + P_{23}^{k}(s)y_3^{(k)}(s)\} = g_2(s)$$

$$\text{For } i=3 \quad \sum_{k=0}^{2}\{P_{31}^{k}(s)y_1^{(k)}(s) + P_{32}^{k}(s)y_2^{(k)}(s) + P_{33}^{k}(s)y_3^{(k)}(s)\} = g_3(s)$$

In addition, this system can be expressed in matrix form:

$$\sum_{k=0}^{2}\left\{\begin{pmatrix} P_{11}^{k} & P_{12}^{k} & P_{13}^{k} \\ P_{21}^{k} & P_{22}^{k} & P_{23}^{k} \\ P_{31}^{k} & P_{32}^{k} & P_{33}^{k} \end{pmatrix}\begin{pmatrix} y_1^{(k)}(s) \\ y_2^{(k)}(s) \\ y_3^{(k)}(s) \end{pmatrix}\right\} = \begin{pmatrix} g_1(s) \\ g_2(s) \\ g_3(s) \end{pmatrix} \tag{4}$$

$$P_k(s) = \begin{pmatrix} P_{11}^{k} & P_{12}^{k} & P_{13}^{k} \\ P_{21}^{k} & P_{22}^{k} & P_{23}^{k} \\ P_{31}^{k} & P_{32}^{k} & P_{33}^{k} \end{pmatrix}, \quad Y^{(k)}(s) = \begin{pmatrix} y_1^{(k)}(s) \\ y_2^{(k)}(s) \\ y_3^{(k)}(s) \end{pmatrix}, \quad G(s) = \begin{pmatrix} g_1(s) \\ g_2(s) \\ g_3(s) \end{pmatrix}$$

The system can be written in the closed matrix form as follows using the previous equalities:

$$\sum_{k=0}^{2}\{P_k(s)Y^{(k)}(s)\} = G(s) \tag{5}$$

Assume that there is an approximate solution in the form trimmed Taylor series (6) of this equation system (5) at sorting points as:

$$s_r = a + \frac{b-a}{N} r, \quad r = 0,1,\ldots,N \quad (a \le s \le b)$$

$$y_j(s) \cong \sum_{n=0}^{N} a_{jn} s^n \tag{6}$$

where N is an arbitrary integer. For

$$X(s) = \begin{pmatrix} 1 & s & \ldots & s^n \end{pmatrix}, \quad A_j = \begin{pmatrix} a_{j0} \\ a_{j1} \\ \vdots \\ a_{jN} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

the approximate solution (6) and its derivatives can be expressed:

$$y_j(s) = X(s) A_j, \quad j = 1,2,3$$

$$y_j^{(k)}(s) = X(s) B^k A_j, \quad j = 1,2,3 \text{ and } k = 0,1,2$$

The derivatives of the approximate solution can be clearly written for the values $j = 1,2,3$:

$$\left.\begin{aligned} y_1^{(k)}(s) &= X(s) B^k A_1 \\ y_2^{(k)}(s) &= X(s) B^k A_2 \\ y_3^{(k)}(s) &= X(s) B^k A_3 \end{aligned}\right\} \tag{7}$$

$$Y^{(k)}(s) = \begin{pmatrix} y_1^{(k)}(s) \\ y_2^{(k)}(s) \\ y_3^{(k)}(s) \end{pmatrix} = \begin{pmatrix} X(s) B^k A_1 \\ X(s) B^k A_2 \\ X(s) B^k A_3 \end{pmatrix} = \begin{pmatrix} X(s) & 0 & 0 \\ 0 & X(s) & 0 \\ 0 & 0 & X(s) \end{pmatrix} \begin{pmatrix} B^k & 0 & 0 \\ 0 & B^k & 0 \\ 0 & 0 & B^k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

$$\overline{X}(s) = \begin{pmatrix} X(s) & 0 & 0 \\ 0 & X(s) & 0 \\ 0 & 0 & X(s) \end{pmatrix}, \quad \overline{B}^k = \begin{pmatrix} B^k & 0 & 0 \\ 0 & B^k & 0 \\ 0 & 0 & B^k \end{pmatrix}, \quad A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

with the help of the previous equations, the system (7) can be expressed in the closed matrix form:

$$Y^{(k)}(s) = \overline{X}(s) \overline{B}^k A, \quad k = 0,1,2 \tag{8}$$

If the equality (8) is written in eq. (5), the following expression is obtained:

$$\sum_{k=0}^{2} \{P_k(s) \overline{X}(s) \overline{B}^k A\} = G(s)$$

The following equation is obtained if s_r points are used here.

$$\sum_{k=0}^{2}\{P_k(s_r)\overline{X}(s_r)\overline{B}^k A\} = G(s_r), \quad r = 0,1,2,...,N \tag{9}$$

This equality can be written briefly: $\sum_{k=0}^{2} P_k \overline{X}\,\overline{B}^{k^*} A = G$. Here, the matrices $P_k, \overline{X}, \overline{B}^{k^*}$, G are defined:

$$P_k = \begin{pmatrix} P_k(s_0) & 0 & \cdots & 0 \\ 0 & P_k(s_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & P_k(s_N) \end{pmatrix}, \quad \overline{X} = \begin{pmatrix} \overline{X}(s_0) & 0 & \cdots & 0 \\ 0 & \overline{X}(s_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \overline{X}(s_N) \end{pmatrix}$$

$$\overline{B}^{k^*} = \begin{pmatrix} \overline{B}^k \\ \overline{B}^k \\ \vdots \\ \overline{B}^k \end{pmatrix}, G = \begin{pmatrix} G(s_0) \\ G(s_1) \\ \vdots \\ G(s_N) \end{pmatrix}.$$

On the other hand for $W = \sum_{k=0}^{2} P_k \overline{X}\,\overline{B}^{k^*}$ the following equality is obvious:

$$WA = G \Rightarrow (W;G) = A \tag{10}$$

In addition,

$$y_1(a) = \lambda_0 \Rightarrow X(a)A_1 = \lambda_0, y_1'(a) = X(a)BA_1 = \lambda_1$$

$$y_2(a) = \mu_0 \Rightarrow X(a)A_2 = \mu_0, y_2'(a) = X(a)BA_2 = \mu_1$$

$$y_3(a) = \delta_0 \Rightarrow X(a)A_3 = \delta_0, y_3'(a) = X(a)BA_3 = \delta_1$$

the matrix equation of the conditions can be written as follows using the previous equations.

$$\begin{pmatrix} X(a) & 0 & 0 \\ 0 & X(a) & 0 \\ 0 & 0 & X(a) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \mu_0 \\ \delta_0 \end{pmatrix}$$

$$\begin{pmatrix} X(a)B & 0 & 0 \\ 0 & X(a)B & 0 \\ 0 & 0 & X(a)B \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \mu_1 \\ \delta_1 \end{pmatrix}$$

The matrix equation of conditions can be written in closed form:

$$U_0 = \begin{pmatrix} X(a) & 0 & 0 \\ 0 & X(a) & 0 \\ 0 & 0 & X(a) \end{pmatrix}, \quad A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, \quad \theta_0 = \begin{pmatrix} \lambda_0 \\ \mu_0 \\ \delta_0 \end{pmatrix} \Rightarrow U_0 A = \theta_0 \Rightarrow (U_0;\theta_0) = A$$

$$U_1 = \begin{pmatrix} X(a)B & 0 & 0 \\ 0 & X(a)B & 0 \\ 0 & 0 & X(a)B \end{pmatrix}, \quad A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, \quad \theta_1 = \begin{pmatrix} \lambda_1 \\ \mu_1 \\ \delta_1 \end{pmatrix} \Rightarrow U_1 A = \theta_1 \Rightarrow \left(U_1;\theta_1\right) = A$$

If this last two equations are used in the expression (10), the $\widetilde{W}A = \widetilde{G}$ matrix is obtained. The unknowns matrix A is obtained from this increased matrix expression. If the components of this matrix are used in eq. (6), an approximate solution of the system of eq. (3) is obtained:

$$y_1(s) = a_{10}s^0 + a_{11}s^1 + a_{12}s^2 + ... + a_{1N}s^N$$

$$y_2(s) = a_{20}s^0 + a_{21}s^1 + a_{22}s^2 + ... + a_{2N}s^N$$

$$y_3(s) = a_{30}s^0 + a_{31}s^1 + a_{32}s^2 + ... + a_{3N}s^N.$$

The solution of differential equation system characterizing curve of constant breadth according to normal

The variables in the differential equation system (2) can be written:

$$\lambda = y_1, \quad \mu = y_2, \quad \delta = y_3$$

Thus, the system of eqs. (2) can be expressed in the form of a general system of differential equations:

$$\sum_{k=0}^{2}\sum_{j=1}^{3} P_{ij}^k(s) y_j^{(k)}(s) = h_i(s), \quad i = 1,2,3 \tag{11}$$

Here, the coefficient functions of the variables and their derivatives can be written explicitly as follows for the changing values of the i.

– *For i* = 1; $P_{11}^0(s) = -k_{n1}(s)$ $P_{12}^0(s) = 0$ $P_{13}^0(s) = k_{n2}(s)$

$P_{11}^1(s) = 0$ $P_{12}^1(s) = -1 + k_{n1}(s)$ $P_{13}^1(s) = 0$

$P_{11}^2(s) = 1$ $P_{12}^2(s) = 0$ $P_{13}^2(s) = 0$

– *For i* = 2; $P_{21}^0(s) = 0$ $P_{22}^0(s) = -\dfrac{k_{n2}^2(s)}{k_{n1}(s)} - k_{n1}(s)$ $P_{23}^0(s) = -\left(\dfrac{k_{n2}(s)}{k_{n1}(s)}\right)'$

$P_{21}^1(s) = 1 + k_{n1}(s)$ $P_{22}^1(s) = 0$ $P_{23}^1(s) = -\dfrac{k_{n2}(s)}{k_{n1}(s)} - k_{n2}(s)$

$P_{21}^2(s) = 0$ $P_{22}^2(s) = 1$ $P_{23}^2(s) = 0$

– *For i* = 3; $P_{31}^0(s) = k_{n2}(s)$ $P_{32}^0(s) = \left(\dfrac{k_{n2}(s)}{k_{n1}(s)}\right)'$ $P_{33}^0(s) = -\dfrac{k_{n2}^2(s)}{k_{n1}(s)}$

$P_{31}^1(s) = 0$ $P_{32}^1(s) = \dfrac{k_{n2}(s)}{k_{n1}(s)} + k_{n2}(s)$ $P_{33}^1(s) = 0$

$P_{31}^2(s) = 0$ $P_{32}^2(s) = 0$ $P_{33}^2(s) = 1$

with the help of these equalities, the components of matrix $P_k(\mathrm{s})$ are obtained:

$$P_{11}^k(s) = -k_{n1}(s) \qquad P_{12}^k(s) = -1 - k_{n1}(s) \qquad P_{13}^k(s) = k_{n2}(s)$$

$$P_{21}^k(s) = 1 + k_{n1}(s) \qquad P_{22}^k(s) = 1 - \frac{k_{n2}^2(s)}{k_{n1}(s)} - k_{n1}(s) \qquad P_{23}^k(s) = -\left(\frac{k_{n2}(s)}{k_{n1}(s)}\right)' - \frac{k_{n2}(s)}{k_{n1}(s)} - k_{n2}(s)$$

$$P_{31}^k(s) = k_{n2}(s) \qquad P_{32}^k(s) = \left(\frac{k_{n2}(s)}{k_{n1}(s)}\right)' + \frac{k_{n2}(s)}{k_{n1}(s)} + k_{n2}(s) \qquad P_{33}^k(s) = 1 - \frac{k_{n2}^2(s)}{k_{n1}(s)}$$

Further, the matrices $P_k(s), Y^k(s)$ and $H(s)$ are:

$$P_k(s) = \begin{pmatrix} P_{11}^k & P_{12}^k & P_{13}^k \\ P_{21}^k & P_{22}^k & P_{23}^k \\ P_{31}^k & P_{32}^k & P_{33}^k \end{pmatrix}, \quad Y^{(k)}(s) = \begin{pmatrix} y_1^{(k)}(s) \\ y_2^{(k)}(s) \\ y_3^{(k)}(s) \end{pmatrix}, \quad H(s) = \begin{pmatrix} h_1(s) \\ h_2(s) \\ h_3(s) \end{pmatrix} = \begin{pmatrix} -\left(\dfrac{1}{k_{n1}(s)}\right)' \\ -2 \\ 0 \end{pmatrix}$$

Using these matrices, the equation system (11) can be expressed as closed form:

$$\sum_{k=0}^{2} \{P_k(s) Y^{(k)}(s)\} = H(s) \tag{12}$$

Assume that there is an approximate solution in the form trimmed Taylor series (13) of this equation system (12) at sorting points:

$$s_0 = 0, s_1 = \pi, s_2 = 2\pi, \quad 0 \le s \le 2\pi$$

$$y_j(s) \cong \sum_{n=0}^{2} a_{jn} s^n \tag{13}$$

The following matrices are open for $N = 2$:

$$X(s) = \begin{pmatrix} 1 & s & s^2 \end{pmatrix}, \quad A_j = \begin{pmatrix} a_{j0} \\ a_{j1} \\ a_{j2} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

In that case, the approximate solution (13) and its derivatives can be expressed:

$$y_j(s) = X(s) A_j, \quad j = 1,2,3$$

$$y_j^{(k)}(s) = X(s) B^k A_j, \quad j = 1,2,3 \text{ and } k = 0,1,2$$

The derivatives of the approximate solution can be written clearly for the values $j = 1,2,3$.

$$\left.\begin{aligned} y_1^{(k)}(s) &= X(s) B^k A_1 \\ y_2^{(k)}(s) &= X(s) B^k A_2 \\ y_3^{(k)}(s) &= X(s) B^k A_3 \end{aligned}\right\} \tag{14}$$

The system (14) can be expressed in the closed matrix form:

$$Y^{(k)}(s) = \overline{X}(s)\overline{B}^{k} A, \qquad k = 0,1,2. \tag{15}$$

If this last equality is written in eq. (12), the following expression is obtained:

$$\sum_{k=0}^{2} \{P_k(s)\overline{X}(s)\overline{B}^{k^*} A\} = H(s) \tag{16}$$

Here, using the ordering points $s_0 = 0, s_1 = \pi, s_2 = 2\pi;\ P_k, \overline{X}, \overline{B}^{k^*}, H$ matrices are obtained:

$$P_k = \begin{pmatrix} P_k(0) & 0 & 0 \\ 0 & P_k(\pi) & 0 \\ 0 & 0 & P_k(2\pi) \end{pmatrix},\ \overline{X} = \begin{pmatrix} \overline{X}(0) & 0 & 0 \\ 0 & \overline{X}(\pi) & 0 \\ 0 & 0 & \overline{X}(2\pi) \end{pmatrix},\ \overline{B}^{k^*} = \begin{pmatrix} \overline{B}^{k} \\ \overline{B}^{k} \\ \overline{B}^{k} \end{pmatrix},\ H = \begin{pmatrix} H(0) \\ H(\pi) \\ H(2\pi) \end{pmatrix}$$

Now the W matrix can be calculated with the help of:

$$W = \sum_{k=0}^{2} P_k \overline{X}\overline{B}^{k^*}$$

The following equality is clear:

$$WA = H \Rightarrow (W;H) = A \tag{17}$$

On the other hand:

$$y_1(0) = \lambda_0 \Rightarrow X(0)A_1 = a_{10} = \lambda_0,\quad y_1'(0) = X(0)BA_1 = a_{11} = \lambda_1$$

$$y_2(0) = \mu_0 \Rightarrow X(0)A_2 = a_{20} = \mu_0,\quad y_2'(0) = X(0)BA_2 = a_{21} = \mu_1$$

$$y_3(0) = \delta_0 \Rightarrow X(0)A_3 = a_{30} = \delta_0,\quad y_3'(0) = X(0)BA_3 = a_{31} = \delta_1$$

the matrix equation of the conditions can be written as follows using the previous equations:

$$\begin{pmatrix} X(0) & 0 & 0 \\ 0 & X(0) & 0 \\ 0 & 0 & X(0) \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \mu_0 \\ \delta_0 \end{pmatrix}$$

$$\begin{pmatrix} X(0)B & 0 & 0 \\ 0 & X(0)B & 0 \\ 0 & 0 & X(0)B \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \mu_1 \\ \delta_1 \end{pmatrix}.$$

In addition, these matrix equations can be written in closed form:

$$U_0 = \begin{pmatrix} X(0) & 0 & 0 \\ 0 & X(0) & 0 \\ 0 & 0 & X(0) \end{pmatrix}, A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, \theta_0 = \begin{pmatrix} \lambda_0 \\ \mu_0 \\ \delta_0 \end{pmatrix} \Rightarrow U_0 A = \theta_0 \Rightarrow (U_0;\theta_0) = A$$

$$U_1 = \begin{pmatrix} X(0)B & 0 & 0 \\ 0 & X(0)B & 0 \\ 0 & 0 & X(0)B \end{pmatrix}, A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, \theta_1 = \begin{pmatrix} \lambda_1 \\ \mu_1 \\ \delta_1 \end{pmatrix} \Rightarrow U_1 A = \theta_1 \Rightarrow \left(U_1;\theta_1\right) = A$$

If this last two equalities are used in the expression (17), $\widetilde{W}A = \widetilde{H} \Rightarrow \left(\widetilde{W};\widetilde{H}\right) = A$ is obtained. Then the $\tilde{W}$ matrix is found:

$$\widetilde{W} = \begin{pmatrix} -k_{n1} & 0 & 2 & 0 & a & 0 & k_{n2} & 0 & 0 \\ 0 & -a & 0 & c & 0 & 2 & -b & d & 0 \\ k_{n2} & 0 & 0 & b & f & 0 & e & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

where a,b,c,d,e,f values are calculated:

$$a = -1 - k_{n1}(s), \quad b = \left[\frac{k_{n2}(s)}{k_{n1}(s)}\right]', \quad \mathrm{c} = \frac{k_{n2}^2(s) - k_{n1}^2(s)}{k_{n1}(s)}, \quad d = -\frac{k_{n2}(s)}{k_{n1}(s)} - k_{n2}(s)$$

$$e = -\frac{k_{n2}^2(s)}{k_{n1}(s)}, \quad f = \frac{k_{n2}(s) + k_{n2}^2(s)}{k_{n1}(s)}$$

Finally, with the help of the equation $A = \widetilde{W}^{-1}\widetilde{H}$, components of matrix of unknowns A are calculated:

$$a_{10} = \lambda_0, \quad a_{11} = \lambda_1, \quad a_{12} = \frac{-\left[\dfrac{1}{k_{n1}(0)}\right]'}{2} + \frac{k_{n1}(0)}{2}\lambda_0 - \frac{k_{n2}(0)}{2}\delta_0 + \frac{1 + k_{n1}(0)}{2}\mu_1$$

$$a_{20} = \mu_0, \quad a_{21} = \mu_1$$

$$a_{22} = -1 - \frac{\dfrac{k_{n2}^2(0) - k_{n1}^2(0)}{k_{n1}(0)}}{2}\mu_0 - \frac{\left[\dfrac{k_{n2}(0)}{k_{n1}(0)}\right]'}{2}\delta_0 + \frac{1 + k_{n1}(0)}{2}\lambda_1 + \frac{\dfrac{k_{n2}(0)}{k_{n1}(0)} + k_{n2}(0)}{2}\delta_1$$

$$a_{30} = \delta_0, \; a_{31} = \delta_1$$

$$a_{32} = -\frac{k_{n2}(0)}{2}\lambda_0 - \frac{\left[\dfrac{k_{n2}(0)}{k_{n1}(0)}\right]'}{2}\mu_0 + \frac{\dfrac{k_{n2}^2(0)}{k_{n1}(0)}}{2}\delta_0 - \frac{\dfrac{k_{n2}(0) + k_{n2}^2(0)}{k_{n1}(0)}}{2}\mu_1$$

If these components of the unknown matrix are used in eq. (13), the solution set of the equation system is obtained:

$$y_1 = \lambda = a_{10} + a_{11}s^1 + a_{12}s^2$$

$$\lambda = \left\{ \frac{-\left[\dfrac{1}{k_{n1}(0)}\right]'}{2} + \frac{k_{n1}(0)}{2}\lambda_0 - \frac{k_{n2}(0)}{2}\delta_0 + \frac{1+k_{n1}(0)}{2}\mu_1 \right\} s^2 + \lambda_1 s + \lambda_0$$

$$y_2 = \mu = a_{20} + a_{21}s^1 + a_{22}s^2$$

$$\mu = \left\{ -1 - \frac{\dfrac{k_{n2}^2(0) - k_{n1}^2(0)}{k_{n1}(0)}}{2}\mu_0 - \frac{\left[\dfrac{k_{n2}(0)}{k_{n1}(0)}\right]'}{2}\delta_0 + \frac{1+k_{n1}(0)}{2}\lambda_1 + \frac{\dfrac{k_{n2}(0)}{k_{n1}(0)} + k_{n2}(0)}{2}\delta_1 \right\} \mathrm{s}^2 + \mu_1 s + \mu_0$$

$$y_3 = \delta = a_{30} + a_{31}s^1 + a_{32}s^2$$

$$\delta = \left\{ -\frac{k_{n2}(0)}{2}\lambda_0 - \frac{\left[\dfrac{k_{n2}(0)}{k_{n1}(0)}\right]'}{2}\mu_0 + \frac{\dfrac{k_{n2}^2(0)}{k_{n1}(0)}}{2}\delta_0 - - \frac{\dfrac{k_{n2}(0) + k_{n2}^2(0)}{k_{n1}(0)}}{2}\mu_1 \right\} s^2 + \delta_1 s + \delta_0$$

Conclusion and suggestions

In this study, the coefficients which determine the curve of constant breadth according to normal are calculated approximately.

Knowing the approximate value of these coefficients is very important in order to create such a curve in space or to determine if a given curve is suitable for this type.

In addition, the approximate values of the solutions obtained for $s_0 = 0$ and $s_1 = 2\pi$ are equal. This is an important point for a curve to be a closed curve.

In this study, the concept of curve of constant breadth is approached with a different interpretation. More specifically, in this study instead of the tangents at the opposite points where the breadth has a constant value on the curve, the situation of parallel and opposite direction normal vectors is examined.

The curves obtained by this idea are called *curve of constant breadth according to normal vector*. In this direction, firstly, a differential equation system with variable coefficients is obtained which characterizes these curves. Then, in order to solve this type of equation systems, Taylor matrix method based on collocation points is given. Finally, an approximate solution of the system of differential equation obtained is calculated by this method given.

In this study, the concept of constant breadth is interpreted according to the normal vector. With similar thinking, the concept of curve of constant breadth according to binormal vector or darboux vector can be mentioned. In addition, solutions of differential equations or equation systems to be obtained from this interpretation can be made by different methods. Thus, the concept of constant breadth in the differential geometry can be discussed in more detail in different ways.

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