CONFORMABLE FRACTIONAL DERIVATIVE AND ITS APPLICATION TO FRACTIONAL KLEIN-GORDON EQUATION

by

Kangle WANG and Shaowen YAO*

School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, China

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This paper adopts conformable fractional derivative to describe the fractional Klein-Gordon equations. The conformable fractional derivative is a new simple well-behaved definition. The fractional complex transform and variational iteration method are used to solve the fractional equation. The result shows that the proposed technology is very powerful and efficient for fractional differential equations.

Key words: *conformable fractional derivative, fractional complex transform, fractional Klein-Gordon equation, variational iteration method*

Introduction

In the past two decades, the fractional derivative has gained considerable attention of physicists, mathematicians, and engineers. Many kinds of definitions of fractional derivatives have been proposed [1-4]. The most popular are:

– The Riemann-Liouville definition is [1, 4]:

$$\mathrm{D}_x^{\alpha}[f(x)]=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_0^x (x-t)^{n-\alpha-1}f(t)\mathrm{d}t$$

– The Caputo's definition is [1, 4]:

$$\mathrm{D}_x^{\alpha}[f(x)]=\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\mathrm{d}t$$

The previous two definitions satisfy the property that the fractional derivative is a linear. However, they have lost some of the basic properties that usual derivatives have such as the product rule and the chain rule. Recently, the authors in [5, 6] define the conformable fractional derivative, which is a new well-behaved simple definition. The new definition of fractional derivative successfully overcomes some of the setbacks of the traditional definitions. So, the conformable fractional derivative is a powerful tool to describe all kinds of nature phenomena.

In this paper, the fractional Klein-Gordon equations are described in the sense of the conformable fractional derivative. We use the He's variational iteration method (VIM) [7] and fractional complex transform method [8-10] to solve the fractional Klein-Gordon equations. The aim of the paper is that the fractional complex transform can convert the fractional differ-

* Corresponding author, e-mail: ysw140917@163.com

ential equation into a classical ordinary differential equation, and that we use the VIM to solve the differential equation.

We consider the fractional Klein-Gordon equation:

$$u_t^{2\alpha} - u_{xx} = -F(u)$$

with the initial conditions:

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

where $F(u)$ is a linear or non-linear function. The equation has attracted much attention in studying solitons and condensed matter physics, in investigating the interaction of solitons in collisionless plasma, the recurrence of initial states, and in examining the non-linear wave equations [11].

Conformable fractional derivative

Let a function $f:[0,\infty) \to R$. Then the conformable fractional derivative of f of order α is defined [5]:

$$\mathrm{D}_t^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0, \alpha \in (0,1)$. If f is α-differentiable and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then we define:

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t)$$

Let $0 < \alpha < 1$ and $n \in \{1,2,3,\ldots\}$. Then the conformable fractional derivative of order n is given [6]:

$$\mathrm{D}^{n\alpha} f = \mathrm{D}^\alpha \mathrm{D}^\alpha \mathrm{D}^\alpha \ldots \mathrm{D}^\alpha f$$

The VIM

Consider the following non-linear differential equation:

$$Lu + Nu = g(x) \tag{1}$$

where N is a non-linear operator, L is a linear operator, and $g(x)$ is a homogeneous term. With use of the VIM [7], a correct functional for eq. (1) is structured:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{Lu_n(\xi) + N\hat{u}_n(\xi) - g(\xi)\}\mathrm{d}\xi \tag{2}$$

where λ is a Lagrange multiplier, which can be identified optimally via variational theory, and the second term on the right is called the correction, and $\hat{u}_n$ is considered as a restricted variation, *i. e.* $\delta\hat{u}_n = 0$.

Fractional complex transform

Consider the following general fractional differential equation:

$$\begin{cases} f(u, u_t{}^\alpha, u_x{}^\beta, u_y{}^\gamma, u_z{}^\lambda, u_t{}^{2\alpha}, u_x{}^{2\beta}, u_y{}^{2\gamma}, u_z{}^{2\lambda} \ldots) = 0 \\ 0 < \alpha \le 1, 0 < \beta \le 1, 0 < \gamma \le 1, 0 < \lambda \le 1, \end{cases} \tag{3}$$

where $u_t^\alpha = \partial^\alpha u/\partial t^\alpha$ denotes conformable fractional derivation, and u is a continuous (but not necessarily differentiable) function.

The fractional complex transform reads:

$$T = \frac{pt^\alpha}{\Gamma(1+\alpha)}, \quad X = \frac{qx^\beta}{\Gamma(1+\beta)}, \quad Y = \frac{ky^\gamma}{\Gamma(1+\gamma)}, \quad Z = \frac{lz^\lambda}{\Gamma(1+\lambda)}$$

where $p, q, k,$ and l are the unknown constants. Using the basic properties of the conformable fractional derivation and above transform, we have:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = p\frac{\partial u}{\partial T}, \quad \frac{\partial^\beta u}{\partial x^\beta} = q\frac{\partial u}{\partial X}, \quad \frac{\partial^\gamma u}{\partial y^\gamma} = k\frac{\partial u}{\partial Y}, \quad \frac{\partial^\lambda u}{\partial z^\lambda} = l\frac{\partial u}{\partial Z}$$

Therefore, we can easily convert the fractional differential equations into the classical ordinary differential equations.

Numerical applications

Consider the non-linear fractional Klein-Gordon equation:

$$u_t^{2\alpha} - u_{xx} = -u^2 \tag{4}$$

subject to the initial conditions:

$$u(x,0) = 1 + \sin(x), \quad u_t(x,0) = 0 \tag{5}$$

With the aid of the fractional complex transform $T = t^\alpha/\Gamma(1+\alpha)$, eq. (4) convert into the following form:

$$u_{TT} - u_{xx} = -u^2 \tag{7}$$

subject to the initial conditions:

$$u(x,0) = 1 + \sin(x), \quad u_T(x,0) = 0 \tag{8}$$

Using the VIM to solve eq. (7), we construct a correction functional:

$$u_{n+1}(x,T) = u_n(x,T) + \int_0^T \lambda\{u_{n\tau\tau}(x,\tau) - \hat{u}_{nxx}(x,\tau) + \hat{u}_n{}^2(x,\tau)\}\mathrm{d}\tau \tag{9}$$

The stationary conditions can be obtained:

$$\lambda'' - \lambda = 0, \quad 1 - \lambda'\big|_{\tau=T} = 0, \quad \lambda\big|_{\tau=T} = 0$$

We have:

$$\lambda(\tau) = \tau - T$$

Therefore, we obtain the iteration formula:

$$u_{n+1}(x,T) = u_n(x,T) + \int_0^T (\tau - T)\{u_{n\tau\tau}(x,\tau) - u_{nxx}(x,\tau) + u_n{}^2(x,\tau)\}\mathrm{d}\tau \tag{10}$$

Using the iteration formula eq. (9) with the initial approximation:

$$u_0(x,T)=u(x,0)+Tu_T(x,0)=1+\sin(x)$$

we have the following results:

$$u_1(x,T)=1+\sin x-\frac{T^2}{2!}(1+3\sin x+\sin^2 x)$$

$$u_2(x,T)=1+\sin x-\frac{T^2}{2!}(1+3\sin x+\sin^2 x)+\frac{T^4}{4!}(11+12\sin x+2\sin^2 x)\sin x$$

$$u_3(x,T)=1+\sin x-\frac{T^2}{2!}(1+3\sin x+\sin^2 x)+\frac{T^4}{4!}(11+12\sin x+2\sin^2 x)\sin x+$$
$$+\frac{T^6}{6!}(18-57\sin x-160\sin^2 x-82\sin^3 x-10\sin^4 x) \tag{11}$$

Substituting eq. (6) into eq. (11), we obtain the solution of eq. (4):

$$u(x,T)=1+\sin x-\frac{\left[\frac{t^\alpha}{\Gamma(1+\alpha)}\right]^2}{2!}(1+3\sin x+\sin^2 x)+\frac{\left[\frac{t^\alpha}{\Gamma(1+\alpha)}\right]^4}{4!}(11+12\sin x+2\sin^2 x)\sin x+$$
$$+\frac{\left[\frac{t^\alpha}{\Gamma(1+\alpha)}\right]^6}{6!}(18-57\sin x-160\sin^2 x-82\sin^3 x-10\sin^4 x)+\ldots$$

Conclusion

Based on the conformable fractional derivative, we combined the fractional complex transform and VIM to find the approximate solution of the fractional Klein-Gordon equation. The result shows that the proposed method is very powerful, efficient and easy mathematical methods for solving the linear and non-linear fractional differential equations in science and engineering.

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Nomenclature

t – time, [s]
x – space co-ordinate, [m]

Greek symbol

α – fractional order, [–]

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