AN ANALYTICAL SOLUTION FOR SOLVING A NEW WAVE EQUATION WITHIN LORENZO-HARTLEY KERNEL

by

Feng GAO [a,b]

[a] State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou, China
[b] School of Mechanics and Civil Engineering, China University of Mining and Technology, Xuzhou, China

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In this article we investigate the general fractional-order derivatives of the Riemann-Liouville type via Lorenzo-Hartley kernel, general fractional-order integrals and the new general fractional-order wave equation defined on the definite domain with the analytical soluton.

Key words: *general fractional-order derivative, general fractional-order integral, general fractional-order wave equation, analytical solution*

Introduction

Theory of general fractional-order calculus, as new mathematical tool, has successfully used to investigate the mathematical model in mathematical physics. The general fractional derivatives via the special kernels have the properties, such as memory and integro-differential represtations with the non-singular behaviors, were developed in [1]. For example, these kernels, proposed by Caputo and Fabrizio [2], Mittag-Leffler [3], Rabotnov [4], Miller and Ross [5], and so on, have been investigated:

$$\Omega\left(\lambda t^{\alpha}\right)=\begin{cases} e^{-\lambda x} & [2] \\ E_{\alpha}\left(\lambda t^{\alpha}\right)=\sum_{\kappa=0}^{\infty}\frac{\lambda^{\kappa}t^{\kappa\alpha}}{\Gamma\left(\kappa\alpha+1\right)} & [3] \\ G_{\alpha,\upsilon}\left(\lambda t^{\alpha}\right)=\sum_{\kappa=0}^{\infty}\frac{\lambda^{\kappa}t^{\kappa\alpha-1+\upsilon}}{\Gamma\left(\kappa\alpha+\upsilon\right)} & [4] \\ \mathfrak{M}_{\alpha}\left(\lambda t\right)=t^{\alpha}\sum_{\kappa=0}^{\infty}\frac{\lambda^{\kappa}t^{\kappa}}{\Gamma\left(\kappa+1+\alpha\right)} & [5] \end{cases}$$

The tasks have been paid attention to many researchers in the mathematical physics in [6, 7].

Recently, the Lorenzo and Hartley have proposed a special function (called the Lorenzo-Hartley function, see [8]). The general fractional-order calculus operators via the Loren-

Author’s e-mail: fgao@cumt.edu.cn

zo-Hartley function were investigated in [1]. The main goal of the article is to model the wave equation within general fractional derivative via the Lorenzo-Hartley kernel. We propose the wave equation within general fractional derivative of the Riemann-Liouville type via Lorenzo-Hartley kernel and investigate the analytical solution with the special function.

Basic theory of the general fractional calculus via Lorenzo-Hartley kernel

In this section, we give the concepts and properties of the general fractional-order derivatives and general fractional integrals. In order to show the properties of them, we start with the Lorenzo-Hartley function.

The Lorenzo-Hartley function can be expressed [1]:

$$\Theta_\gamma\left(\nu t^\gamma\right)=\sum_{\kappa=0}^{\infty}\frac{\nu^\kappa t^{(\kappa+1)\gamma-1}}{\Gamma\left[(\kappa+1)\gamma\right]} \tag{1}$$

and its Laplace transform [1, 8]:

$$\Im\left\{\Theta_\gamma\left(\nu t^\gamma\right)\right\}=\sum_{\kappa=0}^{\infty}\frac{\nu^\kappa}{s^{(\kappa+1)\gamma}}=s^{-\gamma}\left(1-\nu s^{-\gamma}\right)^{-1}\quad\left(\left|\nu s^{-\gamma}\right|<1\right) \tag{2}$$

where the Laplace transform operator is denoted [1]:

$$\Im\left[l(t)\right]=l(s)=\int_0^\infty \mathrm{e}^{-st}l(t)\,\mathrm{d}t \tag{3}$$

General fractional-order derivatives of the Riemann-Liouville type within Lorenzo-Hartley kernel

The left-sided general fractional-order derivative of the Riemann-Liouville type within the Lorenzo-Hartley kernel is given [1]:

$$_{\mathrm{RLY}}\mathrm{D}_{a+}^{\gamma,\nu}h(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t\Theta_\gamma\left[-\nu(t-\tau)^\gamma\right]h(\tau)\,\mathrm{d}\tau \tag{4}$$

and the right-sided general fractional-order derivative of the Riemann-Liouville type within the Lorenzo-Hartley kernel:

$$_{\mathrm{RLY}}\mathrm{D}_{b-}^{\gamma,\nu}h(t)=-\frac{\mathrm{d}}{\mathrm{d}t}\int_t^b\Theta_\gamma\left[-\nu(\tau-t)^\gamma\right]h(\tau)\,\mathrm{d}\tau \tag{5}$$

The left-sided general fractional-order derivative of the Riemann-Liouville type within the Lorenzo-Hartley kernel is given [1]:

$$_{\mathrm{RLY}}\mathrm{D}_{a+}^{\gamma,n,\nu}h(t)=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^t\Theta_\gamma\left[-\nu(\tau-t)^\gamma\right]h(\tau)\,\mathrm{d}\tau \tag{6}$$

and the right-sided general fractional-order derivative of the Riemann-Liouville type within the Lorenzo-Hartley kernel:

$$ {}_{\mathrm{RLY}}\mathrm{D}_{b-}^{\gamma,n,\nu}h(t)=(-1)^{n}\frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}}\int_{t}^{\infty}\Theta_{\gamma}\left[-\nu(t-\tau)^{\gamma}\right]h(\tau)\mathrm{d}\tau \tag{7}$$

where n is positive integer numbers.

When $a=0$, the general fractional-order derivatives of the Riemann-Liouville type within the Lorenzo-Hartley kernel is given [1]:

$$ {}_{\mathrm{RLY}}\mathrm{D}_{0+}^{\gamma,\nu}h(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int_{0}^{t}\Theta_{\gamma}\left[-\nu(t-\tau)^{\gamma}\right]h(\tau)\mathrm{d}\tau \tag{8}$$

and

$$ {}_{\mathrm{RLY}}\mathrm{D}_{0+}^{\gamma,n,\nu}h(t)=\frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}}\int_{0}^{t}\Theta_{\gamma}\left[-\nu(\tau-t)^{\gamma}\right]h(\tau)\mathrm{d}\tau \tag{9}$$

General fractional-order integrals

The left-sided general fractional-order integral is defined [1]:

$$ {}_{\mathrm{R}}I_{a+}^{\gamma,n,\nu}g(t)=\int_{a}^{t}(t-\tau)^{n-\gamma-1}E_{\gamma,n-\gamma}^{-1}\left[-\nu(t-\tau)^{\gamma}\right]g(\tau)\mathrm{d}\tau \tag{10}$$

and the right-sided general fractional-order integral:

$$ {}_{\mathrm{R}}I_{b-}^{\gamma,n,\nu}g(t)=(-1)^{n}\int_{t}^{b}(\tau-t)^{n-\gamma-1}E_{\gamma,n-\gamma}^{-1}\left[-\nu(\tau-t)^{\gamma}\right]g(\tau)\mathrm{d}\tau \tag{11}$$

For $n=1$, the left-sided general fractional-order integral is defined [1]:

$$ {}_{\mathrm{R}}I_{a+}^{\gamma,\nu}g(t)={}_{\mathrm{R}}I_{a+}^{\gamma,1,\nu}g(t)=\int_{a}^{t}(t-\tau)^{-\gamma}E_{\gamma,1-\gamma}^{-1}\left[-\nu(t-\tau)^{\gamma}\right]g(\tau)\mathrm{d}\tau \tag{12}$$

and the right-sided general fractional-order integral:

$$ {}_{\mathrm{R}}I_{b-}^{\gamma,\nu}g(t)={}_{\mathrm{A}}^{\mathrm{R}}I_{b-}^{\gamma,1,\nu}g(t)=-\int_{t}^{b}(\tau-t)^{-\gamma}E_{\gamma,1-\gamma}^{-1}\left[-\nu(\tau-t)^{\gamma}\right]g(\tau)\mathrm{d}\tau \tag{13}$$

For $a=0$, the general fractional-order integrals are defined [1]:

$$ {}_{\mathrm{R}}I_{0+}^{\gamma,n,\nu}g(t)=\int_{0}^{t}(t-\tau)^{n-\gamma-1}E_{\gamma,n-\gamma}^{-1}\left[-\nu(t-\tau)^{\gamma}\right]g(\tau)\mathrm{d}\tau \tag{14}$$

and

$$ {}_{\mathrm{R}}I_{0+}^{\gamma,\nu}g(t)={}_{\mathrm{R}}I_{0+}^{\gamma,1,\nu}g(t)=\int_{0}^{t}(t-\tau)^{-\gamma}E_{\gamma,1-\gamma}^{-1}\left[-\nu(t-\tau)^{\gamma}\right]g(\tau)\mathrm{d}\tau \tag{15}$$

The Laplace transforms of the general fractional calculus operators

The Laplace transforms of the general fractional derivatives and general fractional integrals are given [1]:

$$L\left\{{}_{\mathrm{RLY}}\mathrm{D}_{0+}^{\gamma,n,\nu} g(t)\right\} = s^{n-\gamma}\left(1+\nu s^{-\gamma}\right)^{-1} g(s) - \sum_{\mu=0}^{n-1} s^{n-\mu-1}\left(\mathrm{D}^{\mu}\left[{}_{\mathrm{R}}I_{0+}^{\gamma,n,\nu} g(+0)\right]\right) \tag{16}$$

$$L\left\{{}_{\mathrm{RLY}}\mathrm{D}_{0+}^{\gamma,\nu} g(t)\right\} = s^{1-\gamma}\left(1+\nu s^{-\gamma}\right)^{-1} g(s) - {}_{\mathrm{R}}I_{0+}^{\gamma,\nu} g(+0) \tag{17}$$

$$\Im\left\{{}_{\mathrm{R}}I_{0+}^{\gamma,n,\nu} g(t)\right\} = s^{\gamma-n}\left(1+\nu s^{-\gamma}\right) g(s) \tag{18}$$

and

$$\Im\left\{{}_{\mathrm{R}}I_{0+}^{\gamma,\nu} g(t)\right\} = s^{\gamma-1}\left(1+\nu s^{-\gamma}\right) g(s) \tag{19}$$

A new wave equation within general fractional derivative via Lorenzo-Hartley kernel

Let us consider the new wave equation in the 1-D cases containing the general fractional derivative via Lorenzo-Hartley kernel defined on the definite domain:

$${}_{\mathrm{RLY}}\partial_{0+}^{\gamma,2,\nu}\Re(x,t) = \frac{\partial^2}{\partial x^2}\Re(x,t) \quad (2 > x > 0,\ t > 0) \tag{20}$$

with the initial and boundary conditions:

$$\Re(0,t) = 0, \quad \Re(2,t) = 0, \quad {}_{\mathrm{R}}I_{0+}^{\gamma,2,\nu}\Re(x,+0) = 0, \quad {}_{\mathrm{R}}I_{0+}^{\gamma,\nu}\Re(x,+0) = \sin(\pi x) \tag{21}$$

where

$${}_{\mathrm{RLY}}\partial_{0+}^{\gamma,2,\nu}\Re(x,t) = \frac{\partial^2}{\partial t^2}\int_0^t \Theta_\gamma\left[-\nu(\tau-t)^\gamma\right]\Re(x,\tau)\,\mathrm{d}\tau \tag{22}$$

and

$${}_{\mathrm{R}}I_{0+}^{\gamma,\nu}\Re(x,+0) = {}_{\mathrm{R}}I_{0+}^{\gamma,1,\nu}\Re(x,+0) \tag{23}$$

With use of the Laplace transform, we easily obtain:

$$\frac{\partial^2}{\partial x^2}\Re(x,s) = s^{1-\gamma}\left(1+\nu s^{-\gamma}\right)^{-1}\Re(x,s) - \sin(\pi x) \tag{24}$$

Setting the general solution of eq. (24) given:

$$\Re(x,s) = \aleph_1\sin(\pi x) + \aleph_2\cos(\pi x) \tag{25}$$

where $\aleph_1$ and $\aleph_2$ are the constants, we have:

$$\left[-\pi^2 - s^{1-\gamma}\left(1+\nu s^{-\gamma}\right)^{-1}\right]\left[\aleph_1\sin(\pi x) + \aleph_2\cos(\pi x)\right] = -\sin(\pi x) \tag{26}$$

which leads to

$$\aleph_2 = 0 \quad \text{and} \quad \aleph_1 = \frac{1}{\pi^2 + s^{1-\gamma}\left(1+\nu s^{-\gamma}\right)^{-1}} \tag{27}$$

Thus, we have:

$$\Re(x,s)=\frac{\sin(\pi x)}{\pi^2+s^{1-\gamma}\left(1+\nu s^{-\gamma}\right)^{-1}}=\frac{\sin(\pi x)}{\pi^2}\left[\frac{1}{1+\frac{1}{\pi^2}s^{1-\gamma}\left(1+\nu s^{-\gamma}\right)^{-1}}\right] \tag{28}$$

Thus, we have:

$$\Re(x,s)=\frac{\sin(\pi x)}{\pi^2}\sum_{i=0}^{n}\left(-\frac{1}{\pi^2}\right)^i s^{(1-\gamma)i}\left(1+\nu s^{-\gamma}\right)^{-i},\quad \left|\frac{1}{\pi^2}s^{1-\gamma}\left(1+\nu s^{-\gamma}\right)^{-1}\right|<1 \tag{29}$$

Thus, we get the analytical solution for the new wave equation in the 1-D cases containing the general fractional derivative via Lorenzo-Hartley kernel defined on the definite domain as follows:

$$\Re(x,t)=\frac{\sin(\pi x)}{\pi^2}\sum_{i=0}^{n}\left(-\frac{1}{\pi^2}\right)^i t^{\gamma i-i-1}E^{i}_{\gamma,\gamma i-i}\left(-\nu t^{\gamma}\right) \tag{30}$$

Conclusion

In the present work, we have investigated the theory of the general fractional calculus via Lorenzo-Hartley kernel. Moreover, the general fractional-order model for the wave defined on the definite domain within general fractional derivative via Lorenzo-Hartley kernel was proposed and its solution were presented with the aid of the Laplace transforms and the special functions. The obtained results are presented to describe the general fractional-order models arising in mathematical physics.

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Nomenclature

x – space co-ordinate, [m]
t – time co-ordinate, [s]

Greek symbol

γ – fractional order, [–]

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