AN ANALYTICAL SOLUTION FOR SOLVING A NEW GENERAL FRACTIONAL-ORDER MODEL FOR WAVE IN MINING ROCK

by

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In this paper, we consider the general fractional-order derivatives of the Liouville-Sonine-Caputo and Liouville-Sonine type containing the Lorenzo-Hartley kernel. A general fractional-order model for the wave equation with the analytical solution is discussed in detail. The general fractional-order formula is accurate and efficient for description of the complex, power-law and memory behaviors for the mining rock.

Key words: *general fractional-order derivative, Lorenzo-Hartley kernel, wave equation, analytical solution, mining rock*

Introduction

The linear and non-linear models for the wave motion have been successfully observed an investigated in the mining rock [1-3]. For example, the linear Guyer-McCall-Boitnott equation of the motion describe 1-D wave propagation in the mining rock was reported [4]. The linear McCall equation of the motion describe 1-D wave propagation in the mining rock was considered [5]. The linear McCall-Guyer wave equation of the motion in the mining rock was presented [6].

Recently, the general fractional-order derivatives of the Liouville-Sonine and Liouville-Sonine-Caputo types containing the Lorenzo-Hartley kernel was proposed [7]. The main goal of the paper is to the linear McCall equation of the motion describe 1-D wave propagation in the mining rock in the sense of the general fractional-order derivative of the Liouville-Sonine-Caputo type.

General fractional-order derivatives containing the Lorenzo-Hartley kernel

In this section, we investigate the general fractional-order calculus, inclusive of the general fractional-order derivatives and integrals [7].

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General fractional-order integral operators containing the Lorenzo-Hartley kernel

The left-sided general fractional-order integral operator containing the Lorenzo-Hartley kernel are expressed [7]:

$$ {}_{\mathrm{H}}I_{a+}^{\alpha,\gamma} k(t) = \int_a^t \aleph_\alpha \left[-\gamma (t-\tau)^\alpha \right] k(\tau) \mathrm{d}\tau \tag{1} $$

and the right-sided general fractional-order integral operator containing the Lorenzo-Hartley kernel:

$$ {}_{\mathrm{H}}I_{b-}^{\alpha,\gamma} k(t) = \int_t^b \aleph_\alpha \left[-\gamma (\tau-t)^\alpha \right] k(\tau) \mathrm{d}\tau \tag{2} $$

respectively, where the Lorenzo-Hartley function is defined [8]:

$$ \aleph_\alpha \left(\gamma t^\alpha \right) = \sum_{\kappa=0}^{\infty} \frac{\gamma^\kappa t^{(\kappa+1)\alpha-1}}{\Gamma\left[(\kappa+1)\alpha \right]} \tag{3} $$

If the Laplace transform operator is defined [7]:

$$ \wp\left[\iota(t) \right] = \iota(s) = \int_0^\infty \mathrm{e}^{-st} \iota(t) \mathrm{d}t \tag{4} $$

then the Laplace transform of eq. (3) can be given [7]:

$$ \wp\left\{ \aleph_\alpha \left(\gamma t^\alpha \right) \right\} = s^{-\alpha} \left(1 - \gamma s^{-\alpha} \right)^{-1} \quad \left(\left| v s^{-\alpha} \right| < 1 \right) \tag{5} $$

When $a = 0$, the general fractional-oder integral operator containing the Lorenzo-Hartley kernel are expressed [7]:

$$ {}_{\mathrm{H}}I_{0+}^{\alpha,\gamma} k(t) = \int_0^t \aleph_\alpha \left[-\gamma (t-\tau)^\alpha \right] k(\tau) \mathrm{d}\tau \tag{6} $$

General fractional-order derivatives of the Liouville-Sonine type

The left-sided general fractional-order derivative of the Liouville-Sonine type containing the Lorenzo-Hartley kernel is defined [7]:

$$ {}_{\mathrm{H}}^{\mathrm{LS}}\mathrm{D}_{a+}^{\alpha,\gamma} k(t) = {}_{\mathrm{H}}I_{a+}^{\alpha,\gamma} \left[k^{(1)}(t) \right] = \int_a^t \aleph_\alpha \left[-\gamma (t-\tau)^\alpha \right] k^{(1)}(\tau) \mathrm{d}\tau \tag{7} $$

and the right-sided general fractional-order derivative of the Liouville-Sonine type containing the Lorenzo-Hartley kernel [7]:

$$ {}_{\mathrm{H}}^{\mathrm{LS}}\mathrm{D}_{b-}^{\alpha,\lambda} f(t) = {}_{\mathrm{H}}I_{b-}^{\alpha,\gamma} \left[-f^{(1)}(t) \right] = -\int_t^b \aleph_\alpha \left[-\gamma (\tau-t)^\alpha \right] f^{(1)}(\tau) \mathrm{d}\tau \tag{8} $$

When $a = 0$, eq. (5) can be re-written [7]:

$$ {}_{\mathrm{H}}^{\mathrm{LS}}\mathrm{D}_{0+}^{\alpha,\gamma} k(t) = {}_{\mathrm{H}}I_{0+}^{\alpha,\gamma} \left[k^{(1)}(t) \right] = \int_0^t \aleph_\alpha \left[-\gamma (t-\tau)^\alpha \right] k^{(1)}(\tau) \mathrm{d}\tau \tag{9} $$

and the Laplace transform of eq. (9) [7]:

$$ \wp\left\{ {}_{\mathrm{H}}^{\mathrm{LS}}\mathrm{D}_{0+}^{\alpha,\gamma} k(t) \right\} = s^{-\alpha} \left(1 - \gamma s^{-\alpha} \right)^{-1} \left[s k(s) - k(0) \right] \tag{10} $$

General fractional-order derivatives of the Liouville-Sonine-Caputo type

The left-sided general fractional-order derivative of the Liouville-Sonine-Caputo type containing the Lorenzo-Hartley kernel is defined [7]:

$$ {}^{\mathrm{LSC}}_{\ \mathrm{H}}\mathrm{D}^{\alpha,\kappa,\gamma}_{a+}k(t) = {}_{\mathrm{H}}I^{\alpha,\gamma}_{a+}\left[k^{(\kappa)}(t)\right] = \int_a^t \aleph_\alpha\left[-\gamma(t-\tau)^\alpha\right]k^{(\kappa)}(\tau)\mathrm{d}\tau \tag{11} $$

and the right-sided general fractional-order derivative of the Liouville-Sonine-Caputo type containing the Lorenzo-Hartley kernel [7]:

$$ {}^{\mathrm{LSC}}_{\ \mathrm{H}}\mathrm{D}^{\alpha,\lambda}_{b-}k(t) = {}_{\mathrm{H}}I^{\alpha,\gamma}_{b-}\left[(-1)^\kappa k^{(\kappa)}(t)\right] = (-1)^\kappa \int_t^b \aleph_\alpha\left[-\gamma(\tau-t)^\alpha\right]k^{(\kappa)}(\tau)\mathrm{d}\tau \tag{12} $$

When $a = 0$, eq. (8) becomes [7]:

$$ {}^{\mathrm{LSC}}_{\ \mathrm{H}}\mathrm{D}^{\alpha,\kappa,\gamma}_{0+}k(t) = {}_{\mathrm{H}}I^{\alpha,\gamma}_{0+}\left[k^{(\kappa)}(t)\right] = \int_0^t \aleph_\alpha\left[-\gamma(t-\tau)^\alpha\right]k^{(\kappa)}(\tau)\mathrm{d}\tau \tag{13} $$

and the Laplace transform of eq. (13) [7]:

$$ \wp\left\{{}^{\mathrm{LSC}}_{\ \mathrm{H}}\mathrm{D}^{\alpha,\kappa,\gamma}_{0+}k(t)\right\} = s^{-\alpha}\left(1-\gamma s^{-\alpha}\right)^{-1}\left[s^\kappa k(s) - \sum_{j=1}^{\kappa} s^{\kappa-j}k^{(j-1)}(0)\right] \tag{14} $$

General fractional-order integrals

The left-sided general fractional-order integral is defined [7]:

$$ {}^{\mathrm{R}}_{\mathrm{H}}I^{\alpha,\kappa,\gamma}_{a+}k(t) = \int_a^t (t-\tau)^{\kappa-\alpha-1}E^{-1}_{\alpha,\kappa-\alpha}\left[-\gamma(t-\tau)^\alpha\right]k(\tau)\mathrm{d}\tau \tag{15} $$

and the right-sided general fractional-order integral [7]:

$$ {}^{\mathrm{R}}_{\mathrm{H}}I^{\alpha,\kappa,\gamma}_{b-}k(t) = (-1)^\kappa \int_t^b (\tau-t)^{\kappa-\alpha-1}E^{-1}_{\alpha,\kappa-\alpha}\left[-\gamma(\tau-t)^\alpha\right]k(\tau)d\tau \tag{16} $$

When $a = 0$, eq. (15) can be given [7]:

$$ {}^{\mathrm{R}}_{\mathrm{H}}I^{\alpha,\kappa,\gamma}_{0+}k(t) = \int_0^t (t-\tau)^{\kappa-\alpha-1}E^{-1}_{\alpha,\kappa-\alpha}\left[-\gamma(t-\tau)^\alpha\right]k(\tau)d\tau \tag{17} $$

with the Laplace transform [7]:

$$ \wp\left\{{}^{\mathrm{R}}_{\mathrm{H}}I^{\alpha,\kappa,\gamma}_{0+}k(t)\right\} = s^{\alpha-\kappa}\left(1+\gamma s^{-\alpha}\right)k(s) \tag{18} $$

For more relations between the general fractional-order derivatives and integrals, [7].

A general fractional-order wave model with the analytical solution

We now consider the general fractional-order wave model within the general fractional-order derivative of the Liouville-Sonine-Caputo type containing the Lorenzo-Hartley kernel:

$$ {}^{\mathrm{LSC}}_{\ \mathrm{H}}\partial^{\alpha,2,\gamma}_{0+}\Pi(x,t) = \frac{\partial^2\Pi(x,t)}{\partial x^2} + \sin(\pi x) \quad (0 < x < 1, t > 0) \tag{19} $$

with the initial and boundary conditions:

$$ \Pi(0,t) = 0,\ \Pi(1,t) = 0,\ \Pi(x,0) = 0,\ \text{and}\ \frac{\partial\Pi(x,0)}{\partial x} = 0 \tag{20} $$

where the general fractional-order partial derivative is defined:

$$ {}_{\mathrm{H}}^{\mathrm{LSC}}\partial_{0+}^{\alpha,2,\gamma}\Pi(x,t) = {}_{\mathrm{H}}I_{0+}^{\alpha,\gamma}\left[\Pi^{(2)}(x,t)\right] = \int_0^t \aleph_\alpha\left[-\gamma(t-\tau)^\alpha\right]\Pi^{(2)}(x,\tau)\,\mathrm{d}\tau \tag{21} $$

Making use of the Laplace transform of eq. (19), we obtain:

$$ \frac{\partial^2\Pi(x,s)}{\partial x^2} - s^{2-\alpha}\left(1+\gamma s^{-\alpha}\right)^{-1}\Pi(x,s) = \frac{\sin(\pi x)}{s} \tag{22} $$

Suppose that:

$$ \Pi(x,s) = \mu_1 \mathrm{e}^{x\sqrt{s^{2-\alpha}\left(1+\gamma s^{-\alpha}\right)^{-1}}} + \mu_2 \mathrm{e}^{-x\sqrt{s^{2-\alpha}\left(1+\gamma s^{-\alpha}\right)^{-1}}} + \mu_3\sin(\pi x) + \mu_4\cos(\pi x) \tag{23} $$

where μ_1, μ_2, μ_3, and μ_4 are the constants, then:

$$ \left[-\pi^2 - s^{2-\alpha}\left(1+\gamma s^{-\alpha}\right)^{-1}\right]\left[\mu_3\sin(\pi x) + \mu_4\cos(\pi x)\right] = \frac{\sin(\pi x)}{s} \tag{24} $$

which leads:

$$ \mu_3 = \frac{1}{s}\frac{1}{-\pi^2 - s^{2-\alpha}\left(1+\gamma s^{-\alpha}\right)^{-1}},\quad \mu_4 = 0 \tag{25} $$

Thus, we obtain:

$$ \Pi(x,s) = \mu_1 \mathrm{e}^{x\sqrt{s^{2-\alpha}\left(1+\gamma s^{-\alpha}\right)^{-1}}} + \mu_2 \mathrm{e}^{-x\sqrt{s^{2-\alpha}\left(1+\gamma s^{-\alpha}\right)^{-1}}} + \frac{1}{s}\frac{1}{-\pi^2 - s^{2-\alpha}\left(1+\gamma s^{-\alpha}\right)^{-1}}\sin(\pi x) \tag{26} $$

which from eq. (20) implies:

$$ \mu_1 = 0,\ \mu_2 = 0 $$

and

$$ \Pi(x,s) = \frac{1}{s}\frac{1}{-\pi^2 - s^{2-\alpha}\left(1+\gamma s^{-\alpha}\right)^{-1}}\sin(\pi x) \tag{27} $$

With the use:

$$ \frac{1}{1+\dfrac{1}{\pi^2}s^{2-\gamma}\left(1+\nu s^{-\alpha}\right)^{-1}} = $$

$$ = \sum_{i=0}^{n}\left(-\frac{1}{\pi^2}\right)^i s^{(2-\gamma)i}\left(1+\nu s^{-\gamma}\right)^{-i},\quad \left|\frac{1}{\pi^2}s^{2-\gamma}\left(1+\nu s^{-\gamma}\right)^{-1}\right| < 1 \tag{28} $$

eq. (27) can be re-written:

$$ \Pi(x,s) = -\frac{\sin(\pi x)}{\pi^2}\sum_{i=0}^{n}\left(-\frac{1}{\pi^2}\right)^i s^{(2-\alpha)i-1}\left(1+\gamma s^{-\alpha}\right)^{-i} \tag{29} $$

Therefore, the analytical solution of eq. (19) takes the form:

$$ \Pi(x,t) = -\frac{\sin(\pi x)}{\pi^2}\sum_{j=0}^{n}\left(-\frac{1}{\pi^2}\right)^j t^{(2-\alpha)j}\left(1+\gamma s^{-\alpha}\right)^{-j} E^i_{\alpha,\alpha i-2i+1}\left(-\gamma t^\alpha\right) \tag{30} $$

Conclusion

We have investigated the general fractional-order calculus containing the Lorenzo-Hartley kernel. A general fractional-order model for the wave equation containing the general fractional-order derivative of the Liouville-Sonine-Caputo type containing the Lorenzo-Hartley kernel with the analytical solution is presented with the use of the Laplace transform. The result is accurate and efficient for designing of the complex, power-law and memory behaviors for the mining rock.

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Nomenclature

t – time co-ordinate, [m]
x – space co-ordinate, [m]

Greek symbol

α – fractional order, [–]

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