NEW GENERAL CALCULI WITH RESPECT TO ANOTHER FUNCTIONS APPLIED TO DESCRIBE THE NEWTON-LIKE DASHPOT MODELS IN ANOMALOUS VISCOELASTICITY

by

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Original scientific paper
https://doi.org/10.2298/TSCI180921260Y

In this article, we address the general derivatives and integrals with respect to another function for the first time. We consider the new perspective in anomalous viscoelasticity containing the general derivatives with respect to another functions containing the power-law, exponential, and logarithmic functions. The results are accurate and efficient in the descriptions of the complex behaviors of the materials.

Key words: *general derivatives, general integrals, general calculi, anomalous viscoelasticity, dashpot*

Introduction

The classical calculus (the well-known Newton-Leibniz calculus) of the functions with the integer order and the variable can be extended since the order becomes (I) any fractional-order and (II) any function order, and the variables of the functions can suggested as (III) the functions, or both are considered (see [1, 2]). When the condition (I) is valid, the fractional calculus of constant order has been presented in [3, 4]. When the condition (II) is given, the fractional calculus of variable order has been proposed in [5]. When (I) and (III) are considered, the fractional calculus of constant order with respect to another function have been reported in [6]. When (II) and (III) are employed, the fractional calculus of variable order with respect to another function has been presented in [7]. For more information other definitions of the calculi, see [8].

As applications of the Newton-Leibniz calculus, Newton proposed the dashpot element (called the Newtonian dashpot element) is given [9]:

$$\sigma(\tau) = \gamma \mathbb{D}^{(1)} \varepsilon(\tau)$$

where γ is the material parameter, ε – the strain, σ – the stress, and τ – the time. As a generalization of the Newtonian dashpot element, the fractional-order models in rheology were proposed in [10, 11] due to the Nutting behaviors in materials [12].

The idea that new calculi consist of the Newton-Leibniz calculus and (III) are has been not considerable. The goals of the paper are to present the general calculi with respect to another functions containing the power-law, exponential, and logarithmic functions, and to present the new applications to the Newton-like dashpot models in anomalous viscoelasticity.

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The general calculi with respect to another functions

Classical calculus

The classical derivative (the well-known Newton-Leibniz derivative) is defined:

$$\mathbb{D}^{(1)}\Theta(\tau)=\frac{\mathrm{d}\Theta(\tau)}{\mathrm{d}\tau}=\lim_{\Delta\tau\to 0}\frac{\Theta(\tau+\Delta\tau)-\Theta(\tau)}{\Delta\tau} \tag{1}$$

The classical integral (the well-known Newton-Leibniz integral) is defined:

$$\mathbb{I}^{(1)}\Phi(\tau)=\int_0^\tau \Phi(\tau)\,\mathrm{d}\tau \tag{2}$$

The relations between eqs. (1) and (2) are given:

$$\Theta(\tau)=\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau \Theta(\tau)\,\mathrm{d}\tau \tag{3}$$

and

$$\Theta(\tau)=\int_0^\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\Theta(\tau)\,\mathrm{d}\tau+\Theta(0) \tag{4}$$

The general calculus with respect to another function

Let $h^{(1)}(\tau)>0$. The general derivatives and integrals with respect to another functions are presented as follows.

The general derivative with respect to another function is defined:

$$\mathbb{D}^{(1)}_{\tau,h}\Theta(\tau)=\left(\frac{1}{\frac{\mathrm{d}h(\tau)}{\mathrm{d}\tau}}\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\Theta(\tau)=\left(\frac{\mathrm{d}\tau}{\mathrm{d}h(\tau)}\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\Theta(\tau)=\frac{\mathrm{d}}{\mathrm{d}h(\tau)}\Theta(\tau)=\frac{1}{h^{(1)}(\tau)}\frac{\mathrm{d}\Theta(\tau)}{\mathrm{d}\tau} \tag{5}$$

The general integral with respect to another function is defined:

$${}_0\mathbb{I}^{(1)}_{\tau,h}\Phi(\tau)=\int_0^\tau \Phi(\tau)h^{(1)}(\tau)\,\mathrm{d}\tau \tag{6}$$

The general derivative of higher order with respect to another function is defined:

$$\mathbb{D}^{(n)}_{\tau,h}\Theta(\tau)=\left(\frac{1}{h^{(1)}(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n\Theta(\tau) \tag{7}$$

Their relationships between eqs. (5) and (6) are presented:

$$\Theta(\tau)=\frac{1}{h^{(1)}(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau \Theta(\tau)h^{(1)}(\tau)\,\mathrm{d}\tau \tag{8}$$

and

$$\Theta(\tau)=\int_0^\tau\left[\left(\frac{1}{h^{(1)}(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\Theta(\tau)\right]h^{(1)}(\tau)\mathrm{d}\tau+\Theta(0) \tag{9}$$

More generally, the general integral with respect to another function is defined:

$$_a\mathbb{I}^{(1)}_{\tau,h}\Phi(\tau)=\int_a^\tau\Phi(\tau)h^{(1)}(\tau)\mathrm{d}\tau \tag{10}$$

In this case, we have their relationships:

$$\Theta(\tau)=\frac{1}{h^{(1)}(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_a^\tau\Theta(\tau)h^{(1)}(\tau)\mathrm{d}\tau \tag{11}$$

and

$$\Theta(\tau)=\int_a^\tau\left[\left(\frac{1}{h^{(1)}(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\Theta(\tau)\right]h^{(1)}(\tau)\mathrm{d}\tau+\Theta(a)=\int_a^\tau\frac{\mathrm{d}\Theta(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau+\Theta(a) \tag{12}$$

Remark that for $h(\tau)=\tau$, eqs. (5) and (6) become eqs. (1) and (2), respectively.

The general calculus with respect to power-law function

The general derivative with respect to power-law function, denoted as $h(\tau)=\tau^\alpha$ $(\tau\neq 0)$, is defined:

$$\mathbb{D}^{(1)}_{\tau,\tau^\alpha}\Theta(\tau)=\frac{1}{\alpha\tau^{\alpha-1}}\frac{\mathrm{d}\Theta(\tau)}{\mathrm{d}\tau} \tag{13}$$

where α are any real numbers.

The general integral with respect to power-law function is defined:

$$_0\mathbb{I}^{(1)}_{\tau,\tau^\alpha}\Phi(\tau)=\alpha\int_0^\tau\Phi(\tau)\tau^{\alpha-1}\mathrm{d}\tau \tag{14}$$

The general derivative of higher order with respect to power-law function is defined:

$$\mathbb{D}^{(n)}_{\tau,\tau^\alpha}\Theta(\tau)=\left(\frac{1}{\alpha\tau^{\alpha-1}}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n\Theta(\tau) \tag{15}$$

Their relationships between eqs. (13) and (14) are presented:

$$\Theta(\tau)=\frac{1}{\tau^{\alpha-1}}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau\Theta(\tau)\tau^{\alpha-1}\mathrm{d}\tau \tag{16}$$

and

$$\Theta(\tau)=\int_0^\tau\frac{\mathrm{d}\Theta(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau+\Theta(0) \tag{17}$$

More generally,

$$\Theta(\tau)=\frac{1}{\tau^{\alpha-1}}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{a}^{\tau}\Theta(\tau)\tau^{\alpha-1}\mathrm{d}\tau \tag{18}$$

and

$$\Theta(\tau)=\int_{a}^{\tau}\left[\left(\frac{1}{\tau^{\alpha-1}}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\Theta(\tau)\right]\tau^{\alpha-1}\mathrm{d}\tau+\Theta(a)=\int_{a}^{\tau}\frac{\mathrm{d}\Theta(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau+\Theta(a) \tag{19}$$

Remarek that Chen suggested that the power-law function as the Hausdorff measure the with the aid of the hypotheses of fractal invariance and fractal equivalence [13].

The general calculus with respect to exponential function

The general derivative with respect to exponential function, denoted as $h(\tau)=\mathrm{e}^{\lambda\tau}$ with real number λ, is defined:

$$\mathbb{D}_{\tau,e^{\lambda\tau}}^{(1)}\Theta(\tau)=\frac{1}{\lambda e^{\lambda\tau}}\frac{\mathrm{d}\Theta(\tau)}{\mathrm{d}\tau} \tag{20}$$

The general integral with respect to exponential function is defined:

$$_{0}\mathbb{I}_{\tau,e^{\lambda\tau}}^{(1)}\Theta(\tau)=\int_{0}^{\tau}\Theta(\tau)\lambda\mathrm{e}^{\lambda\tau}\mathrm{d}\tau \tag{21}$$

The general derivative of higher order with respect to another function is defined:

$$\mathbb{D}_{\tau,\mathrm{e}^{\lambda\tau}}^{(n)}\Theta(\tau)=\left(\frac{1}{\lambda\mathrm{e}^{\lambda\tau}}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^{n}\Theta(\tau) \tag{22}$$

Their relationships between eqs. (20) and (21) are given:

$$\Theta(\tau)=\frac{1}{\mathrm{e}^{\lambda\tau}}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{0}^{\tau}\Theta(\tau)\mathrm{e}^{\lambda\tau}\mathrm{d}\tau \tag{23}$$

and

$$\Theta(\tau)=\int_{0}^{\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\Theta(\tau)\mathrm{d}\tau+\Theta(0) \tag{24}$$

Generally, the general integral with respect to exponential function is defined:

$$_{a}\mathbb{I}_{\tau,\mathrm{e}^{\lambda\tau}}^{(1)}\Theta(\tau)=\int_{a}^{\tau}\Theta(\tau)\lambda\mathrm{e}^{\lambda\tau}\mathrm{d}\tau \tag{25}$$

$$\Theta(\tau)=\frac{1}{\mathrm{e}^{\lambda\tau}}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{a}^{\tau}\Theta(\tau)\mathrm{e}^{\lambda\tau}\mathrm{d}\tau \tag{26}$$

and

$$\Theta(\tau)=\int_{a}^{\tau}\left[\left(\frac{1}{\mathrm{e}^{\lambda\tau}}\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\Theta(\tau)\right]\mathrm{e}^{\lambda\tau}\mathrm{d}\tau+\Theta(a)=\int_{a}^{\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\Theta(\tau)\mathrm{d}\tau+\Theta(a) \tag{27}$$

The general calculus with respect to logarithmic function

The general derivative with respect to logarithmic function is defined:

$$\mathbb{D}_{\tau,\ln\tau}^{(1)}\Theta(\tau)=\tau\frac{\mathrm{d}\Theta(\tau)}{\mathrm{d}\tau} \tag{28}$$

The general integral with respect to logarithmic function is defined:

$${}_0\mathbb{I}_{\tau,\ln\tau}^{(1)}\Theta(\tau)=\int_0^\tau\frac{1}{\tau}\Theta(\tau)\mathrm{d}\tau \tag{29}$$

The general derivative of higher order with respect to logarithmic function is defined:

$$\mathbb{D}_{\tau,\ln\tau}^{(n)}\Theta(\tau)=\left(\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n\Theta(\tau) \tag{30}$$

Their relationships between eqs. (28) and (29) can be written:

$$\Theta(\tau)=\left(\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\int_0^\tau\frac{1}{\tau}\Theta(\tau)\mathrm{d}\tau=\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau\frac{1}{\tau}\Theta(\tau)\mathrm{d}\tau \tag{31}$$

and

$$\Theta(\tau)=\int_0^\tau\left(\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\Theta(\tau)\frac{1}{\tau}\mathrm{d}\tau+\Theta(0)=\int_0^\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\Theta(\tau)\mathrm{d}\tau+\Theta(0) \tag{32}$$

More generally,

$${}_a\mathbb{I}_{\tau,\ln\tau}^{(1)}\Theta(\tau)=\int_a^\tau\frac{1}{\tau}\Theta(\tau)\mathrm{d}\tau \tag{33}$$

$$\Theta(\tau)=\left(\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\int_a^\tau\frac{1}{\tau}\Theta(\tau)\mathrm{d}\tau=\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\int_a^\tau\frac{1}{\tau}\Theta(\tau)\mathrm{d}\tau \tag{34}$$

and

$$\Theta(\tau)=\int_a^\tau\left(\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\Theta(\tau)\frac{1}{\tau}\mathrm{d}\tau+\Theta(a)=\int_a^\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\Theta(\tau)\mathrm{d}\tau+\Theta(a) \tag{35}$$

The Newton-like dashpot elements containing the general derivatives with respect to another functions

Model 1. The Newton-like dashpot element containing the general derivative with respect to another function is given:

$$\sigma(\tau)=\frac{\gamma}{h^{(1)}(\tau)}\frac{\mathrm{d}\varepsilon(\tau)}{\mathrm{d}\tau}=\gamma\mathbb{D}_{\tau,h}^{(1)}\varepsilon(\tau) \tag{36}$$

where γ is the material parameter, ε – the strain, σ – the stress, and τ – the time.

When $\sigma(0)=\sigma_0$, eq. (36) can be presented:

$$\sigma_0=\gamma\mathbb{D}_{\tau,h}^{(1)}\varepsilon(\tau) \tag{37}$$

with the solution

$$\varepsilon(\tau)=\int_0^\tau \frac{\sigma_0}{\gamma} h^{(1)}(\tau)\,\mathrm{d}\tau \tag{38}$$

Model 2. The Newton-like dashpot element containing the general derivative with respect to power-law function is given [14]:

$$\sigma(\tau)=\gamma \mathbb{D}_{\tau,\tau^\alpha}^{(1)} \varepsilon(\tau) \tag{39}$$

where γ is the material parameter, ε – the strain, σ – the stress, and τ – the time.

When $\sigma(0)=\sigma_0$, eq. (39) can be written:

$$\sigma_0=\gamma \mathbb{D}_{\tau,\tau^\alpha}^{(1)} \varepsilon(\tau) \tag{40}$$

with the solution

$$\varepsilon(\tau)=\frac{\sigma_0}{\gamma}\tau^\alpha \tag{41}$$

Model 3. The Newton-like dashpot element containing the general derivative with respect to exponential function is given:

$$\sigma(\tau)=\gamma \mathbb{D}_{\tau,e^{\lambda\tau}}^{(1)} \varepsilon(\tau) \tag{42}$$

where γ is the material parameter, ε – the strain, σ – the stress, and τ – the time.

When $\sigma(0)=\sigma_0$, eq. (42) can be given:

$$\sigma_0=\gamma \mathbb{D}_{\tau,e^{\lambda\tau}}^{(1)} \varepsilon(\tau) \tag{43}$$

with the solution

$$\varepsilon(\tau)=\frac{\sigma_0}{\gamma}\mathrm{e}^{\lambda\tau} \tag{44}$$

Model 4. The Newton-like dashpot element containing the general derivative with respect to logarithmic function is considered:

$$\sigma(\tau)=\gamma \mathbb{D}_{\tau,\ln\tau}^{(1)} \varepsilon(\tau) \tag{45}$$

where γ is the material parameter, ε – the strain, σ – the stress, and τ – the time.

When $\sigma(0)=\sigma_0$, eq. (45) can be presented:

$$\sigma_0=\gamma \mathbb{D}_{\tau,\ln\tau}^{(1)} \varepsilon(\tau) \tag{46}$$

with the solution

$$\varepsilon(\tau)=\frac{\sigma_0}{\gamma}\ln\tau \tag{47}$$

Conclusion

In this work, as the extended version of the well-known Newton-Leibniz calculus, we have proposed the general calculi with respect to another functions, containing the general derivative and integral with respect to another functions, such as power-law, exponential, and logarithmic functions for the first time. The Newton-like dashpot models in anomalous visco-elasticity were considered in detail. The formulae could be used to model the complex problems for the solid materials in the deep underground engineering.

Acknowledgment

This work was supported by the financial support of the 333 Project of Jiangsu Province, People's Republic of China (Grant No. BRA2018320), and the Yue-Qi Scholar of the China University of Mining and Technology (Grant No. 102504180004). This paper is devoted to the 110th. Anniversary of the 1909 Founding of China University of Mining and Technology.

Nomenclature

x – space co-ordinate, [m]

Greek symbols

γ – material parameter, [Pa s]

$\varepsilon(\tau)$ – strain, [–]

$\sigma(\tau)$ – stress, [Pa]

τ – time, [s]

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Paper submitted: September 21, 2018
Paper revised: January 13, 2019
Paper accepted: February 18, 2019